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k-CONNECTIVITY

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Abstract
The notion of some connectedness is introduced and it is compared with usual connected sets.

Subsets of a topological space that are called i-connected have been introduced by J. Knop & M. Wróbel in [2] in 2006. The reason of introducing this notion is based upon the fact that a subset of the space $\mathbb{R}$ with nonempty interior is i-connected if and only if it is connected.

Definition 1 (J. Knop & M. Wróbel – 2006) A subset A of a topological space X is called to be i-connected if it is connected and $\operatorname{Int} A$ is connected.

We want to define the so called k-connected subsets of a topological space, which are a little different from i-connected sets, but have some interesting properties.

Definition 2 *A subset A of a topological space X is called to be k-connected if its interior is connected and $A \subset \overline{\operatorname{Int} A}$.*

Of course, each k-connected set is i-connected and connected as well, but not conversely.

Moreover, let us remark that each open connected set is k-connected as well.

It is quite easy to see that for Euclidean space of real numbers $\mathbb{R}$, a subset A with a nonempty interior is k-connected if and only if it is connected.

As we could observe, k-connected sets are similar to i-connected ones, so it is worth to compare these kinds of sets.

Theorem 1 *A subset A of a topological space is k-connected if and only if it can be represented in the form*

$$A = B \cup C$$

such that C is open and connected and

$$B \cap C = \varnothing, \quad B \subset \overline{C}.$$

P r o o f. 1. Suppose first that a set A is k-connected. Let $C = \operatorname{Int} A$ and $B = A \setminus C$. Then, of course, $\operatorname{Int} B = \varnothing$ and $B \cap C = \varnothing$. Since $A \subset \overline{\operatorname{Int} A}$, then $B \subset \overline{C}$.

2. Suppose now that a set A has the form $A = B \cup C$, where C is open and connected and

$$B \cap C = \varnothing, \quad B \subset \overline{C}.$$

Then

$$C = \operatorname{Int} C \subset \operatorname{Int} A,$$

consequently

$$\overline{C} \subset \overline{\operatorname{Int} A}$$

and

$$A = B \cup C \subset \overline{C} \cup C = \overline{C} \subset \overline{\operatorname{Int} A}.$$

Two considered possibilities complete the proof. □

Theorem 2 *If A is a k-connected subset of a topological space and C fulfils the following inclusions*

$$A \subset C \subset \overline{A},$$

then C is also a k-connected set.

P r o o f. Since $\operatorname{Int} A$ is a connected set and

$$\operatorname{Int} A \subset \operatorname{Int} C \subset \operatorname{Int} \overline{A} \subset \overline{A} \subset \overline{\overline{\operatorname{Int} A}} = \overline{\operatorname{Int} A},$$

then $\operatorname{Int} C$ is a connected set. Moreover,

$$C \subset \overline{A} = \overline{\overline{\operatorname{Int} A}} = \overline{\operatorname{Int} A} \subset \overline{\operatorname{Int} C},$$

which proves the theorem. □

Theorem 3 *If $\{A_s : s \in \mathcal{S}\}$ is a class of k-connected subsets of a topological space and there exists an index s_0 in $\mathcal{S}$ such that*

$$\operatorname{Int} A_s \cap \operatorname{Int} A_{s_0} \neq \varnothing$$

for all s from $\mathcal{S}$, then the set $\bigcup_{s \in \mathcal{S}} A_s$ is also a k-connected subset of X.

P r o o f. Each of the sets $\operatorname{Int} A_s$ is connected, so is A_s. In view of our assumptions, the sets $\bigcup_{s \in \mathcal{S}} A_s$ and $\bigcup_{s \in \mathcal{S}} \operatorname{Int} A_s$ are connected as well. Since each of the sets A_s is k-connected, then $A_s \subset \overline{\operatorname{Int} A_s}$ and applying standard properties of interior and closure operations one can infer from the definition of k-connected sets that

$$\operatorname{Int}\left(\bigcup_{s \in \mathcal{S}} \operatorname{Int} A_s\right) = \bigcup_{s \in \mathcal{S}} \operatorname{Int} A_s \subset \bigcup_{s \in \mathcal{S}} \overline{\operatorname{Int} A_s} \subset \overline{\bigcup_{s \in \mathcal{S}} \operatorname{Int} A_s}.$$

From this and connectedness of the set $\bigcup_{s \in \mathcal{S}} \operatorname{Int} A_s$ we can infer that the set $\operatorname{Int}\left(\bigcup_{s \in \mathcal{S}} A_s\right)$ is connected.

Moreover,

$$\bigcup_{s\in\mathcal{S}} A_s \subset \bigcup_{s\in\mathcal{S}} \overline{\operatorname{Int} A_s} \subset \overline{\bigcup_{s\in\mathcal{S}} \operatorname{Int} A_s} \subset \overline{\operatorname{Int}\left(\bigcup_{s\in\mathcal{S}} A_s\right)},$$

which completes the proof. □

As usual, one can ask whether a continuous image of a k-connected set is also k-connected. The following theorem gives the positive answer under some additional condition.

Theorem 4 *If X and Y are topological spaces and a map $f : X \longrightarrow Y$ is an open and continuous injection of X into Y, then $f(A)$ is a k-connected subset of Y whenever A is a k-connected subset of X.*

P R O O F. If A is a k-connected subset of the space X, then it can be represented in the form

$$A = B \cup C,$$

where C is open, connected and

$$B \cap C = \varnothing, \quad B \subset \overline{C}.$$

Then

$$f(A) = f(B \cup C) = f(B) \cup f(C).$$

The set $f(C)$ is open, connected and

$$f(B) \cap f(C) = \varnothing, \quad f(B) \subset f\left(\overline{C}\right) \subset \overline{f(C)}.$$

In view of Theorem 1, the set $f(A)$ is k-connected. □

There is no difficulty to construct topological spaces and a continuous surjection for which the image of k-connected sets is not k-connected. So, openness of such a map cannot be omitted in the previous theorem.

If a topological space X fulfils the condition: for each element x of X there exists an open and connected set U such that $x \in U$, then we can define the notion of k-component of a point. Theorem 3 allows us to consider the biggest k-connected set containing a point x. It is called the *k-component* of this point in topological space X. It happens that in such spaces k-components coincide with components (in the usual sense). Let us notice that this condition is a bit weaker than local connectedness of the space.

Theorem 5 *If X is a topological space such that for each element x of X there exists an open and connected set U such that $x \in U$, then k-component of any point x is equal to the component of this point (in the usual sense).*

P R O O F. Since each k-connected set is connected (in the usual sense), then k-component of a point in any topological space is contained in the (usual) component of that point. It is left then that in topological spaces fulfilling our condition, the component of a point is contained in its k-component. Let C be the component of a point x_0. For each point x from the set C there exists an open and connected set U_x such that $x \in U_x$. From the definition of components it follows that $U_x \subset C$ for each point x from C. Thus the component C is a union of all sets U_x, $x \in C$. Then the set C is open, hence it is k-connected and containing the component of the point x_0. □

Now we will consider some sufficient conditions for topological spaces in which the class of k-connected sets and the class of connected sets with nonempty interior coincide. Let us denote this class of topological spaces by $\mathscr{K}$.

Theorem 6 *If X is a topological space from $\mathscr{K}$, then*

$$x \in \operatorname{Int}\left(\overline{U} \cup \overline{V}\right)$$

for each disjoint open and connected sets U and V and x belonging to $\overline{U} \cap \overline{V}$.

P R O O F. Let us suppose in the contrary that there exist disjoint, open and connected sets U, V and a point x such that

$$x \in \overline{U} \cap \overline{V}, \quad x \notin \operatorname{Int}\left(\overline{U} \cup \overline{V}\right).$$

Since the sets U, V are not separated from $\{x\}$, then the set E, where $E = U \cup V \cup \{x\}$, is connected. Its interior equals $U \cup V$, which is not connected. In this way we obtained a connected set which is not k-connected. Contradiction. □

Similarly, one can prove the next theorem:

Theorem 7 *If X is a topological space from $\mathscr{K}$ and U, V, and W are pairwise disjoint open and connected subsets of X, then*

$$\overline{U} \cap \overline{V} \cap \overline{W} = \varnothing.$$

Let us remind the notions of cut points and strong cut points of a topological space.

Definition 3 *Let X be a connected topological space. A point x from X is called a cut point if the set $X \setminus \{x\}$ is not connected.*

Definition 4 *Let X be a connected topological space. A point x from X is called a strong cut point if the set $X \setminus \{x\}$ has two components.*

As a corollary we can obtain the following theorem:

Theorem 8 *If X is a topological space from $\mathscr{K}$ and x is a cut point of the space X, then $X \setminus \{x\}$ has two components.*

We can formulate this theorem in other words: Each cut point of a space X from the class $\mathscr{K}$ is a strong cut point.

References

[1] J.L. Kelley. *General Topology*. Springer, New York - Heidelberg - Berlin, 1955.

[2] J. Knop, M. Wróbel. Some properties of i-connected sets. *Annales Academiae Paedagogicae Cracoviensis, Studia Mathematica,* VI, Folia 45, 51–56, 2007.