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COMPARISON OF FUZZY NUMBERS RANKING METHODS

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Abstract
The paper compares fuzzy numbers ranking methods used in recent papers. We present modified attributes and a canonical representation of any fuzzy number by a trapezoidal one. Moreover, fuzzy numbers with the same attributes are ranked with the use of new attribute describing a difference between the considered fuzzy number and its canonical representation.

1. Introduction

Since fuzzy numbers do not form a natural linear order, different comparison approaches are used. We remind a few methods of ranking which can be found in the literature. Particularly we examine the method proposed in [8]. We present an example which shows why fuzzy numbers cannot be compared absolutely, but only relatively, what causes an impossibility of creation the unique ranking for two fixed fuzzy numbers in this method. We also compare it with another method with metric approach described in [7]. Finally we examine conformity of preference relation introduced in fuzzy numbers family with the usual order on real line, what leads to a description of certain equivalence relation. We modify attributes defined in [2] and [3], and also introduce new attribute in order to rank fuzzy numbers which belong to the same equivalence class.

2. Fuzzy numbers

First, we remaind necessary facts on fuzzy numbers.

Definition 1 ([5]). By fuzzy number we call a convex fuzzy set $A : \mathbb{R} \to [0,1]$ (i.e. $\underset{x,y\in\mathbb{R}}{\forall} \underset{\lambda\in(0,1)}{\forall} A(\lambda x + (1-\lambda)y) \geq A(x) \wedge A(y))$ fulfilling the conditions:
A is upper semicontinuous, $\bigvee A = 1$, supp$A$ is bounded.
The family of all fuzzy numbers will be denoted by FN.

After Goetschel and Voxman we characterize fuzzy numbers by the following

Theorem 1 ([6]). *Let* $\underline{a}, \overline{a} : [0,1] \longrightarrow \mathbb{R}$ *fulfill following conditions:*

$$\underline{a} \text{ is increasing and bounded,} \tag{1}$$

$$\overline{a} \text{ is decreasing and bounded,} \tag{2}$$

$$\underline{a}(1) \leq \overline{a}(1), \tag{3}$$

$$\underline{a}, \overline{a} \text{ are left continuous in } [0,1], \tag{4}$$

$$\underline{a}, \overline{a} \text{ are right continuous in } 0. \tag{5}$$

Then $A : \mathbb{R} \longrightarrow [0,1]$ *defined as*

$$A(x) = \sup\{\alpha : \underline{a}(\alpha) \leq x \leq \overline{a}(\alpha)\}, x \in \mathbb{R}$$

is a fuzzy number given in parametric form $(\underline{a}, \overline{a})$.
Moreover, if $A = (\underline{a}, \overline{a})$ *is a fuzzy number, then* $\underline{a}$ *and* $\overline{a}$ *fulfill conditions* (1) – (5).

This characterization makes calculations of arithmetical operations more clear. Moreover, it gives a simple method of representing fuzzy numbers by their arms, specially useful for those which are not continuous (or even crisp).

We can unambiguously signify trapezoidal fuzzy numbers $T = (\underline{t}, \overline{t})$ (this means those which have linear functions $\underline{t}, \overline{t}$) by a quadruple $T = \langle a, b, c, d\rangle$, where $a \leq b \leq c \leq d$, $T(a) = T(d) = 0$, $T(b) = T(c) = 1$. Special class of trapezoidal fuzzy numbers are symmetrical trapezoidal fuzzy numbers. Here it is enough to use three parameters and signify symmetrical trapezoidal fuzzy number by

$$T_s = \langle m - c - d, m - c, m + c, m + c + d\rangle. \tag{6}$$

A triangular fuzzy number $\hat{A}$ is a special case of trapezoidal fuzzy number $A = \langle a_1, a_2, a_3, a_4 \rangle$, where $a_2 = a_3$. We will signify a triangular fuzzy number by a triple $\hat{A} = \langle \hat{a_1}, \hat{a_2}, \hat{a_3} \rangle$ where $\hat{a_1} = a_1$, $\hat{a_2} = a_2 = a_3$, $\hat{a_3} = a_4$.

Example 1. Let $A \in \text{FN}$, $A = (2\alpha + \frac{1}{2}, 2\frac{1}{2}), \alpha \in [0, 1]$.

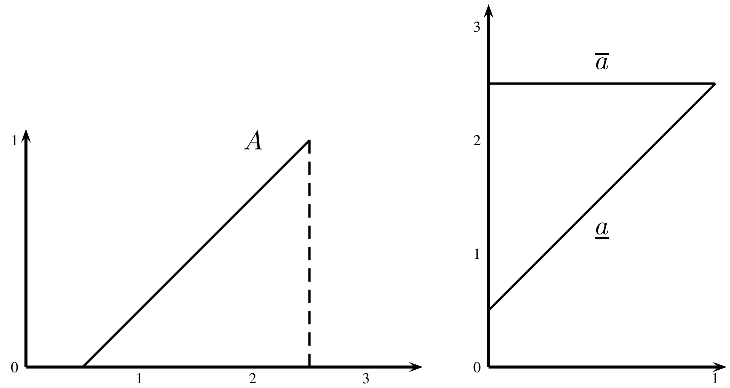

Figure 1: A fuzzy number A and its parametric representation $(\underline{a}, \overline{a})$.

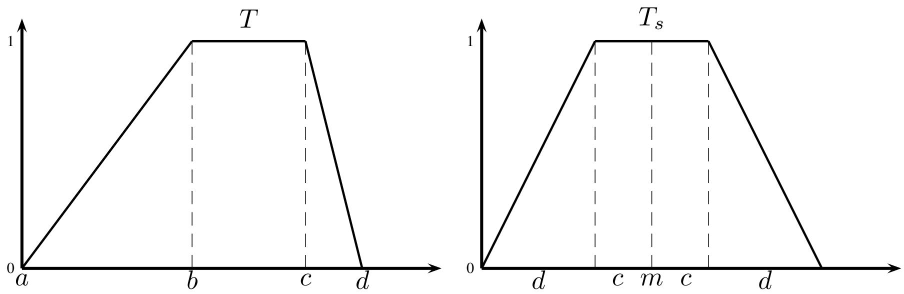

Figure 2: Trapezoidal and symmetrical trapezoidal fuzzy numbers.

3. Ranking methods

In this section we remind two different methods of fuzzy numbers ranking. The first one was presented in [8].

Let $A, B \in \text{FN}$ be such that $\text{supp}A, \text{supp}B \neq \{m\}$. We need preference functions given as follows:

$$G_A(x) := \frac{\int_x^\infty A(t)dt}{\int_{\text{supp}A} A(t)dt}, \quad G_B(x) = \frac{\int_x^\infty B(t)dt}{\int_{\text{supp}B} B(t)dt}$$

which evaluate the fuzzy numbers A, B at each point $x \in \mathbb{R}$.

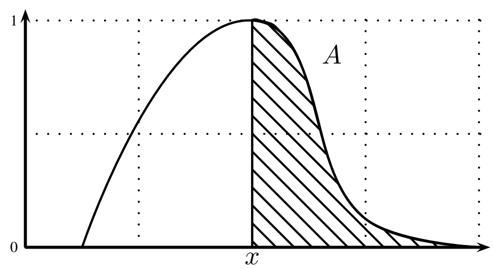

Figure 3: $G_A(x)$ is the ratio of shaded area over the total area.

We define the sets

$\Omega_A := \{x \in \mathbb{R} : G_A(x) > G_B(x)\}$, $\Omega_B = \{x \in \mathbb{R} : G_B(x) > G_A(x)\}$,

$\Omega := \text{supp}A \cup \text{supp}B$.

Let us notice that Ω_A is a set composed of separated intervals, because A is upper semicontinuous; similarly Ω_B. Therefore, let $|\Omega|, |\Omega_A|, |\Omega_B|$ be lengths of sums of the component intervals of $\Omega, \Omega_A, \Omega_B$, respectively. Now we can calculate the preference ratio for A, B as follows:

$$R(A) := \frac{|\Omega_A|}{|\Omega|}, \quad R(B) := \frac{|\Omega_B|}{|\Omega|}.$$

In other words, $R(A)$ represents the percentage of Ω that A is more preferred then B. Using preference ratios we define the relation

$$A \prec_1 B \Leftrightarrow R(A) < R(B). \tag{7}$$

This method, having other advantages, fails very important property of comparison - transitivity.

Example 2. Relation $\prec_1$ is not transitive. Let $A, B, C \in \mathrm{FN}$ be triangular fuzzy numbers given in parametric form:
$A = (19\alpha+1, 20)$, $B = (11, -9\alpha+20)$, $C = (1, -30\alpha+31)$, for $\alpha \in [0, 1]$.
We will show that $A \prec_1 B$, $B \prec_1 C$, $C \prec_1 A$.

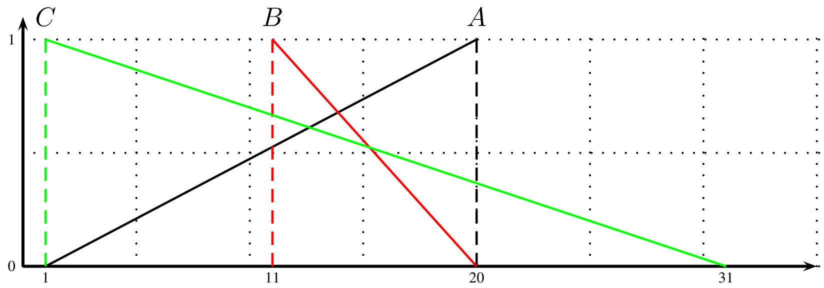

Figure 4: Fuzzy numbers $A = \langle 1, 20, 20\rangle$, $B = \langle 11, 11, 20\rangle$, $C = \langle 1, 1, 31\rangle$.

For triangular numbers, it is easy to calculate that

$$\int_{\mathrm{supp}A} A(t)dt = \int_0^{20} A(t)dt = \frac{1}{2}\cdot 1 \cdot 20 = 10,$$

$$\int_{\mathrm{supp}B} B(t)dt = \int_{11}^{20} B(t)dt = \frac{1}{2}\cdot 1 \cdot 9 = 4.5,$$

$$\int_{\mathrm{supp}C} C(t)dt = \int_1^{31} C(t)dt = \frac{1}{2}\cdot 1 \cdot 30 = 15.$$

By the same procedure, we calculate

$$\int_x^{\infty} A(t)dt = \int_x^{20} A(t)dt = \int_0^{20} A(t)dt - \int_0^{x} A(t)dt = 10 - \frac{1}{2}x \cdot \frac{1}{20}x =$$

$$\frac{1}{40}(-x^2+400),$$

$$\int_x^{\infty} B(t)dt = \int_x^{20} B(t)dt = \frac{1}{2}\cdot(20-x)\cdot(-\frac{1}{9}x+\frac{20}{9}) = \frac{1}{18}(x^2-40x+400),$$

$$\int_x^{\infty} C(t)dt = \int_x^{31} C(t)dt = \frac{1}{2}\cdot(31-x)\cdot(-\frac{1}{30}x+\frac{31}{30}) = \frac{1}{60}(x^2-62x+961).$$

Thus,

$$G_A(x) = \begin{cases} 1, & x \in (-\infty, 0), \\ \frac{1}{400}(-x^2 + 400), & x \in [0, 20], \\ 0, & x \in (20, \infty), \end{cases}$$

$$G_B(x) = \begin{cases} 1, & x \in (-\infty, 11), \\ \frac{1}{81}(x^2 - 40x + 400), & x \in [11, 20], \\ 0, & x \in (20, \infty), \end{cases}$$

$$G_C(x) = \begin{cases} 1, & x \in (-\infty, 1), \\ \frac{1}{900}(x^2 - 62x + 961), & x \in [11, 31], \\ 0, & x \in (31, \infty). \end{cases}$$

It is obvious that $A \prec_1 B$. All over the interval $(0, 11)$ we have $G_B(x) = 1 > G_A(x)$, thus $|\Omega_B| \geq 11$. Because of $\Omega = [0, 20] \cup [10, 20] = [0, 20]$ (so $|\Omega| = 20$), we have $R(B) = \frac{|\Omega_B|}{|\Omega|} > \frac{1}{2}$, this means that $R(B) > R(A)$. Now we consider fuzzy numbers B, C. Obviously, $G_B(x) < G_C(x)$ for $x \in (20, 31)$ and $G_C(x) < G_B(x)$ for $x \in (1, 11)$.
Let us find points inside interval $[11, 20]$ in which $G_B(x) = G_C(x)$.
$G_B(x) = G_C(x) \Leftrightarrow \frac{1}{81}(x^2 - 40x + 400) = \frac{1}{900}(x^2 - 62x + 961)$ for $x \in [11, 20]$. After calculations we obtain the point $x_1 = 15.285$ and the second one, which is outside the interval $[11, 20]$. Therefore, $\Omega_B = [1, 15.285]$ which means that $|\Omega_B| = 14.285$. Because $|\Omega| = 30$, we obtain $R(B) < \frac{1}{2} < R(C)$. Thus, $B \prec_1 C$. Similarly, we observe that for A, C there is $G_A(x) < G_C(x)$ for $x \in (0, 1) \cup (20, 31)$.
Let us find points inside interval $[20, 31]$ in which $G_A(x) = G_C(x)$.
$G_A(x) = G_C(x) \Leftrightarrow \frac{1}{400}(-x^2 + 400) = \frac{1}{900}(x^2 - 62x + 961)$ for $x \in [11, 20]$. After calculations we obtain the points $x_1 = 1.04$ and $x_2 = 18.03$. Therefore, $\Omega_C = [0, 1.04] \cup [18.03, 31]$ and $\Omega_A = [1.04, 18.03]$ which means that $|\Omega_C| = 14.01$ and $|\Omega_A| = 16.99$. Because $|\Omega| = 31$, we obtain $R(C) < \frac{1}{2} < R(A)$. Thus, $C \prec_1 A$.

Another method of fuzzy numbers comparison was presented in [7].

Definition 2 ([7]). A fuzzy number $H_\Psi : \mathbb{R} \to [0, 1]$ is called the lower horizon of the given familly $\Psi \subset \mathrm{FN}$ if $\sup(\mathrm{supp} H_\Psi) \leq \inf(\mathrm{supp} A)$, for any $A \in \Psi$.

Definition 3 ([7])**.** Let $A, B \in \mathrm{FN}$, $H_{\{A,B\}}$ denote a fixed lower horizon of $\{A, B\}$ and d is a metric in FN. The relation $\prec_2$ is given by

$$A \prec_2 B \Leftrightarrow d(A, H_{\{A,B\}}) \leq d(B, H_{\{A,B\}}). \tag{8}$$

Quite natural is that

Remark 1. *Relation $\prec_2$ depends on the choice of metric d. For example, discreet metric d_r gives $A \prec_2 B$ for each $A, B \in \mathrm{FN}$, because $d_r(A, H_{A,B}) = 1 = d_r(B, H_{A,B})$.*

Let us consider the metric $\delta_{p,q}$ (cf.[7] Definition 2.2, Theorem 2.1)

$$\delta_{p,q}(A, B) := \sqrt[p]{(1-q)\int_0^1 |\underline{a}(\alpha) - \underline{b}(\alpha)|^p d\alpha + q\int_0^1 |\overline{a}(\alpha) - \overline{b}(\alpha)|^p d\alpha}, \tag{9}$$

for $1 \leq p < \infty$, $A, B \in \mathrm{FN}$. The interesting question is: does the relation $\prec_2$ depend on the choice of horizon, for metric $\delta_{p,q}$ with fixed p, q. It is proved that

Theorem 2 ([7])**.** *The relation $\prec_2$ based on metric $\delta_{1,q}$ does not depend on the choice of any horizon.*

Unfortunately, the general answer to this question is negative.

Example 3. Let $p = 2$, $q = \frac{1}{2}$ in (9). We have

$$\delta_{2,\frac{1}{2}}(A, B) = \sqrt[2]{\frac{1}{2}(\int_0^1 (\underline{a}(\alpha) - \underline{b}(\alpha))^2 d\alpha + \int_0^1 (\overline{a}(\alpha) - \overline{b}(\alpha))^2 d\alpha)}.$$

The relation $\prec_2$ based on metric $\delta_{2,\frac{1}{2}}$ depends on the choice of horizon. Let $A = (\frac{1}{2} + \alpha, 2\frac{1}{2} - \alpha)$, $B = (2\alpha, 2)$, $H_1 = (0, 0)$, $H_2 = (2\alpha - 2, 0)$. Then

$$d(A, H_1) = \sqrt{\frac{31}{12}} < \sqrt{\frac{32}{12}} = d(B, H_1) \Leftrightarrow A \prec_1 B,$$

$$d(A, H_2) = \sqrt{\frac{49}{12}} > \sqrt{\frac{48}{12}} = d(B, H_2) \Leftrightarrow B \prec_1 A.$$

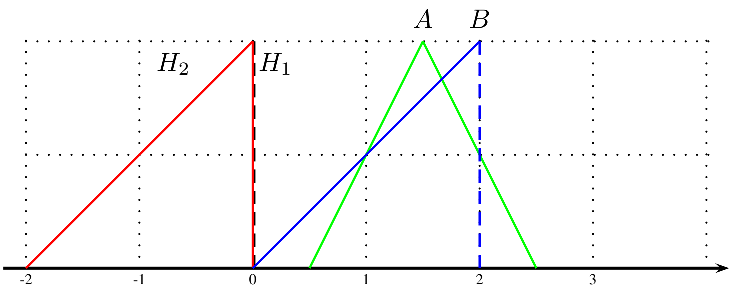

Figure 5: Fuzzy numbers $A = (\frac{1}{2} + \alpha, 2\frac{1}{2} - \alpha)$, $B = (2\alpha, 2)$ and horizons $H_1 = (0, 0)$, $H_2 = (2\alpha - 2, 0)$.

From this example we can see that the choice of the horizon can decide on the final result. Particularly, if we consider at least three fuzzy numbers, we need to fix horizon, what is a necessary condition for $\prec_2$ being transitive relation. It forms the new problem: how to choose the horizon. We will not investigate it here but will rather propose another method of ranking, using a special kind of description of fuzzy numbers.

4. Characterizing indexes for fuzzy number

In papers [2] and [3] the authors introduce three attributes for fuzzy numbers, which we remind here.

Definition 4 ([3]). Let $A = (\underline{a}, \overline{a})$ be a fuzzy number, $A' : \mathbb{R} \to [0, 1]$ be a function $A'(x) := 1 - A(x)$, and $s : [0, 1] \to [0, 1]$ be an increasing function which fulfills $s(0) = 0, s(1) = 1$. *Value, ambiguity* and *fuzzines* of A are the functions $Val, Amb, Fuz : \mathrm{FN} \to \mathbb{R}$ given as follows:

$$Val(A) := \int_0^1 s(\alpha)[\overline{a}(\alpha) + \underline{a}(\alpha)]d\alpha,$$

$$Amb(A) := \int_0^1 s(\alpha)[\overline{a}(\alpha) - \underline{a}(\alpha)]d\alpha,$$

$$Fuz(A) := \int_0^1 s(\alpha)[\overline{a}(0) - \underline{a}(0)]d\alpha - \Big[\int_{\frac{1}{2}}^1 s(\alpha)[\underline{a'}(\alpha) - \underline{a}(0)]d\alpha$$
$$+ \int_{\frac{1}{2}}^1 s(\alpha)[\overline{a}(\alpha) - \underline{a}(\alpha)]d\alpha + \int_{\frac{1}{2}}^1 s(\alpha)[\overline{a}(0) - \overline{a'}(\alpha)]d\alpha$$
$$+ \int_0^{\frac{1}{2}} s(\alpha)[\underline{a}(\alpha) - \underline{a}(0)]d\alpha + \int_0^{\frac{1}{2}} s(\alpha)[\overline{a'}(\alpha) - \overline{a'}(\alpha)]d\alpha$$
$$+ \int_0^{\frac{1}{2}} s(\alpha)[\overline{a}(0) - \overline{a}(\alpha)]d\alpha\Big].$$

These three attributes give us an information about a location and a shape of fuzzy number, which they describe. The authors suggest to use these attributes for representation of fuzzy number by trapezoidal fuzzy number having the same attributes. However, there are examples showing that not every fuzzy number can be represented in this way. It was only proved that if A is a concave function, then A is representable by a trapezoidal fuzzy number.
We modify attributes Val, Amb, Fuz in this way, that using new attributes Val^*, Amb^*, Fuz^* we can represent every fuzzy number by the trapezoidal one.

Definition 5. $Val^*, Amb^*, Fuz^* : \mathrm{FN} \to \mathbb{R}$ are the functions given as:

$$Val^*(A) := \frac{1}{2}\int_0^1 [\overline{a}(\alpha) + \underline{a}(\alpha)]d\alpha, \tag{10}$$

$$Amb^*(A) := \int_0^1 [\overline{a}(\alpha) - \underline{a}(\alpha)]d\alpha, \tag{11}$$

$$Fuz^*(A) := \frac{Amb^*(A) - (\overline{a}(1) - \underline{a}(1))}{Amb^*(A)}. \tag{12}$$

Remark 2. *If* $A \in \mathrm{FN}$ *has exactly one point* $x \in \mathbb{R}$*, where* $A(x) = 1$ *(i.e.* $\overline{a}(1) = \underline{a}(1)$*), then* $Fuz^*(A) = 1$.

Remark 3. *If* $A \in \mathrm{FN}, A(x) = \chi_{[a,b]}$ *is a characteristic function, then* $Fuz^*(A) = 0$.

For symmetrical trapezoidal fuzzy number

$$T_s = \langle m-c-d, m-c, m+c, m+c+d\rangle$$

it is easy to calculate that

$$Val^*(T_s) = m,\ Amb^*(T_s) = 2c+d,\ \ Fuz^*(T_s) = \tfrac{d}{2c+d}.$$

Solving for c and d, we get

$$m = Val^*(T_s), \tag{13}$$

$$c = \frac{Amb^*(T_s)(1-Fuz^*(T_s))}{2}, \tag{14}$$

$$d = Fuz^*(T_s)\cdot Amb^*(T_s). \tag{15}$$

Let us notice that (differently than in [3]) $\frac{Amb^*(A)(1-Fuz^*(A))}{2} \geq 0$ and $Fuz^*(A)\cdot Amb^*(A) \geq 0$, for every $A \in$ FN. Thus, it is possible to represent any fuzzy number (not only with concave membership function) by symmetrical trapezoidal fuzzy number.

Example 4 (cf. [3], Example 3)**.** Let $A = (2\alpha-2, -\sqrt{\alpha}+1)$. We have

$$Val^*(A) = \frac{1}{2}\int_0^1 [-\sqrt{\alpha}+1+2\alpha-2]d\alpha = -\frac{1}{3},$$

$$Amb^*(A) = \int_0^1 [-\sqrt{\alpha}+1-(2\alpha-2)]d\alpha = 1\frac{1}{3},$$

$$Fuz^*(A) = 1,$$

on the basis of Remark 2. From (13)-(15), we get

$$m = -\frac{1}{3}, \quad c = 0, \quad d = 1\frac{1}{3}.$$

Thus, from (6) the symmetrical trapezoidal representation of A (actually a symmetrical triangular fuzzy number) is $T_s = \langle -1\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1\rangle$.

We can also easy represent triangular and trapezoidal fuzzy numbers by symmetrical ones.

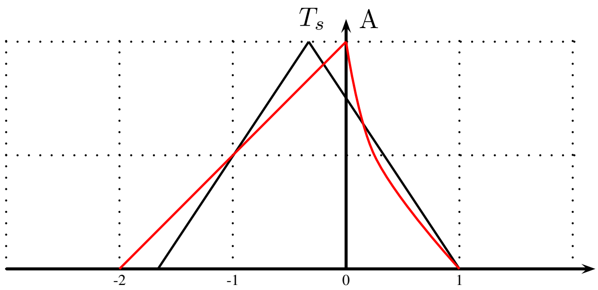

Figure 6: Fuzzy number A and its symmetrical trapezoidal representation T_s.

Corollary 1. *Let $A = \langle a_1, a_2, a_3, a_4 \rangle$ be a trapezoidal fuzzy number. Then*

$$Val^*(A) = \frac{1}{4}(a_1 + a_2 + a_3 + a_4),$$

$$Amb^*(A) = \frac{1}{2}(-a_1 - a_2 + a_3 + a_4),$$

$$Fuz^*(A) = \frac{(a_4 - a_1) - (a_3 - a_2)}{(a_4 - a_1) + (a_3 - a_2)}.$$

Corollary 2. *Let $\hat{A} = \langle \hat{a}_1, \hat{a}_2, \hat{a}_3 \rangle$ be a triangular fuzzy number. Then*

$$Val^*(\hat{A}) = \frac{1}{4}(\hat{a}_1 + 2\hat{a}_2 + \hat{a}_3),$$

$$Amb^*(\hat{A}) = \frac{1}{2}(\hat{a}_3 - \hat{a}_1),$$

$$Fuz^*(\hat{A}) = 1.$$

5. Equivalence relation in FN

In this section, we introduce equivalence relation in the set of fuzzy numbers using attributes Val^*, Amb^*, Fuz^*.

Definition 6. Let $A, B \in$ FN. A, B are equivalent ($A \cong B$) iff $(Val^*(A), Amb^*(A), Fuz^*(A)) = (Val^*(B), Amb^*(B), Fuz^*(B))$.

Theorem 3. *If $A, B \in FN$ and $A \cong B$, then*

$$\int_0^1 (\underline{a}(\alpha) - \underline{b}(\alpha))d\alpha = 0, \int_0^1 (\overline{a}(\alpha) - \overline{b}(\alpha))d\alpha = 0. \tag{16}$$

Proof. Let $A, B \in$ FN. Then $Val^*(A) = Val^*(B)$ and $Amb^*(A) = Amb^*(B)$. This means that

$$\frac{1}{2}\int_0^1 [\overline{a}(\alpha) + \underline{a}(\alpha)]d\alpha = \frac{1}{2}\int_0^1 [\overline{b}(\alpha) + \underline{b}(\alpha)]d\alpha,$$

$$\int_0^1 [\overline{a}(\alpha) - \underline{a}(\alpha)]d\alpha = \int_0^1 [\overline{b}(\alpha) - \underline{b}(\alpha)]d\alpha.$$

Thus,

$$\int_0^1 [(\overline{a}(\alpha) + \underline{a}(\alpha)) - (\overline{b}(\alpha) + \underline{b}(\alpha))]d\alpha = 0$$

and

$$\int_0^1 [(\overline{a}(\alpha) - \underline{a}(\alpha)) - (\overline{b}(\alpha) - \underline{b}(\alpha))]d\alpha = 0.$$

Adding and subtracting these integrals, we get (16). □

Theorem 4. *Let $A = \langle a_1, a_2, a_3 \rangle$ be a triangular fuzzy number. Then a triangular fuzzy number B is equivalent to A (in the sense of relation $\cong$) iff $B = \langle a_1 + a_2 - b, b, a_2 + a_3 - b \rangle$, $b \in [\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]$.*

Proof. ($\Rightarrow$) Let $A = \langle a_1, a_2, a_3 \rangle$, $B = \langle b_1, b, b_3 \rangle$ and $A \cong B$. Since

$$\begin{cases} \frac{1}{4}(a_1 + 2a_2 + a_3) & = \frac{1}{4}(b_1 + 2b + b_3), \\ \frac{1}{2}(a_3 - a_1) & = \frac{1}{2}(b_3 - b_1), \end{cases}$$

then solving the above system with respect to b_1 and b_3 with a parameter b, we get

$$\begin{cases} b_1 & = a_1 + a_2 - b \\ b_3 & = a_3 + a_2 - b \end{cases}$$

Because $b_1 \leq b_2 \leq b_3$, so $b \in [\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]$.
($\Leftarrow$) Let $A = \langle a_1, a_2, a_3 \rangle$, $B = \langle a_1+a_2-b, b, a_2+a_3-b \rangle$ for $b \in [\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]$. From Corollary 2

$$Val^*(B) = \frac{1}{4}(a_1+a_2-b+2b+a_2+a_3-b) = \frac{1}{4}(a_1+2a_2+a_3) = Val^*(A),$$

$$Amb^*(B) = \frac{1}{2}(a_2 + a_3 - b - (a_1 + a_2 - b)) = \frac{1}{2}(a_3 - a_1) = Amb^*(A).$$

From Remark 2 $Fuz^*(B) = 1 = Fuz^*(A)$. Thus $A \cong B$.

□

The next example describes the families of triangular and trapezoidal fuzzy numbers with the same attributes Val^*, Amb^*, Fuz^*.

Example 5. Let $A = \langle 1, 3, 4 \rangle$ be a triangular fuzzy number. As a result of Theorem 4, we can specify all triangular fuzzy numbers equivalent to A given by a triple $\langle a_1 + a_2 - b, b, a_2 + a_3 - b \rangle$ for $b \in [\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]$. In particular, we have B_L for $b = \frac{a_1+a_2}{2}$ and B_U for $b = \frac{a_2+a_3}{2}$. For $b = \frac{a_1+a_2}{2} = 2$ we get $B_L = \langle 2, 2, 5 \rangle$ and for $b = \frac{a_2+a_3}{2} = 3.5$ we get $B_U = \langle 0.5, 3.5, 3.5 \rangle$.

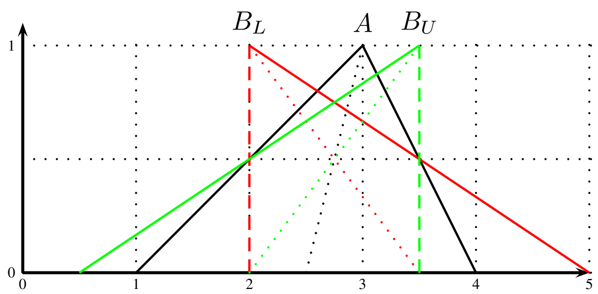

Figure 7: An example of three equivalent triangular fuzzy numbers.

Similarly as in the proof of Theorem 4, on the strength of Corollary 1 we get

Theorem 5. *Let $A = \langle a_1, a_2, a_3, a_4 \rangle$ be a trapezoidal fuzzy number. Then a trapezoidal fuzzy number B is equivalent to A (in the sense of relation $\cong$) iff $B = \langle b, a_1 + a_2 - b, a_1 + a_3 - b, a_4 - a_1 + b \rangle$ for $b \in [\frac{2a_1+a_3-a_4}{2}, \frac{a_1+a_2}{2}]$.*

Example 6. Let $A = \langle 1, 2, 3, 4 \rangle$ be a trapezoidal fuzzy number. As a result of Theorem 5, we can specify all trapezoidal fuzzy numbers equivalent to A given by a quadruple $\langle b, a_1 + a_2 - b, a_1 + a_3 - b, a_4 - a_1 + b \rangle$ for $b \in [\frac{2a_1+a_3-a_4}{2}, \frac{a_1+a_2}{2}]$. In particular we have B_L for $b = \frac{a_1+a_2}{2}$ and B_U for $b = \frac{2a_1+a_3-a_4}{2}$. For $b = \frac{a_1+a_2}{2} = 1.5$ we get $B_L = \langle 1.5, 1.5, 2.5, 4.5 \rangle$ and for $b = \frac{2a_1+a_3-a_4}{2} = 0.5$ we get $B_U = \langle 0.5, 2.5, 3.5, 3.5 \rangle$.

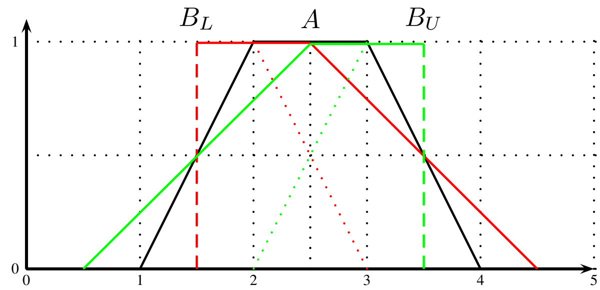

Figure 8: Example of three equivalent trapezoidal fuzzy numbers.

Here we present some properties of attributes Val^*, Amb^*, Fuz^*.

Lemma 1. *Let $A, B \in$ FN. If $Amb^*(A) = Amb^*(B)$ and $Fuz^*(A) = Fuz^*(B)$, then $\overline{a}(1) - \underline{a}(1) = \overline{b}(1) - \underline{b}(1)$.*

Proof. Let $A, B \in$ FN. From $Fuz^*(A) = Fuz^*(B)$, we have

$$\frac{Amb^*(A) - (\overline{a}(1) - \underline{a}(1))}{Amb^*(A)} = \frac{Amb^*(B) - (\overline{b}(1) - \underline{b}(1))}{Amb^*(B)}.$$

Since $Amb^*(A) = Amb^*(B)$, then $\overline{a}(1) - \underline{a}(1) = \overline{b}(1) - \underline{b}(1)$. □

Similarly, we get

Lemma 2. *Let $A, B \in$ FN. If $Amb^*(A) = Amb^*(B)$ and $Fuz^*(A) > Fuz^*(B)$, then $\overline{a}(1) - \underline{a}(1) < \overline{b}(1) - \underline{b}(1)$.*

Lemma 3. *Let $A, B \in$ FN$, t > 0$. Then*

$$Val^*(A + B) = Val^*(A) + Val^*(B), \quad Val^*(tA) = tVal^*(A).$$

Proof. Let $A, B \in$ FN. Then also $A + B \in$ FN. One can calculate that

$$\begin{aligned} Val^*(A+B) &= \frac{1}{2}\int_0^1 [\overline{(a+b)}(\alpha) + \underline{(a+b)}(\alpha)]d\alpha \\ &= \frac{1}{2}\int_0^1 [\overline{a}(\alpha) + \overline{b}(\alpha) + \underline{a}(\alpha) + \underline{b}(\alpha)]d\alpha \\ &= \frac{1}{2}\int_0^1 [\overline{a}(\alpha) + \underline{a}(\alpha)]d\alpha + \frac{1}{2}\int_0^1 [\overline{b}(\alpha) + \underline{b}(\alpha)]d\alpha \\ &= Val^*(A) + Val^*(B). \end{aligned}$$

Let $t > 0$. Then also $tA \in \mathrm{FN}$. Similarly, one can calculate that

$$\begin{aligned} Val^*(tA) &= \frac{1}{2}\int_0^1 [\overline{(ta)}(\alpha) + \underline{(ta)}(\alpha)]d\alpha \\ &= \frac{1}{2}\int_0^1 t[\overline{a}(\alpha) + \underline{a}(\alpha)]d\alpha = tVal^*(A). \end{aligned}$$

□

Lemma 4. *Let* $A, B \in \mathrm{FN}, t > 0$. *Then*

$$Amb^*(A+B) = Amb^*(A) + Amb^*(B), \quad Amb^*(tA) = tAmb^*(A).$$

Proof. Let $A, B \in \mathrm{FN}$. Then

$$\begin{aligned} Amb^*(A+B) &= \int_0^1 [\overline{(a+b)}(\alpha) - \underline{(a+b)}(\alpha)]d\alpha \\ &= \int_0^1 [\overline{a}(\alpha) + \overline{b}(\alpha) - (\underline{a}(\alpha) + \underline{b}(\alpha))]d\alpha \\ &= \int_0^1 [\overline{a}(\alpha) - \underline{a}(\alpha)]d\alpha + \int_0^1 [\overline{b}(\alpha) - \underline{b}(\alpha)]d\alpha = Amb^*(A) + Amb^*(B). \end{aligned}$$

Similarly,

$$Amb^*(tA) = \int_0^1 [\overline{(ta)}(\alpha) + \underline{(ta)}(\alpha)]d\alpha = \int_0^1 t[\overline{a}(\alpha) + \underline{a}(\alpha)]d\alpha = tAmb^*(A).$$

□

Lemma 5. *Let* $A, B \in \mathrm{FN}$. *If* $Amb^*(A) = Amb^*(B) > 0$, *then*

$$Fuz^*(A+B) \leq Fuz^*(A) + Fuz^*(B).$$

Proof.

$$\begin{aligned} Fuz^*(A+B) &= \frac{Amb^*(A+B) - [\overline{(a+b)}(1) - \underline{(a+b)}(1)]}{Amb^*(A+B)} \\ &= \frac{Amb^*(A) + Amb^*(B) - [\overline{a}(1) + \overline{b}(1) - (\underline{a}(1) + \underline{b}(1))]}{Amb^*(A) + Amb^*(B)} \\ &= \frac{Amb^*(A) - [\overline{a}(1) - \underline{a}(1)]}{Amb^*(A) + Amb^*(B)} + \frac{Amb^*(B) - [\overline{b}(1) - \underline{b}(1)]}{Amb^*(A) + Amb^*(B)} \\ &\leq Fuz^*(A) + Fuz^*(B). \end{aligned}$$

□

Now we define a new attribute for fuzzy number, which describe the difference between an original fuzzy number and its symmetrical trapezoidal representation.

Definition 7. Let $T_s = (\underline{t_s}, \overline{t_s})$ be a symmetrical trapezoidal representation of fuzzy number $A = (\underline{a}, \overline{a})$. The *difference* of A is a function $Dif^* : \mathrm{FN} \to \mathbb{R}$ given as follows $Dif^*(A) := \overline{a}(1) - \overline{t_s}(1)$.

Obviously, on the strength of Lemma 1,

Remark 4. *If $A = (\underline{a}, \overline{a}) \in \mathrm{FN}$, $T_s = (\underline{t_s}, \overline{t_s})$ is symmetrical trapezoidal representation of A, then $Dif^*(A) = \underline{a}(1) - \underline{t_s}(1)$.*

Using attributes $Val^*, Amb^*, Fuz^*, Dif^*$ we define new ranking for fuzzy numbers.

Definition 8. Let $A, B \in \mathrm{FN}$. Then $A \prec_3 B$ iff A, B fulfill one of the following conditions:

$$Val^*(A) < Val^*(B), \tag{17}$$

$$Val^*(A) = Val^*(B), Amb^*(A) < Amb^*(B), \tag{18}$$

$$Val^*(A) = Val^*(B), Amb^*(A) = Amb^*(B), Fuz^*(A) > Fuz^*(B), \tag{19}$$

$$A \cong B, Dif^*(A) < Dif^*(B). \tag{20}$$

Remark 5. *Fuzzy numbers A, B with all identical attributes $Val^*(A) = Val^*(B)$, $Amb^*(A) = Amb^*(B)$, $Fuz^*(A) = Fuz^*(B)$, $Dif^*(A) = Dif^*(B)$ are indistinguishable in the sense of relation $\prec_3$.*

Theorem 6. *The relation $\prec_3$ forms a strict partial order in* FN *(i.e. $\prec_3$ is asymmetric, irreflexive and transitive).*

Proof. Let $A, B \in \mathrm{FN}$ and $A \prec_3 B$. Directly from Definition 8 we obtain four cases:

A1.) if (17), then $Val^*(B) \nleq Val^*(A)$, thus it is not $(B \prec_3 A)$.

A2.) if (18), then $Val^*(B) = Val^*(A)$, but $Amb^*(B) \nleq Amb^*(A)$, thus it is not $(B \prec_3 A)$.

A3.) if (19), then $Val^*(B) = Val^*(A)$, $Amb^*(B) = Amb^*(A)$, but $Fuz^*(B) \ngeq Fuz^*(A)$, thus it is not $(B \prec_3 A)$.

A4.) if (20), then $B \cong A$, but $Dif^*(B) \nless Dif^*(A)$, thus it is not $(B \prec_3 A)$. This means that $(A \prec_3 B) \Rightarrow\sim (B \prec_3 A)$ ($\prec_3$ is asymmetric).

Immediately we get $\sim (A \prec_3 A)$ ($\prec_3$ is irreflexive).
For showing $((A \prec_3 B)$ and $(B \prec_3 C)) \Rightarrow (A \prec_3 C)$ (transitivity), we need to check 16 cases. We show here only three of them. It is easy to check the remaining cases similarly.
Let $(A \prec_3 B)$ and $(B \prec_3 C)$.

T1.) if A, B fulfill (17) and B, C fulfill (17), then A, C fulfill (17), thus $A \prec_3 C$.

T2.) if A, B fulfill (17) and B, C fulfill (18), then A, C fulfill (17), thus $A \prec_3 C$.

T3.) if A, B fulfill (19) and B, C fulfill (19), then A, C fulfill (19), thus $A \prec_3 C$.

□

The order $\prec_3$ fulfills two basic properties.

Theorem 7. *Let* $A, B, C \in \mathrm{FN}$. *If* $A \prec_3 B$, *then* $A + C \prec_3 B + C$.

Proof. Let $A, B, C \in \mathrm{FN}$ and $A \prec_3 B$. We need to check four cases:

I. $Val^*(A) < Val^*(B)$. Then, from Lemma 3,

$$Val^*(A+C) = Val^*(A)+Val^*(C) < Val^*(B)+Val^*(C) = Val^*(B+C).$$

Thus, $A + C \prec_3 B + C$.

II. $Val^*(A) = Val^*(B)$, $Amb^*(A) < Amb^*(B)$. Then, from Lemma 3,

$$Val^*(A+C) = Val^*(B+C)$$

and from Lemma 4

$$Amb^*(A+C) = Amb^*(A) + Amb^*(C) < Amb^*(B) + Amb^*(C) = Amb^*(A+C).$$

Thus, $A + C \prec_3 B + C$.

III. $Val^*(A) = Val^*(B)$, $Amb^*(A) = Amb^*(B)$, $Fuz^*(A) > Fuz^*(B)$. From Lemma 3 and 4,

$$Val^*(A+C) = Val^*(B+C), \quad Amb^*(A+C) = Amb^*(B+C).$$

$$Fuz^*(A+C) = \frac{Amb^*(A) + Amb^*(C) - [\overline{(a+c)}(1) - \underline{(a+c)}(1)]}{Amb^*(A) + Amb^*(C)}$$

$$= \frac{Amb^*(B) + Amb^*(C) - [\overline{a}(1) + \overline{b}(1) + \overline{c}(1) - \underline{a}(1) - \underline{b}(1) - \underline{c}(1) - \overline{b}(1) + \underline{b}(1)]}{Amb^*(B) + Amb^*(C)}$$

$$= Fuz^*(B+C) - \frac{(\overline{a}(1) - \underline{a}(1)) - (\overline{b}(1) - \underline{b}(1))}{Amb^*(B) + Amb^*(C)}.$$

On the basis of Lemma 2, $\frac{(\overline{a}(1) - \underline{a}(1)) - (\overline{b}(1) - \underline{b}(1))}{Amb^*(B) + Amb^*(C)} < 0$, so we get $Fuz^*(A+C) > Fuz^*(B+C)$. Thus, $A + C \prec_3 B + C$.

IV. $A \cong B$, $Dif^*(A) < Dif^*(B)$. From Lemma 1, 3 and 4, we have $(A+C) \cong (B+C)$. Let $T_s = (\underline{t_s}, \overline{t_s})$ be a symmetrical trapezoidal representation of $A + C$ and $B + C$. Then

$$\begin{aligned} Dif^*(A+C) &= \overline{(a+c)}(1) - \overline{t_s}(1) = \overline{a}(1) + \overline{c}(1) - \overline{t_s}(1) \\ &< \overline{b}(1) + \overline{c}(1) - \overline{t_s}(1) = \overline{(b+c)}(1) - \overline{t_s}(1) \\ &= Dif^*(B+C). \end{aligned}$$

Thus, $A + C \prec_3 B + C$. □

Similarly, the following theorem can be proved:

Theorem 8. *Let* $A, B \in \mathrm{FN}^+$, $t > 0$. *If* $A \prec_3 B$, *then* $tA \prec_3 tB$.

6. Concluding remarks

In this paper we discuss two methods of ranking fuzzy numbers presented in recent papers. The shortcomings of both methods cause us to use modified attributes for representing any fuzzy number by a symmetrical trapezoidal one. Additionally, we create the new attribute which describes the difference between a fuzzy number and its symmetrical trapezoidal representation. By means of all these attributes we form a new ranking method. Then we investigate basic properties, in particular, the transitivity of that ranking.

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