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ESTIMATIONS OF LOSS CHARACTERISICS OF SINGLE-SERVER QUEUEING SYSTEMS WITH NON-HOMOGENEOUS CUSTOMERS

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Abstract.
A non-classical single-server queueing system with non-homogeneous customers having some random space requirement (capacity, volume) can be used as a model of a wide class of computer and communicating systems. We assume that the total customers capacity in the system is limited by some constant value $V > 0$ that is called the value of memory capacity of the system. Service time of a customer generally depends on his capacity. For such systems we determine some estimators of stationary loss characteristics and compare the analitical results with ones obtained by simulation.

1. Introduction

In the present work we investigate single-server queueing systems with non-homogeneous customers and limited memory capacity. This means that

1) each customer is characterized by some non-negative random space requirement (capacity, volume) ζ;

2) the total sum $\sigma(t)$ of capacities of all customers present in the system at arbitrary time moment t is limited by some constant value V $(0 < V \leq \infty)$ that is called the value of memory capacity of the system;

3) the customer's service time ξ and his capacity ζ are generally dependent.

Such systems have been used to model and solve various practical problems occurring in the design of computer and communicating systems. It is clear that they differ from usual classical queueing systems. For example, we can analyze the non-classical system $M/G/1/(\infty, V)$ that differs from the classical system $M/G/1/\infty$ in the sense of assumptions 1–3.

Let

$$F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$$

be the distribution function of the random vector (ζ, ξ). Then

$$L(x) = \mathbf{P}\{\zeta < x\} = F(x, \infty),\ \ B(t) = \mathbf{P}\{\xi < t\} = F(\infty, t)$$

are the distribution functions of customer's capacity and service time, respectively. The memory space is occupied by customer's capacity at the epoch he arrives and is released entirely at the epoch he completes service. The process $\sigma(t)$ is called the total customers capacity.

Total capacity limitation (in the case of $V < \infty$) leads to losses of customers. A customer having capacity x, which arrives at the epoch τ, will be admitted to the system if $\sigma(\tau - 0) + x \leq V$. Otherwise ($\sigma(\tau - 0) + x > V$) the customer will be lost.

Various single-server queueing systems with non-homogeneous (in the sense of assumptions 1–3) customers were analyzed in [1–4].

2. Some analytical results

Suppose that customers intrance flow is Poissonian. Let a be an arrival rate of entrance flow of customers, $\eta(t)$ be a number of customers present in the system at time instant t. We can obtain the exact analytical results only in some particular cases of the system $M/G/1/(\infty, V)$. Let us analyze briefly these cases and the possibilities of analytical estimation if it is impossible to obtain the exact results. We assume that the stationary mode exists for all systems under consideration, i.e. the limits $\eta(t) \Rightarrow \eta$, $\sigma(t) \Rightarrow \sigma$ exist in the sense of a weak convergence, where η and σ are the number of customers and total customers capacity in the system in stationary mode.

2.1. The case of unlimited memory space. Assume that $V = \infty$. Then we have the classical $M/G/1/\infty$ system without losses of customers. For such a system we can obtain the stationary characteristics of total customers capacity (see e.g. [4]).

We shall use the following notation. Denote by

$$\alpha(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qt} dF(x,t)$$

the double Laplace-Stieltjes transfom (LST) of the function $F(x,t)$. Let $\varphi(s) = \alpha(s,0)$ and $\beta(q) = \alpha(0,q)$ be the LST of the functions $L(x)$ and $B(t)$, respectively. Let $D(x) = \mathbf{P}\{\sigma < x\}$ be the distribution function of stationary total customers capacity, $\rho = a\beta_1 < 1$. Denote by $\delta(s) = \int_0^\infty e^{-sx} dD(x)$ the LST of the function $D(x)$. Then we have (see again [4])

$$\delta(s) = (1-\rho)\left[1 + \frac{\varphi(s) - \alpha(s, a - a\varphi(s))}{\beta(a - a\varphi(s)) - \varphi(s)}\right]. \tag{1}$$

Denote by $\delta_i = \mathbf{E}\sigma^i$ the ith moment of total customers capacity σ, $i = 1, 2, \dots$. Let $\varphi_i = \mathbf{E}\zeta^i$, $\beta_i = \mathbf{E}\xi^i$ and $\alpha_{ij} = \mathbf{E}(\zeta^i \xi^j)$ be the ith moments of the random variables ζ, ξ and the mixed $(i+j)$th moment of the random variables ζ and ξ, respectively, $i, j = 1, 2, \dots$.

It follows from (1) that

$$\delta_1 = a\alpha_{11} + \frac{a^2\beta_2\varphi_1}{2(1-\rho)}, \tag{2}$$

$$\delta_2 = a(\alpha_{21} + a\varphi_1\alpha_{12}) + \frac{a^3\beta_2\varphi_1\alpha_{11}}{1-\rho} + \frac{a^2\beta_2\varphi_2}{2(1-\rho)} + \frac{a^3\beta_3\varphi_1^2}{3(1-\rho)} + \frac{a^4\beta_2^2\varphi_1^2}{2(1-\rho)^2}. \tag{3}$$

For many real computer systems (for example, for communicating centers) the customer's service time can be defined by the relation $\xi = c\zeta + \xi_1$, where $c \geq 0$ and the random variables ζ and ξ_1 are independent. Then, if we denote by $\kappa(q)$ the LST of the distribution function of the random variable ξ_1, we have from (1) (see [3])

$$\delta(s) = (1-\rho)\frac{\varphi(ca - ca\varphi(s)) - \varphi(s + ca - ca\varphi(s))}{\varphi(ca - ca\varphi(s)) - \varphi(s)/\kappa(ca - ca\varphi(s))}.$$

Denote by κ_i the ith moment of the random variable ξ_1, $i = 1, 2, \dots$. Then the first and second moments of the random variable σ can be obtained from the relations (2), (3), where

$$\alpha_{11} = \varphi_1\kappa_1 + c\varphi_2,\ \alpha_{21} = \varphi_2\kappa_1 + c\varphi_3,$$

$$\alpha_{12} = \varphi_1\kappa_2 + 2c\varphi_2\kappa_1 + c^2\varphi_3,\ \beta_1 = c\varphi_1 + \kappa_1,$$

$$\beta_2 = c^2\varphi_2 + 2c\varphi_1\kappa_1 + \kappa_2,\ \beta_3 = c^3\varphi_3 + 3c^2\varphi_2\kappa_1 + 3c\varphi_1\kappa_2 + \kappa_3.$$

If the customer's service time is proportional to its capacity ($\xi_1 \equiv 0$, $c > 0$), we have

$$\delta(s) = (1-\rho)\frac{\varphi(ca - ca\varphi(s)) - \varphi(s + ca - ca\varphi(s))}{\varphi(ca - ca\varphi(s)) - \varphi(s)}$$

and $\alpha_{11} = c\varphi_2$, $\alpha_{12} = c^2\varphi_3$, $\beta_1 = c\varphi_1$, $\beta_2 = c^2\varphi_2$, $\beta_3 = c^3\varphi_3$.

If, in addition, the random variable ζ has an exponential distribution with parameter f, i.e. $L(x) = 1 - e^{-fx}$, we obtain

$$\delta(s) = \frac{(1-\rho)(s+f)^3}{[(s+f)^2 + cas](s+f-ca)}.$$

Introduce the following notation:

$$b_1 = \frac{2+\rho-\sqrt{\rho(4+\rho)}}{2},\ b_2 = \frac{2+\rho+\sqrt{\rho(4+\rho)}}{2}.$$

Then from the last relation we can obtain the explicit formula for $D(x)$:

$$D(x) = \begin{cases} 1 + \frac{\rho^2 e^{-(1-\rho)fx}}{1-2\rho} - \\ -\frac{\rho(1-\rho)}{\sqrt{\rho(4+\rho)}}\left(\frac{1-b_1}{1-b_1-\rho}e^{-b_1 fx} - \frac{1-b_2}{1-b_2-\rho}e^{-b_2 fx}\right) & \text{if } \rho \neq \frac{1}{2}, \\ 1 + \frac{1}{9}e^{-2fx} - \frac{1}{3}\left(\frac{11}{6} + \frac{fx}{4}\right)e^{-fx/2} & \text{if } \rho = \frac{1}{2}. \end{cases} \quad (4)$$

The first and second moments of the random variable σ in this case take the form:

$$\delta_1 = \frac{1}{f}\cdot\frac{\rho(2-\rho)}{1-\rho},\ \delta_2 = \frac{1}{f^2}\cdot\frac{2\rho(3+\rho^3-\rho^2-2\rho)}{(1-\rho)^2}.$$

2.2. The case of limited memory space. In this case we can obtain some exact results only if service time has an exponential distribution and does not depend on customer's capacity, i.e. $F(x,t) = L(x)B(t)$, where $B(t) = 1 - e^{-\mu t}$, $\mu > 0$.

Denote by $L_*^k(x)$ the kth order Stieltjes convolution of the distribution function $L(x)$:

$$L_*^{(0)}(x) \equiv 1,$$

$$L_*^{(k)}(x) = \int_0^x L_*^{(k-1)}(x-u)dL(u) = L_*^{(k-1)} * L(x),\ k = 1, 2, \ldots.$$

Then we obtain [4] for the loss probability of the system $M/M/1/(\infty, V)$ under consideration:

$$P = \frac{1-(1-\rho)U(V)}{\rho U(V)},$$

where $U(V) = \sum\limits_{k=0}^{\infty} \rho^k L_*^{(k)}(V)$.

It is clear that the last formula is not convenient for calculations. But, if we assume that customer's capacity has an exponential distribution: $L(x) = 1 - e^{-fx}$, $f > 0$, we obtain

$$P = \begin{cases} \dfrac{1-\rho}{e^{(1-\rho)fV} - \rho} & \text{if } \rho \neq 1, \\ (1 + fV)^{-1} & \text{if } \rho = 1. \end{cases}$$

2.3. Analytical estimations of loss characteristics. Denote by $D_V(x)$ the distribution function of total customers capacity for the system $M/G/1/(\infty, V)$ in stationary mode. Then we can define the following loss characteristics [4]:

a) Loss probability

$$P = 1 - \int_0^V D_V(V - x)dL(x).$$

This characteristic shows the part of customers being lost.

b) Probability that unit of customer's capacity will be lost

$$Q = 1 - \frac{1}{\varphi_1} \int_0^V x D_V(V - x)dL(x).$$

This caracteristic shows the part of customers capacity (or customers information) being lost.

Let $D_\infty(x)$ be the distribution function of the total customers capacity for the system $M/G/1/\infty$ (in the case $V = \infty$) in stationary mode (we assume that other parameters of the systems $M/G/1/(\infty, V)$ and $M/G/1/\infty$ are the same and the stationary mode exists for both systems).

It can be easy shown [4] that $D_\infty(x) \leq D_V(x)$ for all real x. This inequality gives us the possibility to estimate loss characteristics for the system $M/G/1/(\infty, V)$, if the distribution function $D_\infty(x)$ is known. Indeed, from the above inequality we have

$$P = 1 - \int_0^V D_V(V - x)dL(x) \leq 1 - \int_0^V D_\infty(V - x)dL(x) = P^*, \quad (5)$$

$$Q = 1 - \frac{1}{\varphi_1} \int_0^V x D_V(V - x)dL(x) \leq 1 - \frac{1}{\varphi_1} \int_0^V x D_\infty(V - x)dL(x) = Q^*. \quad (6)$$

Then, for very small P, Q (or for rather large V), the values P^* and Q^* differ inessentially from the appropriate values P and Q, because

$D_\infty(x) = \lim\limits_{V\to\infty} D_V(x)$. In general, the values P^* and Q^* may be used as upper limits for P and Q. It follows from (6), (7) that, if we choose the memory volume V so that

$$1 - \int_0^V D_\infty(V-x)dL(x) = P^* \quad \left(1 - \frac{1}{\varphi_1}\int_0^V xD_\infty(V-x)dL(x) = Q^*\right),$$

then we can guarantee that the loss probability (the probability that unit of customer's capacity will be lost) be less than P^* (Q^*).

Unfortunately, we can obtain the explisite formula for $D_\infty(x)$ in some particular cases only. For example, it is possible if customer's capacity has an exponential distribution (with parameter f) and service time is proportional to customer's capacity (with poroportionality coefficient c). Then we obtain from the relation (4):

$$P^* = \begin{cases} \frac{\rho(1-\rho)}{\sqrt{\rho(4+\rho)}}\left(\frac{e^{-b_1 fV}}{1-b_1-\rho} - \frac{e^{-b_2 fV}}{1-b_2-\rho}\right) - \frac{\rho e^{-(1-\rho)fV}}{1-2\rho} & \text{if } \rho \neq \frac{1}{2}, \\ \frac{1}{9}(8e^{-fV/2} + e^{-2fV}) + \frac{1}{6}fVe^{-fV/2} & \text{if } \rho = \frac{1}{2}; \end{cases}$$

$$Q^* = \begin{cases} (1+fV)e^{-fV} + \frac{(1+\rho fV - e^{\rho fV})e^{-fV}}{1-2\rho} + \frac{\rho(1-\rho)}{\sqrt{\rho(4+\rho)}}\times \\ \times\left\{\frac{e^{-b_1 fV} - [1+(1-b_1)fV]e^{-fV}}{(1-b_1)(1-b_1-\rho)} - \frac{e^{-b_2 fV} - [1+(1-b_2)fV]e^{-fV}}{(1-b_2)(1-b_2-\rho)}\right\} & \text{if } \rho \neq \frac{1}{2}, \\ \frac{1}{9}(10e^{-fV/2} - e^{-2fV}) + \frac{1}{4}fVe^{-fV/2} & \text{if } \rho = \frac{1}{2}. \end{cases}$$

Note that the value Q is more objective loss characteristic than P, but its calculation is more complicated.

If it is impossible to obtain the explicit form of $D_\infty(x)$, we can use (see e.g. [2, 4]) a good approximation of this function by the distribution function

$$D^*_\infty(x) = 1 - \rho + \rho\frac{\gamma(p, gx)}{\Gamma(p)},$$

where $\gamma(p, gx) = \int_0^{gx} t^{p-1}e^{-t}dt$ is the incomplete gamma function, $\Gamma(p) = \gamma(p,\infty)$ is the gamma function. The first and second moments of this distribution are equal to $\delta_1^* = \rho p/g$, $\delta_2^* = \rho p(p+1)/g^2$, respectively. The parameters p and g should be chosen so that equalities $\delta_1^* = \delta_1$, $\delta_2^* = \delta_2$ hold. So, we obtain

$$p = \frac{{\delta_1}^2}{\rho\delta_2 - {\delta_1}^2}, \quad g = \frac{\rho\delta_1}{\rho\delta_2 - {\delta_1}^2}.$$

Tables 1 and 2 show the values of the functions $D_\infty(x)$ defined by (4) and $D^*_\infty(x)$ for $\rho = 0.2$ and $\rho = 0.8$.

For determination of the loss probability or (more precisely) the probability P^*, we can also approximate the distribution function

$$\Phi(x) = \int_0^x D_\infty(x-u)dL(u)$$

in the relation (5) by the gamma distribution function $\Phi^*(x) = \frac{\gamma(q,rx)}{\Gamma(q)}$.

Table 1: Distribution functions $D_\infty(x)$ and $D^*_\infty(x)$ for $\rho = 0.2$

x	$D_\infty(x)$	$D^*_\infty(x)$
0.0	0.80000	0.80000
1.0	0.84899	0.85069
2.0	0.90966	0.90866
3.0	0.94961	0.94821
4.0	0.97263	0.97183
6.0	0.99215	0.99225
8.0	0.99778	0.99798
10.0	0.99938	0.99949
15.0	0.99997	0.99999
20.0	0.99999	0.99999

Table 2: Distribution functions $D_\infty(x)$ and $D^*_\infty(x)$ for $\rho = 0.8$

x	$D_\infty(x)$	$D^*_\infty(x)$
0	0.20000	0.20000
2	0.37821	0.38170
5	0.63390	0.63088
10	0.85886	0.85705
15	0.94729	0.94708
20	0.98051	0.98439
25	0.99282	0.99316
30	0.99736	0.99758
50	0.99995	0.99996
60	0.99999	1.00000

The first and second moments of this distribution have the form $f_1^* = q/r$, $f_2^* = q(q+1)/r^2$. The function $\Phi(x)$ is the distribution function of the sum of the independent random variables σ and ζ. Thus, the first and second moments of the distribution function $\Phi(x)$ are equal to $f_1 = \delta_1 + \varphi_1$, $f_2 = \delta_2 + \varphi_2 + 2\delta_1\varphi_1$, where δ_1 and δ_2 can be determined from the relations (2) and (3).

The parameters q, r should be chosen so that equalities $f_1^* = f_1$, $f_2^* = f_2$ hold, whence

$$q = \frac{{f_1}^2}{f_2 - {f_1}^2}, \quad r = \frac{f_1}{f_2 - {f_1}^2}.$$

Then we obtain the approximate equality

$$P^* = 1 - \Phi^*(V). \tag{7}$$

3. Calculation and simulation

In spite of possibility to obtain an analytical estimators of loss characteristics, simulation often seems necessary: 1) to obtain more precise estimations, 2) to estimate the quality of approximate results and 3) to obtain good approximate results when analytical approach is impossible or rather complicated.

Table 3: Probabilities P, Q and P^*, Q^* for $\rho = 0.2$

V	P	P^*	Q	Q^*
0	1.00000	1.00000	1.00000	1.00000
2	0.17106	0.24854	0.44498	0.49844
4	0.04467	0.07076	0.14188	0.16763
6	0.01309	0.02074	0.04346	0.05174
8	0.00387	0.00590	0.01303	0.01538
10	0.00111	0.00172	0.00381	0.00447
12	0.00032	0.00048	0.00113	0.00128
14	0.00009	0.00013	0.00036	0.00036
16	0.00003	0.00004	0.00010	0.00010
18	0.00001	0.00001	0.00002	0.00003

Let us confirm this statement by some examples. We shall anlyze the case $\xi = c\zeta + \xi_1$, where $c \geq 0$ and the random variables ζ and ξ_1 are independent (it is clear that in the case $c = 0$ the service time of a customer is independent on his capacity).

Table 4: Probabilities P, Q and P^*, Q^* for $\rho = 0.8$

V	P	P^*	Q	Q^*
0	1.00000	1.00000	1.00000	1.00000
5	0.09427	0.44515	0.21473	0.53490
10	0.02678	0.17490	0.06241	0.21599
15	0.00899	0.06570	0.02115	0.08181
20	0.00321	0.02434	0.00757	0.03038
25	0.00117	0.00897	0.00280	0.01121
30	0.00043	0.00330	0.00103	0.00413
35	0.00016	0.00122	0.00038	0.00152
40	0.00006	0.00045	0.00015	0.00056
45	0.00002	0.00017	0.00006	0.00021

We can compare the values P^* and Q^* with the values P and Q obtained by simulation for the system $M/M/1/(\infty, V)$ ($\xi_1 \equiv 0$), when $\rho = 0.2$ (see table 3) and $\rho = 0.8$ (see table 4). We can see from these results that the approximate equalities $P \cong P^*$ and $Q \cong Q^*$ are satisfactory only if loss characteristics are very small.

Table 5: Probabilities P and P^* for $\rho = 0.2$

V	P^*	P
0.0	1.00000	1.00000
0.5	0.54190	0.51995
1.0	0.16867	0.10250
1.5	0.04323	0.03689
2.0	0.01004	0.00586
2.5	0.00220	0.00131
3.0	0.00046	0.00021
3.5	0.00009	0.00003
4.0	0.00002	0.00001
4.5	0.00000	0.00000

In more general case of the system $M/G/1/(\infty, V)$, we can obtain the approximate value of P^* using the approximate equality (7).

Let, for example, customer's capacity have uniform distribution on the segment $[0; 2]$ and service time of the customer be proportional to his capacity with proportionality coefficient $c = 1$ ($\xi_1 \equiv 0$). The results for these systems obtained by calculation (P^*) and simulation (P) are presented in table 5 ($\rho = 0.2$) and table 6 ($\rho = 0.8$).

Table 6: Probabilities P and P^* for $\rho = 0.8$

V	P^*	P
0	1.00000	1.00000
1	0.67623	0.30277
2	0.37052	0.10314
4	0.09274	0.01882
6	0.02094	0.00432
8	0.00450	0.00105
10	0.00094	0.00024
12	0.00019	0.00007
14	0.00004	0.00001
16	0.00001	0.00000

Table 7: Probabilities P and P^* for $\rho = 0.2$

V	P^*	P
0	1.00000	1.00000
1	0.22507	0.16866
2	0.05391	0.03801
3	0.01306	0.00911
4	0.00318	0.00229
5	0.00078	0.00054
6	0.00019	0.00014
7	0.00005	0.00003
8	0.00001	0.00001
9	0.00000	0.00000

In practice it is often assumed that customer's capacity ζ has an exponential distribution and the random variable ξ_1 has an exponential distribution too. We can see in tables 7 ($\rho = 0.2$) and 8 ($\rho = 0.8$) the appropriate results for such systems for the case $f = 2$, where f is a parameter of customer's capacity, $\mu = 2$, where μ is a parameter of the random variable ξ_1, and $c = 1$.

Table 8: Probabilities P and P^* for $\rho = 0.8$

V	P^*	P
0	1.00000	1.00000
1	0.68200	0.26203
2	0.43507	0.11892
4	0.16857	0.03418
8	0.02360	0.00407
12	0.00319	0.00053
14	0.00116	0.00019
18	0.00015	0.00002
22	0.00002	0.00000
24	0.00001	0.00000

If in the system under consideration entrance flow is not Poissonian (generally, not Markovian), we cannot obtain any exact result in the case $V < \infty$. However, in this case we can use simulation to obtain the loss probability or the probability that a unit of customer's capacity will be lost.

Let us analyze, for example, the influence of entrance flow on loss characteristcs of the system under consideration. It is interesting to compare the loss characteristics of the system with regular entrance flow and the system with Poisson entrance flow when other parameters (including the value ρ) of these systems are the same. Assume that customer's capacity ζ has an exponential distribution with parameter $f = 1$, the random variable ξ_1 also has an exponential distribution with parameter $\mu = 1$ and the proportionality coefficient c is equal to 1.

In Fig. 1 and Fig. 3 we can see how the loss probability P depends on the capacity volume in the case of regular entrance flow (1) and Poisson flow (2) for $\rho = 0.2$ and $\rho = 0.8$ consequently. In Fig. 2 and Fig. 4 we show a similar dependence for the probability Q that a unit of cusomer's capacity will be lost.

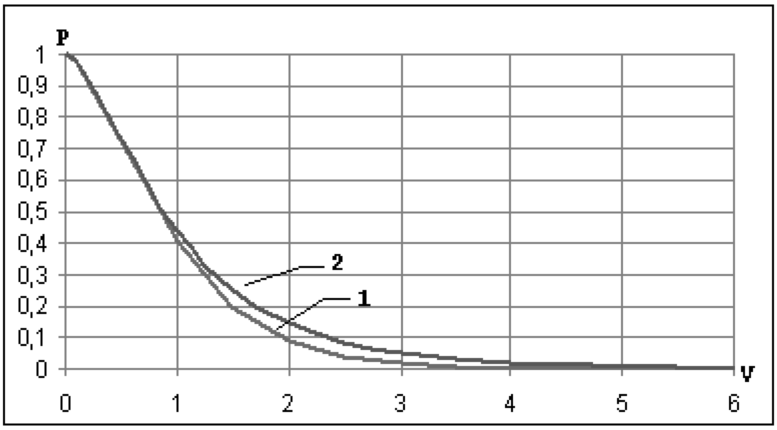

Fig. 1. Probability P for regular (1) and Poisson (2) flow ($\rho = 0.2$)

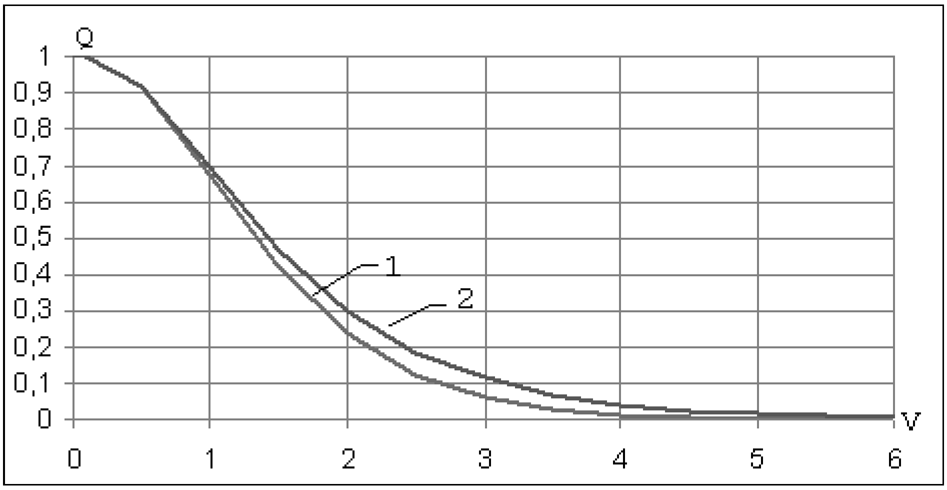

Fig. 2. Probability Q for regular (1) and Poisson (2) flow ($\rho = 0.2$)

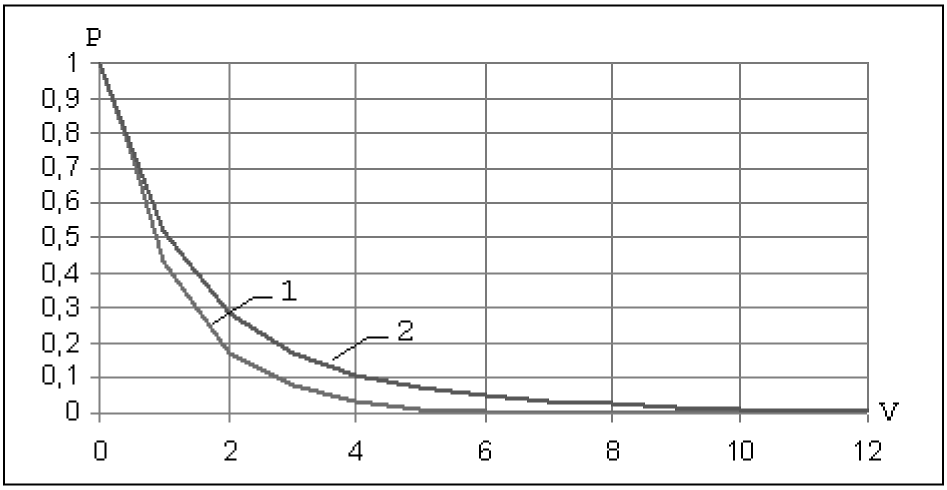

Fig. 3. Probability P for regular (1) and Poisson (2) flow ($\rho = 0.8$)

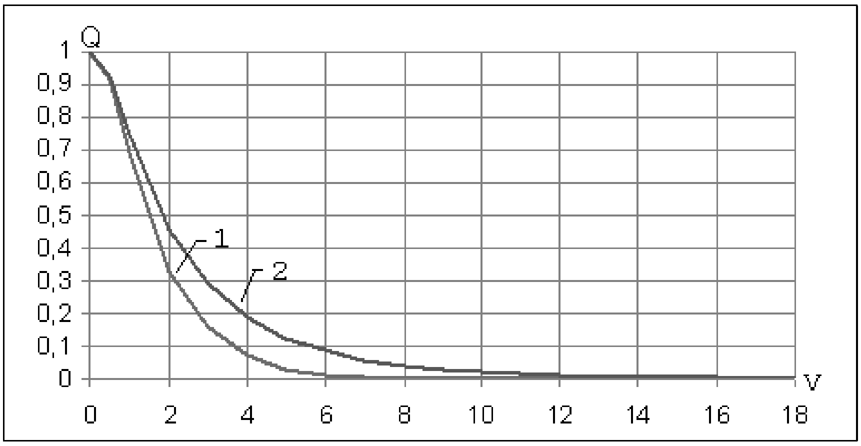

Fig. 4. Probability Q for regular (1) and Poisson (2) flow ($\rho = 0.8$)

4. Conclusion

In this paper we analyze the possibilities of analytical approximation of some loss characteristics for single-server systems with limited total customers capacity. It was shown by some examples that using the upper limits P^* and Q^* instead of the loss probability P and the probability Q that a unit of customer's capacity will be lost, we can guarantee that the appropriate choice of the memory capacity V provides no excess.

However, we should use simulation for more precise estimation of the memory capacity.

References

[1] A. M. Alexandrov, B. A. Kaz. Non-homogeneous demands flow service. *Izvestiya AN SSSR. Tekhnicheskaya Kibernetika*, No 2, 47–53, 1973. (In Russian).

[2] B. Sengupta. The spatial requirement of an $M/G/1$ queue or: how to design for buffer space. *Lect. Notes Contr. Inf. Sci.,* **60**, 547–562, 1984.

[3] O. M. Tikhonenko. *Queueing Models in Computer Systems.* Universitetskoe, Minsk 1990. (In Russian).

[4] O. Tikhonenko. *Metody probabilistyczne analizy systemów informacyjnych.* Akademicka Oficyna Wydawnicza EXIT, Warszawa 2006.