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SOME INEQUALITIES CONNECTED WITH A QUADRATIC FUNCTIONAL EQUATION

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Abstract
Let $(X, +)$ be an Abelian group. One can show that a mapping $f\colon X \to \mathbb{R}$ satisfying the inequality

$$f(x+y) + f(x-y) \leq 2f(x) + 2f(y) \tag{1}$$

for all $x, y \in X$ also satisfies the inequalities

$$f(2x+y) \leq 4f(x) + f(y) + f(x+y) - f(x-y) \tag{2}$$

and

$$f(2x+y) + f(2x-y) \leq 8f(x) + 2f(y) \tag{3}$$

for all $x, y \in X$.

A question of finding sufficient conditions under which the inequalities (1), (2) and (3) are equivalent will be considered. In this note, some properties of the solution of (1) will be proved. We also consider another definition of a subquadratic function given in [1].

1. Introduction

Let X, Y be a vector spaces. It is known that a mapping $f\colon X \to Y$ satisfies

$$f(2x+y) = 4f(x) + f(y) + f(x+y) - f(x-y), \qquad x, y \in X,$$

or

$$f(2x+y) + f(2x-y) = 8f(x) + 2f(y), \qquad x, y \in X,$$

if and only if f satisfies a quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. It is not longer true, if the sign "=" in these equalities will be replaced by "$\leq$".

2. General properties

In this note $(X, +)$ is an Abelian group. We start with the following

Remark 1 *If $f\colon X \to \mathbb{R}$ satisfies the inequality* (1) *for all $x, y \in X$, then f also satisfies the inequalities* (2) *and* (3) *for all $x, y \in X$.*

Proof. Let $x, y \in X$. Since f satisfies the inequality (1), then:

$$f(2x+y) = f(x+y+x) \leq 2f(x+y) + 2f(x) - f(x+y-x) \leq$$

$$\leq 2f(x) + f(x+y) + 2f(x) + 2f(y) - f(x-y) - f(y) =$$

$$= 4f(x) + f(y) + f(x+y) - f(x-y).$$

It is known that if f satisfies (1), then $f(nx) \leq n^2 f(x)$ for all $x \in X$ and for all $n \in \mathbb{N}$ ([3]). Hence,

$$f(2x+y) + f(2x-y) \leq 2f(2x) + 2f(y) \leq 8f(x) + 2f(y). \qquad \square$$

On the other hand, there exists a function f satisfying (2) (or (3)) which is not a solution of (1).

Example 1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} b, & x \neq 0, \\ d, & x = 0, \end{cases}$$

where $b, d \geq 0$ and $3b < d \leq 5b$. It is easy to see that f satisfies (2), *but does not satisfy* (1).

Example 2 *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} b, & x \neq 0, \\ d, & x = 0, \end{cases}$$

where $b, d \geq 0$ *and* $3b < d \leq 9b$. *It is easy to see that* f *satisfies* (3), *but does not satisfy* (1).

One can show that a function $f\colon X \to \mathbb{R}$ satisfying the inequality (3) (or (2)) has similar properties as a function satisfying the inequality defining a subquadratic function (1).

Lemma 1 *If* $f\colon X \to \mathbb{R}$ *satisfies* (3), *then the odd part of* f *is bounded. If, moreover,* $f(0) = 0$, *then* f *is even.*

Proof. On account of (3),

$$f(y) + f(-y) \leq 8f(0) + 2f(y), \qquad y \in X.$$

Therefore,

$$-4f(0) \leq \frac{f(y) - f(-y)}{2}, \qquad y \in X, \tag{4}$$

which means that the odd part of f is an odd function; then it is bounded bilaterally. It follows from (4) that if $f(0) = 0$, then f is even.

Lemma 2 *If* $f\colon X \to \mathbb{R}$ *satisfies* (2), *then the odd part of* f *is bounded. If, moreover,* $f(0) = 0$, *then* f *is even.*

Proof. Setting $x = 0$ in (2), we get

$$f(-y) - f(y) \leq 4f(0), \qquad y \in X,$$

whence our assertion follows easy. □

Remark 2 *Let* $f\colon X \to \mathbb{R}$ *be a function satisfying* (2). *If* $f(0) = 0$, *then* f *satisfies the inequality* (3).

Proof. Since $f(0) = 0$, then by Lemma 2 f is even. Setting $-y$ instead of y in the inequality (2), we get

$$f(2x - y) \leq 4f(x) + f(-y) + f(x - y) - f(x + y) =$$

$$= 4f(x) + f(y) + f(x - y) - f(x + y). \tag{5}$$

Therefore, by (5) and (2), we get the inequality (3).

The next example shows that if f satisfies the inequality (3), this does not mean that f satisfies the inequality (2).

Example 3 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} b, & x \neq 0, \\ d, & x = 0, \end{cases}$$

where $b, d \geq 0$ and $3b < 5b < d \leq 9b$. It is easy to see that f satisfies (3)*, but does not satisfy* (2)*.*

The next Lemma shows that under some additional conditions the inequalities (2) and (3) are equivalent.

Lemma 3 *In the class of subadditive functions satisfying condition $f(0) = 0$, the inequalities* (2) *and* (3) *are equivalent.*

Proof. Assume that f satisfies (2). According to Remark 2, f satisfies (3). Assume now that f satisfies (3). Since $f(0) = 0$, then by Lemma 2 f is even. Every even and subadditive function satisfies the inequality (1). According to Remark 1, f satisfies (2). □

Lemma 4 *In the class of subadditive functions satisfying condition $f(0) = 0$ the inequalities* (1)*,* (2) *and* (3) *are equivalent.*

Proof. By Lemma 3, the inequalities (2) and (3) are equivalent. Assume that f satisfies (1). According to Remark 1, f satisfies (3). Assume now that f satisfies (3). By Lemma 2, f is even. Every even and subadditive function satisfies the inequality (1). □

In order to obtain the main result of this note, we start with some known conditions implying subadditivity of a function $f\colon [0,\infty) \to \mathbb{R}$.

Lemma 5 *Let $f\colon [0,\infty) \to \mathbb{R}$. If the function $g(x) := \frac{1}{x}f(x)$, $x \in (0,\infty)$, is decreasing and $f(0) \geq 0$, then f is subadditive.*

Lemma 6 *Let $f\colon [0,\infty) \to \mathbb{R}$. If f is concave and $f(0) \geq 0$, then f is subadditive.*

Using Lemmas 5, 6 and 4 we can obtain the following two theorems.

Theorem 1 *Let $f\colon [0,\infty) \to \mathbb{R}$. If the function $g(x) := \frac{1}{x}f(x)$, $x \in (0,\infty)$, is decreasing and $f(0) = 0$, then inequalities* (1)*,* (2) *and* (3) *are equivalent.*

Theorem 2 *Let $f\colon [0,\infty) \to \mathbb{R}$. If f is concave and $f(0) = 0$, then inequalities* (1)*,* (2) *and* (3) *are equivalent.*

The last result of this section shows that every superadditive function satisfying the inequality (1) must be equal to zero for every $x \in X$. We shall use the following Lemma.

Lemma 7 *Let $f\colon X \to \mathbb{R}$ be a function satisfying* (3). *If f is a superadditive function, then $f \equiv 0$.*

Proof. By superadditivity, we get $f(0) \leq 0$. On the other hand, if f satisfies (3), then $f(0) \geq 0$. According to Lemma (2), f is even. Therefore, since f is superadditive, we get

$$f(x) + f(x) = f(x) + f(-x) \leq f(x - x) = f(0) = 0.$$

Thus,

$$f(x) \leq 0, \qquad x \in X. \tag{6}$$

On the other hand, since f is superadditive and satisfies (3)

$$4f(x) + 2f(y) \leq 2f(2x) + 2f(y) \leq f(2x + y) + f(2x - y) \leq 8f(x) + 2f(y)$$

for all $x, y \in X$. It implies that

$$0 \leq f(x), \qquad x \in X. \tag{7}$$

By (6) and (7), $f(x) = 0$ for all $x \in X$. □

Lemma 8 *Let $f\colon X \to \mathbb{R}$ be a function satisfying* (1). *If f is a superadditive function, then $f \equiv 0$.*

Proof. According to Remark 1, f satisfies the inequality (3). By Lemma 7, $f(x) = 0$ for all $x \in X$. □

3. The stability of the inequalities (2) and (3) in the sense of Hyers and Ulam

Let $(X, +)$ be an Abelian group. Fix an $\epsilon \geq 0$ and consider a function $f\colon X \to \mathbb{R}$ fulfilling the inequality

$$f(2x + y) \leq 4f(x) + f(y) + f(x + y) - f(x - y) + \epsilon, \qquad x, y \in X.$$

Putting $g(x) := f(x) + \frac{\epsilon}{4}$ for $x \in X$, we observe that g satisfies (2) and, moreover,

$$\mid f(x) - g(x) \mid \leq \frac{\epsilon}{4}, \qquad x \in X.$$

This means that the problem of the stability in the sense of Hyers and Ulam of the inequality (2) has a positive answer.

Similarly, putting $g(x) := f(x) + \frac{\epsilon}{8}$, one can show that the problem of the stability in the sense of Hyers and Ulam of the inequality (3) has a positive answer.

4. Another definition of subquadratic function

According to another definition in [1], a function $f\colon [0,\infty) \to \mathbb{R}$ is said to be subquadratic if for all $x \geq 0$ there exists a constant $c_x \in \mathbb{R}$ such that

$$f(y) - f(x) \leq c_x(y - x) + f(|\, y - x \,|) \tag{8}$$

for all $y \geq 0$.

Remark 3 *If $f\colon [0,\infty) \to \mathbb{R}$ is subquadratic in the sense of* (8)*, then f fulfils* (1) *for all $x, y \in [0,\infty)$ such that $x \geq y$.*

Proof. Let $x \in [0,\infty)$. Then there exists a constant $c_x \in \mathbb{R}$ such that the inequality (8) holds for every $y \in [0,\infty)$. Now take arbitrary $y \in [0,\infty)$ such that $x \geq y$. Setting $x + y$ instead of y in (8), we get

$$f(x + y) \leq c_x y + f(y) + f(x). \tag{9}$$

By the fact that $x \geq y$, we can set $x - y$ instead of y in (8) and then we get

$$f(x - y) \leq -c_x y + f(y) + f(x). \tag{10}$$

Adding (9) and (10) side by side we obtain

$$f(x + y) + f(x - y) \leq 2f(x) + 2f(y)$$

for all $x, y \in [0,\infty)$ such that $x \geq y$. □

Remark 4 *There exists a function fulfilling* (1) *for all $x \geq y \geq 0$, but it is not subquadratic in the sense of* (8).

Proof. Let

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 3, & x = 0. \end{cases}$$

It is easy to see that f satisfies (1). On the other hand, setting 3 instead of y and 1 instead of x in (8), we get

$$f(3) - f(1) - f(2) \leq 2c_1.$$

Thus

$$-\frac{1}{2} \leq c_1. \tag{11}$$

Setting 0 instead of y and 1 instead of x in (8), we obtain that

$$f(0) - f(1) - f(1) \leq -c_1.$$

Thus

$$c_1 \leq -1. \tag{12}$$

Therefore, by (11) and (12), f can not satisfy the condition (8). □

Fix an $\epsilon \geq 0$ and consider a function $f\colon [0, \infty) \to \mathbb{R}$ such that for all $x \geq 0$ there exists a constant $c_x \in \mathbb{R}$ such that

$$f(y) - f(x) \leq c_x(y - x) + f(\mid y - x \mid) + \epsilon$$

for all $y \geq 0$. Putting $g(x) := f(x) + \epsilon$ for $x \in [0, \infty)$, we observe that g satisfies (8) and, moreover,

$$\mid f(x) - g(x) \mid \leq \epsilon, \qquad x \in [0, \infty).$$

This means that the problem of the stability in the sense of Hyers and Ulam of the inequality (8) has a positive answer.

References

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