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FROM EUCLID'S ELEMENTS TO COSSERAT CONTINUA

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Abstract.
The classical continuum theory is based on the assumption that each small particle behaves like a simple material point and ignores the relative motions of constituent parts of this particle. The development of the notion of a point and the development of non-Euclidean geometry is considered. The Cosserat continuum is an example of medium with microstructure, in which "a ponit" has an internal structure. Its motion is determined by the displacement and rotation fields.

1. Introduction

We begin this article from the words of the noted physicist Yuval Ne'eman "The passage in Plutarch about Plato's credo with respect to God's interest in geometry is quoted whenever the birth of modern physics is touched upon, even though the actual realization may differ considerably from Plato's dream. This does not reflect badly on Plato if we consider the state of experimental physics around 400 BC..." [1] (see also [2]).

We can paraphrase these words: "The passage in Plutarch about Plato's credo with respect to God's interest in geometry is quoted whenever the birth of modern mathematics is touched upon, even though the actual realization may differ considerably from Plato's dream. This does not reflect badly on Plato if we consider the state of mathematics around 400 BC..."

2. Development of the notion of a point

Book 1 of Euclid's *Elements* begins with 23 definitions such as point, line, surface, etc. For example:

- *A surface is defined as something that has a length and a width but does not have a hight.*
- *A curve is something that has only a length but does not have a width and a hight.*
- *A point is considered as something that has no parts.*

In mechanics of continua a material point is treated, on the one hand, as a very small particle to apply methods of calculus (the smoothness of continuum means that we can use the notions of continuous and differentiable functions of a point); on the other hand, a material point contains a very large number of atoms to consider continuum instead of discrete lattice. The classical continuum theory is based on the assumption that each small particle behaves like a simple material point and ignores the relative motions of constituent parts of this particle. In other words, the internal structure of a material point is not taken into account.

However, materials used in contemporary high technology are characterized by complex internal structure. The results of experiments also show that "a point" may have an internal structure which can influence the behavior of a medium, hence **a point has parts** (further discussion of a model of material point can be found in [3, 4]).

A list of important examples of continua with microstructure contains: granular media, laminated and layered materials, blocky structures, fibrous materials, consolidated soils, rocks with inhomogeneous microstructure, polycrystalline solids, ceramic composites, functionally graded materials, continua with voids, liquids with nondiffusing gas bubbles, liquid crystals, cracked media, bodies with continuous distribution of dislocations, etc.

The microstructured materials, i.e. materials with irregularities, cannot be described by the classical theories. These facts have forced researchers to build up generalized continuum theories that take into account the internal structure of a small particle. Such a structure can be described using various methods. The microstructure theory of Mindlin [5], the micromorphic theory of Eringen and Suhubi [6], the director theory of Toupin [7], and the multipolar theory of Green and Rivlin [8] should be mentioned.

Worthy of mention are also the early works. Poisson [9] proposed to regard the molecules as little rigid bodies capable of rotation as well as translatory displacements (see also [10]). Voigt [11] introduced a model of the medium with rotational interaction of its particles for studying elastic properties of a crystal.

Mindlin [5] suggested that each element (each "point") of the material is itself a deformable continuum. These continua are fitted together smoothly, so all the simplicity of a field theory results. The macromedium is a collection of particles, whith each of which a micromedium is associated.

Eringen and Suhubi [6] supposed that the material particle ("a point") contains N discrete micromaterial elements. The position vector of a material point in the microelement is expressed as a sum of the position vector of the center of mass and the position vector of a point in the microelement relative to this center of mass. Upon the deformation of the body, because of the rearrangement and relative deformation of the microelements, we obtain the new position vector of the center of mass and the new relative position vector of the material point. The basic assumption underlying the theory of Eringen and Suhubi is the axiom of affine motion according to which the motion of the particle consists of a translation, a rotation about its center of mass, and an affine deformation. Applications of this approach can be found in [12-14].

Duhem [15] noticed that microstructure could be described as effects of direction, and suggested that materials can be considered as sets of points having vectors attached to them, that is as oriented media. Various theories based on this idea were analyzed by Green and Rivlin [8, 16]. Consider a body that is a collection of material points to each of which N vectors called directors are attached. The theory is valid for a greater number of directors, but usually the range 1, 2 or 3 directors is considered.

For example, the theory of liquid crystals proposed by Ericksen [17, 18] corresponds to a choice of only one director. Liquid crystals (a substance that flows as a liquid but maintains some of the ordered structures characteristic of a crystal) have rodlike molecules whose alignment influences their material behavior.

Three main categories have been recognized: nematic, cholesteric, and smectic. Nematic liquid crystals consist of cigar-shaped molecules with their long axis parallel. They maintain their orientation but are free to move in any direction. Cholesteric liquid crystals form thin layers and within each layer the molecules are arranged with thier long axes in the plane of the layer and parallel to each other, as a two-dimensional nematic structures. Smectic

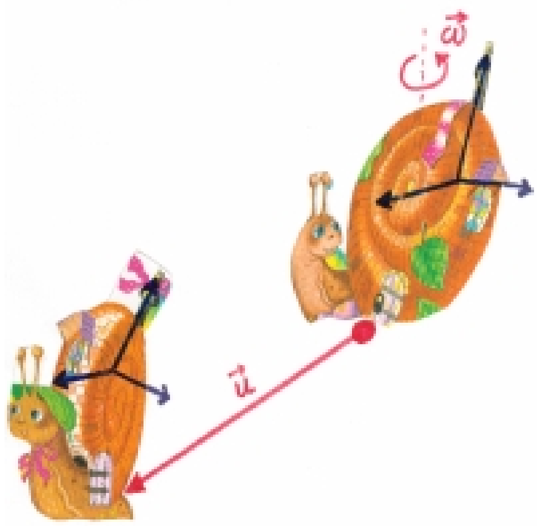

Fig. 1. The motion of a Cosserat continuum.

liquid crystals consist of flat layers of molecules with their long axes oriented perpendicularly to the plane of the layer. During motion the sheets flow freely over each other, but the molecules within each layer remain oriented and do not move between layers [19]. In all these cases the orientation of the molecules can be described by the directors.

A Cosserat medium [20] corresponds to a choice of three independent directors under some additional condition which means that during the deformation process directors can only rotate as the rigid body. The motion of a Cosserat continuum is described by both the displacement vector $\mathbf{u}$ and the rotation vector $\boldsymbol{\omega}$ independent on the displacement field. The situation is schematically shown if Fig. 1.

Additional degrees of freedom connected with the rotation vector cause the appearance of couple-stresses (moment of force per unit of area) in addition to the ordinary stresses (force per unit of area).

The Cosserat continuum is often used for modeling plastic deformation in materials, propagation and interaction of elastic waves in solids with microstructure, for development of theories of rods and shells and for describibg damage of materials (see [21–30], among others).

3. Non-Euclidean geometry

In Euclid's *Elements*, the definitions followed by five postulates. We recall the fifth:

If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

For two thousand years, many attempts were made to prove the parallel postulate from the other four, in particular using the equivalent formulations of the fifth postulate. One of such formulations is known from the time of Proclus (V century), but became recognized as Playfair's axiom after Playfair's commentary (1795) on Euclid's *Elements* in which he proposed the equivalent formulation of the fifth postulate [31]:

Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.

Legendre proved that the fifth postulate is equivalent to the following one:

The sum of the angles of a triangle is equal to two right angles.

In Lobachevsky's "hyperbolic" geometry the fifth postulate is replaced by:

There exist at least two lines parallel to a given line through a given point not on the line.

In Lobachevsky's geometry the sum of the angles of a triangle is less than two right angles.

In Riemann's "spherical" geometry the fifth postulate is replaced by:

Every line through a point not on a given line meets this line.

In Riemann's geometry the sum of the angles of a triangle is greater than two right angles.

History of non-Euclidean geometry can be found, for example, in [32–38].

There are many applications of non-Euclidean geometry in various fields of physics. Apart from well-known Einstein's general theory of relativity, the interesting reader can find other applications in [39–42], among others. In particular, the creation and progress of the continuum theory of imperfections (dislocations and disclinations) in Cosserat continua is closely connected with the use of the ideas and methods of non-Euclidean geometry.

4. Gauge theory of defects in Cosserat continuum

A generalized understanding of media with microstructure, which combines the development of the notion of a point and the non-Euclidean geometry approach, can be achieved using the mathematical structure of fiber bundle (see, for example, [43, 44]).

A structure of differential fiber bundle is a six-tuple

$$(E, B, F, G, \pi, \psi),$$

where the differential manifold E is the total space, the differential manifold B is the base, the differential manifold F is the fiber, the Lie group G is the structural group, the differential map $\pi : E \rightarrow B$ is the projection, and ψ is a family of diffeomorphisms.

Recalling Cosserat results in contemporary language, one can say that within their treatment the properties of the mathematical model are strictly separated. All the geometrical properties are carried by the base manifold, while all the physical properties are embedded within the standard fiber – structural group

$$T(3) \triangleright SO(3),$$

where $T(3)$ is the group of translations, $SO(3)$ is the special orthogonal group, $\triangleright$ is the semi-direct product.

In Fig. 1, the based manifold is made of snails, whereas the shall corresponds to the standard fiber.

In the gauge theory of defects, homogeneity breakdown of action of the group $SO(3)$ leads to apperaing of disclinations and rotation dislocations, while homogeneity breakdown of action of the group $T(3)$ leads to apperance of translational dislocations (see [45-48] and references therein).

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