MAPPINGS PRESERVING SOME DISTANCES

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Abstract
We deal with mappings which preserve unit (some positive) distance, as well as with mappings contracting (expanding) some distance. In particular, we show that a mapping preserving three positive distances is an isometry.

1. Introduction

The theory of isometries has its origin in a paper by S. Mazur and S. Ulam (1932) who proved that every isometry of a normed real vector space onto a normed real vector space is a linear isometry up to translation. Later, A. D. Alexandrov initiated the study of conditions (in terms of preserving certain distances) guaranteeing isometry. F. S. Beckman and D. A. Quarles in [1] proved that every map from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($2 \leq n < \infty$) preserving the unit distance is an isometry.

Let $X$ and $Y$ be normed real vector spaces. A mapping $f : X \to Y$ is called an isometry if $f$ satisfies

$$\|f(x) - f(y)\| = \|x - y\| \text{ for all } x, y \in X.$$  

A distance $\rho > 0$ is said to be contractive (or non-expanding) by $f : X \to Y$ if $\|x - y\| = \rho$ always implies $\|f(x) - f(y)\| \leq \rho$. Similarly, a distance $\rho$ is said to be extensive (or non-shrinking) by $f$ if the
inequality \( \|f(x) - f(y)\| \geq \rho \) is true for all \( x, y \in X \) with \( \|x - y\| = \rho \). We say that \( \rho \) is conservative (or preserved) by \( f \) if \( \rho \) is contractive and extensive by \( f \) simultaneously. Then, a mapping \( f : X \to Y \) is called preserving the distance \( \rho \) if for all \( x, y \) of \( X \) with \( \|x - y\| = \rho \) we have \( \|f(x) - f(y)\| = \rho \).

Consider the following conditions for \( f : X \to Y \) introduced for the first time by Rassias and Šemrl [11]: distance one preserving property (DOPP) and strongly distance one preserving property (SDOPP).

\[
\forall x, y \in X : \quad \|x - y\| = 1 \quad \Rightarrow \quad \|f(x) - f(y)\| = 1 \quad \text{(DOPP)}
\]

\[
\forall x, y \in X : \quad \|x - y\| = 1 \quad \Leftrightarrow \quad \|f(x) - f(y)\| = 1 \quad \text{(SDOPP)}
\]

A number of authors have discussed Alexandrov’s problem under certain additional conditions for a given mapping satisfying DOPP in order to be an isometry and to have posed several interesting and new open problems (cf. [1, 4–10]). Even if \( X, Y \) are normed vector spaces, the above problem is not easy to solve. For example, the following question posed by Rassias has not been answered yet: Is a mapping \( f \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) preserving unit distance necessarily an isometry?

If \( f : X \to Y \) is a contractive mapping, Rassias and Šemrl [11] proved the following:

**Theorem 1.** Let \( X \) and \( Y \) be real normed vector spaces such that one of them has dimension greater than one. Suppose that \( f : X \to Y \) is a mapping satisfying

\[
\|f(x) - f(y)\| \leq \|x - y\| \quad \text{for all} \quad x, y \in X.
\]

Assume also that \( f \) is a surjective mapping satisfying SDOPP. Then \( f \) is a linear isometry up to translation.

In 1985, W. Benz [3] introduced a sufficient condition under which a mapping, with a contractive distance \( \rho \) and an extensive one \( N\rho \), is an isometry (see also [4]):

**Theorem 2.** Let \( X \) and \( Y \) be real normed vector spaces such that \( Y \) is strictly convex\(^1\) and \( X \) has dimension greater than one. Suppose that \( f : X \to Y \) is a mapping satisfying

\[
\|f(x) - f(y)\| \leq \|x - y\| \quad \text{for all} \quad x, y \in X.
\]

Assume also that \( f \) is a surjective mapping satisfying SDOPP. Then \( f \) is a linear isometry up to translation.

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\(^1\) A normed vector space \( Y \) is called strictly convex, if for each pair \( a, b \) of nonzero elements in \( Y \) such that \( \|a + b\| = \|a\| + \|b\| \), it follows that \( a = \gamma b \) for some \( \gamma > 0 \).
$f : X \to Y$ is a mapping and $N \geq 2$ is a fixed integer. If a distance $\rho > 0$ is contractive and $N\rho$ is extensive by $f$, then $f$ is a linear isometry up to translation.

2. Mappings with conservative distances

For real normed vector spaces, we prove the following result based on Theorem 2 above:

**Theorem 3.** Let $X$ and $Y$ be real normed vector spaces such that $Y$ is strictly convex and $X$ has dimension greater than one. Suppose that $f : X \to Y$ is a contractive mapping satisfying DOPP. Then $f$ is a linear isometry up to translation.

**Proof.** From the mentioned theorem of Benz, it follows that if a distance $\rho > 0$ and $N\rho$ are conservative by $f$ for some integer $N \geq 2$, then $f$ is an isometry. By the hypothesis, unit distance is conservative by $f$. We will prove that distance $\frac{1}{2}$ is conservative by $f$ too. Let $x, y \in X$ with $\|x - y\| = \frac{1}{2}$. Then, there exist $z$ of $X$ with $z - x = 2(y - x)$ and $\|z - x\| = 1, \|z - y\| = \frac{1}{2}$. Since $f$ is a contractive mapping, then

$$\forall x, y \in X : \|f(x) - f(y)\| \leq \|x - y\|.$$ 

By a triangle inequality, we have

$$\frac{1}{2} = \|z - y\| \geq \|f(z) - f(y)\| \geq \|f(z) - f(x)\| - \|f(y) - f(x)\| \geq 1 - \|x - y\| = \frac{1}{2}.$$ 

Therefore $\|f(z) - f(y)\| = \frac{1}{2}$. Similarly, for $f(x)$ and $f(y)$ we have

$$\frac{1}{2} = \|x - y\| \geq \|f(x) - f(y)\| \geq \|f(z) - f(x)\| - \|f(z) - f(y)\| \geq 1 - \|z - y\| = \frac{1}{2},$$

it follows that $\|f(x) - f(y)\| = \frac{1}{2}$ and

$$\|f(z) - f(x)\| = \|f(z) - f(y)\| + \|f(y) - f(x)\| = \frac{1}{2} + \frac{1}{2} = 1.$$
Because $Y$ is strictly convex, then for the triple $f(x), f(y), f(z)$ of $Y$ we obtain $f(z) - f(y) = f(y) - f(x)$. Hence,

$$f(y) = \frac{f(x) + f(z)}{2} \quad \text{and} \quad \|f(y) - f(x)\| = \frac{1}{2}.$$ 

So $f$ also preserves distance $\frac{1}{2}$. By Theorem 2, $f$ is a linear isometry up to translation. □

If a mapping $f : X \to Y$ preserves two distances with a noninteger ratio, and $X$ and $Y$ are real normed vector spaces such that $Y$ is strictly convex and $X$ has dimension greater than one, it is an open problem whether or not $f$ must be an isometry (see [9]). However, if $f$ preserves three positive distances, one can prove the following (cf. [12])

**Theorem 4.** Let $X$ and $Y$ be real normed vector spaces such that $Y$ is strictly convex and $X$ has dimension greater than one. Suppose that $f : X \to Y$ satisfies the property that $f$ preserves three distances $\rho, \sigma, \rho + \sigma$, where $\rho$ and $\sigma$ are any positive constants. Then $f$ is a linear isometry up to translation.

**Proof.** (a) Let $x, y \in X$ with $\|x - y\| = 2\rho + \sigma$. Set

$$\tilde{x} = x + \frac{\rho}{2\rho + \sigma}(y - x), \quad \tilde{y} = x + \frac{\rho + \sigma}{2\rho + \sigma}(y - x).$$

Then $\|\tilde{x} - x\| = \|y - \tilde{y}\| = \rho, \|\tilde{x} - \tilde{y}\| = \sigma, \|y - \tilde{x}\| = \|\tilde{y} - x\| = \rho + \sigma$. Since $f$ preserves distances $\rho, \sigma$ and $\rho + \sigma$, then

$$\|f(\tilde{x}) - f(x)\| = \|f(y) - f(\tilde{y})\| = \rho, \|f(\tilde{x}) - f(\tilde{y})\| = \sigma,$$

$$\|f(y) - f(x)\| = \|f(\tilde{y}) - f(x)\| = \rho + \sigma.$$

Hence,

$$\|f(\tilde{y}) - f(x)\| = \|f(\tilde{y}) - f(\tilde{x})\| + \|f(\tilde{x}) - f(x)\| = \rho + \sigma,$$

$$\|f(y) - f(\tilde{x})\| = \|f(y) - f(\tilde{y})\| + \|f(\tilde{y}) - f(\tilde{x})\| = \rho + \sigma.$$

By the hypothesis, $Y$ is strictly convex normed vector space, therefore we obtain

$$f(\tilde{x}) - f(x) = \frac{\rho}{\sigma}(f(\tilde{y}) - f(\tilde{x})), \quad f(\tilde{y}) - f(\tilde{x}) = \frac{\sigma}{\rho}(f(y) - f(\tilde{y})).$$
Thus, we get
\[ f(x) = \frac{\rho + \sigma}{\sigma} f(\tilde{x}) - \frac{\rho}{\sigma} f(\tilde{y}) \quad \text{and} \quad f(y) = \frac{\rho + \sigma}{\sigma} f(\tilde{y}) - \frac{\rho}{\sigma} f(\tilde{x}). \]
Hence \( \| f(x) - f(y) \| = 2\rho + \sigma \) for all \( x, y \in X \), where \( \| x - y \| = 2\rho + \sigma \).
We conclude that \( f \) also preserves the distance \( 2\rho + \sigma \).

(b) Let \( x, y \in X \) with \( \| x - y \| = 2\rho + 2\sigma \). Set
\[ \tilde{x} = x + \frac{\rho + \sigma}{2\rho + 2\sigma} (y - x), \quad \tilde{y} = x + \frac{2\rho + \sigma}{2\rho + 2\sigma} (y - x). \]
Then \( \| \tilde{x} - x \| = \| y - \tilde{x} \| = \rho + \sigma, \quad \| \tilde{x} - \tilde{y} \| = \rho, \quad \| y - \tilde{y} \| = \sigma, \quad \| \tilde{y} - x \| = 2\rho + \sigma \). Since \( f \) preserves distances \( \rho, \sigma, \rho + \sigma \) and \( 2\rho + \sigma \), in a similar way, we obtain that \( \| f(x) - f(y) \| = 2\rho + 2\sigma \). Hence, \( f \) preserves the distance \( 2(\rho + \sigma) \).

By Theorem 2, we conclude that \( f \) is a linear isometry up to translation. \( \square \)

References


