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MAPPINGS PRESERVING SOME DISTANCES

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Abstract
We deal with mappings which preserve unit (some positive) distance, as well as with mappings contracting (expanding) some distance. In particular, we show that a mapping preserving three positive distances is an isometry.

1. Introduction

The theory of isometries has its origin in a paper by S. Mazur and S. Ulam (1932) who proved that every isometry of a normed real vector space onto a normed real vector space is a linear isometry up to translation. Later, A. D. Alexandrov initiated the study of conditions (in terms of preserving certain distances) guaranteeing isometry. F. S. Beckman and D. A. Quarles in [1] proved that every map from $\mathbb{R}^n$ to $\mathbb{R}^n$ $(2 \leq n < \infty)$ preserving the unit distance is an isometry.

Let X and Y be normed real vector spaces. A mapping $f : X \to Y$ is called an *isometry* if f satisfies

$$\|f(x) - f(y)\| = \|x - y\| \quad \text{for all} \quad x, y \in X.$$

A distance $\rho > 0$ is said to be *contractive* (or non-expanding) by $f : X \to Y$ if $\|x - y\| = \rho$ always implies $\|f(x) - f(y)\| \leq \rho$. Similarly, a distance ρ is said to be *extensive* (or non-shrinking) by f if the inequality $\|f(x) - f(y)\| \geq \rho$ is true for all $x, y \in X$ with $\|x - y\| = \rho$. We say that ρ is *conservative* (or preserved) by f if ρ is contractive and extensive by f simultaneously. Then, a mapping $f : X \to Y$ is called preserving the distance ρ if for all x, y of X with $\|x - y\| = \rho$ we have $\|f(x) - f(y)\| = \rho$.

Consider the following conditions for $f : X \to Y$ introduced for the first time by Rassias and Šemrl [11]: *distance one preserving property* (DOPP) and *strongly distance one preserving property* (SDOPP).

$$\forall x, y \in X : \quad \|x - y\| = 1 \quad \Rightarrow \quad \|f(x) - f(y)\| = 1 \qquad \text{(DOPP)}$$

$$\forall x, y \in X : \quad \|x - y\| = 1 \quad \Leftrightarrow \quad \|f(x) - f(y)\| = 1 \qquad \text{(SDOPP)}$$

A number of authors have discussed Alexandrov's problem under certain additional conditions for a given mapping satisfying DOPP in order to be an isometry and to have posed several interesting and new open problems (cf. [1, 4–10]). Even if X, Y are normed vector spaces, the above problem is not easy to solve. For example, the following question posed by Rassias has not been answered yet: *Is a mapping f from $\mathbb{R}^2$ to $\mathbb{R}^3$ preserving unit distance necessarily an isometry?*

If $f : X \to Y$ is a contractive mapping, Rassias and Šemrl [11] proved the following:

Theorem 1. Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|f(x) - f(y)\| \leq \|x - y\| \text{ for all } x, y \in X.$$

Assume also that f is a surjective mapping satisfying SDOPP. Then f is a linear isometry up to translation.

In 1985, W. Benz [3] introduced a sufficient condition under which a mapping, with a contractive distance ρ and an extensive one $N\rho$, is an isometry (see also [4]):

Theorem 2. Let X and Y be real normed vector spaces such that Y is strictly convex[1] and X has dimension greater than one. Suppose that

[1] A normed vector space Y is called strictly convex, if for each pair a, b of nonzero elements in Y such that $\|a + b\| = \|a\| + \|b\|$, it follows that $a = \gamma b$ for some $\gamma > 0$.

$f : X \to Y$ is a mapping and $N \geq 2$ is a fixed integer. If a distance $\rho > 0$ is contractive and $N\rho$ is extensive by f, then f is a linear isometry up to translation.

2. Mappings with conservative distances

For real normed vector spaces, we prove the following result based on Theorem 2 above:

Theorem 3. Let X and Y be real normed vector spaces such that Y is strictly convex and X has dimension greater than one. Suppose that $f : X \to Y$ is a contractive mapping satisfying DOPP. Then f is a linear isometry up to translation.

Proof. From the mentioned theorem of Benz, it follows that if a distance $\rho > 0$ and $N\rho$ are conservative by f for some integer $N \geq 2$, then f is an isometry. By the hypothesis, unit distance is conservative by f. We will prove that distance $\frac{1}{2}$ is conservative by f too. Let $x, y \in X$ with $\|x - y\| = \frac{1}{2}$. Then, there exist z of X with $z - x = 2(y - x)$ and $\|z - x\| = 1$, $\|z - y\| = \frac{1}{2}$. Since f is a contractive mapping, then

$$\forall x, y \in X : \quad \|f(x) - f(y)\| \leq \|x - y\|.$$

By a triangle inequality, we have

$$\begin{aligned}\frac{1}{2} = \|z - y\| \geq \|f(z) - f(y)\| &\geq \|f(z) - f(x)\| - \|f(y) - f(x)\| \\ &\geq 1 - \|x - y\| = \frac{1}{2}\ .\end{aligned}$$

Therefore $\|f(z) - f(y)\| = \frac{1}{2}$. Similarly, for $f(x)$ and $f(y)$ we have

$$\begin{aligned}\frac{1}{2} = \|x - y\| \geq \|f(x) - f(y)\| &\geq \|f(z) - f(x)\| - \|f(z) - f(y)\| \\ &\geq 1 - \|z - y\| = \frac{1}{2}\ ,\end{aligned}$$

it follows that $\|f(x) - f(y)\| = \frac{1}{2}$ and

$$\|f(z) - f(x)\| = \|f(z) - f(y)\| + \|f(y) - f(x)\| = \frac{1}{2} + \frac{1}{2} = 1.$$

Because Y is strictly convex, then for the triple $f(x)$, $f(y)$, $f(z)$ of Y we obtain $f(z) - f(y) = f(y) - f(x)$. Hence,

$$f(y) = \frac{f(x) + f(z)}{2} \quad \text{and} \quad \|f(y) - f(x)\| = \frac{1}{2}\ .$$

So f also preserves distance $\frac{1}{2}$. By Theorem 2, f is a linear isometry up to translation. □

If a mapping $f : X \to Y$ preserves two distances with a noninteger ratio, and X and Y are real normed vector spaces such that Y is strictly convex and X has dimension greater than one, it is an open problem whether or not f must be an isometry (see [9]). However, if f preserves three positive distances, one can prove the following (cf. [12])

Theorem 4. *Let X and Y be real normed vector spaces such that Y is strictly convex and X has dimension greater than one. Suppose that $f : X \to Y$ satisfies the property that f preserves three distances ρ, σ, $\rho + \sigma$, where ρ and σ are any positive constants. Then f is a linear isometry up to translation.*

Proof. (**a**) Let $x, y \in X$ with $\|x - y\| = 2\rho + \sigma$. Set

$$\tilde{x} = x + \frac{\rho}{2\rho + \sigma}(y - x), \quad \tilde{y} = x + \frac{\rho + \sigma}{2\rho + \sigma}(y - x).$$

Then $\|\tilde{x} - x\| = \|y - \tilde{y}\| = \rho$, $\|\tilde{x} - \tilde{y}\| = \sigma$, $\|y - \tilde{x}\| = \|\tilde{y} - x\| = \rho + \sigma$. Since f preserves distances ρ, σ and $\rho + \sigma$, then

$$\|f(\tilde{x}) - f(x)\| = \|f(y) - f(\tilde{y})\| = \rho, \ \|f(\tilde{x}) - f(\tilde{y})\| = \sigma,$$

$$\|f(y) - f(\tilde{x})\| = \|f(\tilde{y}) - f(x)\| = \rho + \sigma.$$

Hence,

$$\|f(\tilde{y}) - f(x)\| = \|f(\tilde{y}) - f(\tilde{x})\| + \|f(\tilde{x}) - f(x)\| = \rho + \sigma,$$

$$\|f(y) - f(\tilde{x})\| = \|f(y) - f(\tilde{y})\| + \|f(\tilde{y}) - f(\tilde{x})\| = \rho + \sigma.$$

By the hypothesis, Y is strictly convex normed vector space, therefore we obtain

$$f(\tilde{x}) - f(x) = \frac{\rho}{\sigma}\big(f(\tilde{y}) - f(\tilde{x})\big), \quad f(\tilde{y}) - f(\tilde{x}) = \frac{\sigma}{\rho}\big(f(y) - f(\tilde{y})\big).$$

Thus, we get

$$f(x) = \frac{\rho+\sigma}{\sigma} f(\tilde{x}) - \frac{\rho}{\sigma} f(\tilde{y}) \quad \text{and} \quad f(y) = \frac{\rho+\sigma}{\sigma} f(\tilde{y}) - \frac{\rho}{\sigma} f(\tilde{x}).$$

Hence $\|f(x) - f(y)\| = 2\rho + \sigma$ for all $x, y \in X$, where $\|x - y\| = 2\rho + \sigma$. We conclude that f also preserves the distance $2\rho + \sigma$.

(**b**) Let $x, y \in X$ with $\|x - y\| = 2\rho + 2\sigma$. Set

$$\tilde{x} = x + \frac{\rho+\sigma}{2\rho+2\sigma}(y - x), \quad \tilde{y} = x + \frac{2\rho+\sigma}{2\rho+2\sigma}(y - x).$$

Then $\|\tilde{x} - x\| = \|y - \tilde{x}\| = \rho + \sigma$, $\|\tilde{x} - \tilde{y}\| = \rho$, $\|y - \tilde{y}\| = \sigma$, $\|\tilde{y} - x\| = 2\rho + \sigma$. Since f preserves distances ρ, σ, $\rho + \sigma$ and $2\rho + \sigma$, in a similar way, we obtain that $\|f(x) - f(y)\| = 2\rho + 2\sigma$. Hence, f preserves the distance $2(\rho + \sigma)$.

By Theorem 2, we conclude that f is a linear isometry up to translation. □

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