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ON SOME PROPERTIES OF CONNECTED FUNCTIONS

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Abstract
We consider some properties of functions defined in a topological space X with values in another topological space Y.

We shall consider some properties of functions defined in a topological space X with values in a topological space Y.

Topological terminology is taken from the books *General Topology* by R. Engelking [1] and *General Topology* by J. L. Kelley [7].

Definition 1 *We say that a function $f : X \longrightarrow Y$ has the Darboux property, if the image of connected subset of X is connected.*

The set of all functions which have the Darboux property is denoted by $\mathscr{D}$.

Definition 2 *We say that a function $f : X \longrightarrow Y$ has the local Darboux property, if for each point x in X and every neighbourhood U of x there exists a connected neighbourhood V such that $f(V)$ is a connected subset of Y.*

The set of all functions which have the local Darboux property is denoted by $\mathscr{D}_l$.

Definition 3 *We say that a function $f : X \longrightarrow Y$ is connected if its graph is a connected set in $X \times Y$.*

The set of all functions which are connected is denoted by $\mathscr{C}$.

Definition 4 *We shall say that a function $f : X \longrightarrow Y$ is strongly connected if $f|E$ is a connected set for each connected subset E of X.*

The set of all functions which are strongly connected is denoted by $\mathscr{C}_s$.

Definition 5 *We shall say that a function $f : X \longrightarrow Y$ is locally strongly connected if for each x in X and its open neighbourhood U there exists an open and connected neighbourhood E of x such that $f|E$ is a connected set in the space $X \times Y$.*

The set of all functions which are strongly connected is denoted by $\mathscr{C}_{ls}$.

By a subgraph of a function $f : X \longrightarrow \mathbb{R}$ we mean the set

$$\{(x, y) \in X \times \mathbb{R} : y < f(x)\},$$

which is denoted by $f(-)$.

By an overgraph of a function $f : X \longrightarrow \mathbb{R}$ we mean the set

$$\{(x, y) \in X \times \mathbb{R} : y > f(x)\},$$

which is denoted by $f(+)$.

We shall make no distinction between a function and its graph.

Definition 6 *We shall say that a function $f : X \longrightarrow \mathbb{R}$ cuts continuum if*

$$f \cap M \neq \emptyset$$

for each continuum M for which $M \cap f(+) \neq \emptyset$ and $M \cap f(-) \neq \emptyset$.

Definitions 3, 4, 5 and 6 define the same class of functions when X and Y are equal to $\mathbb{R}$ with natural topology.

Similarly, definitions 1 and 2 define the same class of functions when X and Y are equal to $\mathbb{R}$ with natural topology.

In the article we shall discuss some properties of those classes and give some sufficient conditions for the space X in which real functions defined in X form the same class.

Immediately from the definitions the next properties follow:

Property 1 *Each continuous function is strongly connected and has the Darboux property.*

Property 2 *Each continuous function defined in a connected space is connected.*

Property 3 *Each continuous function defined in a locally connected space is locally strongly connected and has the local Darboux property.*

Theorem 1 *For every topological space X and Y*

$$\mathscr{C}_s \subset \mathscr{D}.$$

Theorem 2 *If a topological spaces X is connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_s \subset \mathscr{C}.$$

This theorem can be completed to get sufficient condition for a space X to be connected.

Theorem 3 *If a topological space Y has at least two elements and for topological space X*

$$\mathscr{C}_s \subset \mathscr{C},$$

then X is connected.

Theorem 4 *If a topological space X is locally connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_s \subset \mathscr{C}_{ls}, \quad \mathscr{C}_s \subset \mathscr{D}_l, \quad \mathscr{C}_{ls} \subset \mathscr{D}_l, \quad \mathscr{D} \subset \mathscr{D}_l.$$

Theorem 5 *If a topological spaces X is connected and locally connected and Y is an arbitrary topological space, then*

$$\mathscr{C}_{ls} \subset \mathscr{C}.$$

The proof of this theorem can be found in [6]. In the same article there are given examples of real functions defined in $\mathbb{R}^2$ which show that all the considered classes are different.

In the further part of the article we shall discuss under which assumptions some of the considered classes coincide. A few of the properties deal with continuity of functions from those classes; some of theorems involve the condition of D. Gillespie [2] (for real functions of real variable), which is sufficient but not necessary, and other use the condition of P. Long [8] (this condition is also necessary, however for injective functions only).

We say that the sets A and B are separated ([7]) if

$$\overline{A} \cap B = \emptyset = A \cap \overline{B}.$$

Let X and Y be topological spaces. By $\mathscr{B}_x$ we shall denote the class of all open neighbourhoods of the point x from X.

Let $f : X \longrightarrow Y$ be an arbitrary function. The set $C(f, x)$ defined by (see [4])

$$C(f,x) = \bigcap_{U \in \mathscr{B}_x} f(U) \tag{1}$$

is called the *cluster set* of the function f at the point x from X.

Next lemma (in the version for real functions of real variable) is very useful in the theory of Darboux functions.

Lemma 1 *Let X be a locally connected and dense in itself topological space and Y be a locally compact space (or $Y = \mathbb{R}$). If a function $f : X \longrightarrow Y$ has the Darboux property and $f = A \cup B$, where A and B are nonempty and separated sets, then the set K, where $K = \mathrm{proj}_X A$, is perfect and $K = \mathrm{proj}_X B$. Moreover, the sets $K \cap \mathrm{proj}_X A$ and $K \cap \mathrm{proj}_X B$ are dense in K.*

Proof. For simplicity of denotations let us set:

$$A_1 = \mathrm{proj}_X A, \quad B_1 = \mathrm{proj}_X B.$$

Then

$$A_1 \cap B_1 = \emptyset \quad \text{and} \quad A_1 \cup B_1 = X.$$

Hence, A_1 and B_1 are complements for each other; they have the same boundary.

We shall show that the K is a perfect set. Since it is a closed set, then it is sufficient to prove that K is dense in itself. To prove this let us assume that it is not true. Thus, there exist a point x_0 in K and a connected neighbourhood U_0 of x_0 such that

$$(K \cap \{x_0\}) \cap U_0 = \emptyset.$$

For each x in $U_0 \setminus \{x_0\}$ there exists a connected neighbourhood U_x such that

$$U_x \subset A_1 \quad \text{or} \quad U_x \subset B_1.$$

By definition of the sets A and B it follows that

$$f_{\restriction U_x} \subset A \quad \text{or} \quad f_{\restriction U_x} \subset B.$$

There are three possible cases:

1. $f_{\restriction U_x} \subset A$ for each x in $U_0 \setminus \{x_0\}$.
2. $f_{\restriction U_x} \subset B$ for each x in $U_0 \setminus \{x_0\}$.
3. There are x_1 and x_2 in $U_0 \setminus \{x_0\}$ such that $f_{\restriction U_{x_1}} \subset A$ and $f_{\restriction U_{x_1}} \subset B$.

Ad. (1). Since $x_0 \in \mathrm{Fr}\,(A_1)$ and $f_{\restriction U_0 \setminus \{x_0\}} \subset A$, then $x_0 \in B$. In view of properties of cluster sets of connected functions (see [5]), the set $C(f, x)$ is connected. Then the connected set $\{x_0\} \times C(f, x_0)$ has common points with both of the separated sets A and B. Contradiction.

Ad. (2). Similar arguments lead us to a contradiction in this case.

Ad. (3). Let us consider two possibilities:

- the set $U \setminus \{x_0\}$ is connected,
- the set $U \setminus \{x_0\}$ is not connected.

In the first case, the class of sets $\{U_x : x \in U_0 \setminus \{x_0\}\}$ forms an open cover of the set $U_0 \setminus \{x_0\}$.

For every two points of the set $U_0 \setminus \{x_0\}$ there exists a finite sequence $(\xi_1, \ldots, \xi_n)$ of points of the set $U_0 \setminus \{x_0\}$ such that

$$\xi_1 = x_1, \quad \xi_n = x_2$$

and

$$U_{\xi_i} \cap U_{\xi_j} \neq \emptyset \iff |i - j| \leq 1$$

for each i and j from the set $\{1, \ldots, n\}$. Thus,

$$f_{\restriction U_{\xi_1}} \subset A \quad \text{and} \quad f_{\restriction U_{\xi_1}} \cap f_{\restriction U_{\xi_2}} \neq \emptyset.$$

Since $f_{\restriction U_{\xi_2}} \subset A$ or $f_{\restriction U_{\xi_2}} \subset B$, then of course $f_{\restriction U_{\xi_2}} \subset A$.

Continuing this process we can infer that $f_{\restriction U_{\xi_n}} \subset A$ which contradicts to relations $\xi_n = x_2$ and $x_n \in B_1$.

Assume now that the set $U_0 \setminus \{x_0\}$ is not connected. Let V_1 and V_2 be two components of the set $U_0 \setminus \{x_0\}$ containing points x_1 and x_2, respectively.

Of course, $V_1 = \bigcup\limits_{x \in V_1} U_x$.

We shall show that if $x \in V_1$, then $f_{\restriction U_x} \subset A$.

The set V_1 is connected, then for each $\overline{x}$ from V_1 there exists a finite sequence $(\xi_1, \ldots, \xi_n)$ of elements of V_1 such that

$$\xi_1 = x, \quad \xi_n = \overline{x}$$

and

$$U_{\xi_i} \cap U_{\xi_j} \neq \emptyset \iff |i - j| \leq 1.$$

Repeating the arguments from the previous part of the proof, one can show that $f_{\restriction V_1} \subset A$. Similarly, one can get the inclusion $f_{\restriction V_2} \subset B$.

The sets V_1 and V_2 are components of the set $U_0 \setminus \{x_0\}$, where X is a connected and locally connected space. Thus, $V_1 \cup \{x_0\}$ and $V_2 \cup \{x_0\}$ are connected and locally connected subspaces of X, hence in view of properties of cluster sets of Darboux functions

$$f(x_0) \in C(f_{\restriction V_1 \cup \{x_0\}}, x_0) \quad \text{and} \quad f(x_0) \in C(f_{\restriction V_2 \cup \{x_0\}}, x_0).$$

Independently, whether $(x_0, f(x_0)) \in A$ or $(x_0, f(x_0)) \in B$, we obtain the contradiction with the fact that the sets A and B are separated.

In all the cases we have come to contradiction, hence our assumption that the set K is not dense in itself is false.

Let us assume now that the set $K \cap A_1$ is not dense in K.

Then there exists a point x_0 in K and a connected open neighbourhood U_0 of the point x_0 such that

$$(U_0 \cap K \cap A_1) \setminus \{x_0\} = \emptyset.$$

It follows that

$$f_{\restriction U_0 \cup \{x_0\}} \subset B.$$

If $(x_0, f(x_0)) \in A$, then x_0 is an isolated point of the set K, what is impossible because of the set K is dense in itself.

If $(x_0, f(x_0)) \in B$, then $f_{\restriction U_0} \subset B$, hence $x_0 \notin K$, what is also impossible.

In each case, the obtained contradiction proves that the set $K \cap A_1$ is dense in K.

Similarly, one can prove that the set $K \cap B_1$ is dense in K. □

Applying lemma 1 one can prove the next theorem:

Theorem 6 *If X is a locally connected metric space and $f : X \longrightarrow \mathbb{R}$ if a Darboux function of the* I *class of Baire, then f is locally strongly connected.*

Proof. Suppose that there exists an open and connected set U such that the graph of the function $f_{\restriction U}$ is not connected.

Then there are separated sets A and B such that $f_{\restriction U} = A \cup B$. Let

$$A_1 = \text{proj}_X A, \quad B_1 = \text{proj}_X B \quad \text{and} \quad K = \text{Fr}\,(A_1)\,.$$

In view of lemma 1,

$$K = \text{Fr}\,(B)\,,$$

K is a perfect set and $K \cap A_1$, $K \cap B_1$ are dense in K. Since f is of the first class of Baire, then there exists a point of (relative) continuity of the function $f_{\restriction K}$. Let x_0 be that point. There exist sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that

$$a_n \in A_1 \cap K, \quad b_n \in B_1 \cap K, \quad a_n \to x_0, \quad b_n \to x_0$$

and

$$f(a_n) \to f(x_0), \quad f(b_n) \to f(x_0).$$

It follows that

$$(x_0, f(x_0)) \in \overline{A} \cap \overline{B},$$

and since $(x_0, f(x_0))$ belongs to one of the sets A or B, then we get a contradiction with the fact that the sets A and B are separated. □

We can get the next property as an immediate corollary of the previous theorem.

Corollary 1 *If X is a connected and locally connected metric space and a function $f : X \longrightarrow \mathbb{R}$ is a Darboux function of the* I *class of Baire, then f is connected.*

In the further part of the article we shall consider some sufficient conditions for "connected" functions to be continuous.

Theorem 7 *If X is a locally connected topological space, a function $f : X \longrightarrow \mathbb{R}$ has the Darboux property and*

$$\operatorname{Int}\left(\left\{y \in \mathbb{R} : \operatorname{card}\left(f^{-1}(y)\right) \geq \aleph_0\right\}\right) = \emptyset, \tag{2}$$

then f is a continuous function.

Proof. Let us assume that f has the Darboux property and is discontinuous at some point x_0 from X. Then there are real numbers a and b such that

$$a < b, \quad a \in C(f, x_0), \quad b \in C(f, x_0).$$

In view of properties of cluster sets and inverse cluster sets of Darboux functions (see [5]),

$$(a, b) \subset \bigcap_{U \in \mathscr{B}_{x_0}} f(U),$$

where $\mathscr{B}_{x_0}$ is a base of the space X at the point x_0 consisting of connected sets. Then for each point y from (a, b) and neighbourhood U from $\mathcal{B}_{x_0}$ there are points $x_{U,y}$ in U such that

$$y = f(x_{U,y}).$$

Since the set of all such points $x_{U,y}$ is infinite, then

$$(a, b) \subset \left\{y \in \mathbb{R} : \operatorname{card}\left(f^{-1}(y)\right) \geq \aleph_0\right\},$$

which is impossible. □

Since in a locally connected topological space each real strongly connected function has the Darboux property, then:

Corollary 2 *If X is a locally connected topological space, a function $f : X \longrightarrow \mathbb{R}$ is strongly connected and fulfils condition* (2)*, then f is a continuous function.*

Condition (2) has been introduced by D. Gillespie [2], however it is not necessary. See at the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as follows:

$$f(x, y) = x^2 + y^2 \qquad \text{if} \qquad (x, y) \in \mathbb{R}^2.$$

Much more complicated example shows also that condition of Gillespie is not necessary even for real functions of (one) real variable.

We say that a function $f : X \longrightarrow Y$ is closed if the image of any closed set in X is closed in Y.

Theorem 8 *Let X be a locally connected topological space. If a function $f : X \longrightarrow \mathbb{R}$ is closed and has the local Darboux property, then it is continuous.*

Proof. Let us assume that the function f is not continuous. There exists a point x_0 in X such that $C(f, x) \neq \{f(x_0)\}$.

Let y_0 be a point from the set $C(f, x_0)$ which is different from $f(x_0)$. Suppose that $y_0 < f(x_0)$. Let $\mathscr{B}_{x_0}$ be a local base of the space X at the point x_0 which is consisted of connected sets. Let further $\Sigma = \mathbb{N} \times \mathscr{B}_{x_0}$. Consider the following relation in the set Σ:

$$(n_1, U_1) \prec (n_2, U_2) \iff (n_1 \leq n_2 \wedge U_2 \subset U_1).$$

It is obvious that $(\Sigma, \prec)$ is a directed set.

Since $y_0 \in C(f, x_0)$ and $y_0 < f(x_0)$, then for each pair (n, U) from Σ there exists a point $x'_{(n,U)}$ such that

$$x'_{(n,U)} \in U \quad \textit{and} \quad f\left(x'_{(n,U)}\right) < y_0 + \frac{1}{n}.$$

If $n \geq n_0$, where

$$n_0 = \min\left\{k \in \mathbb{N} : y_0 + \frac{1}{k} < f(x_0)\right\},$$

then

$$f\left(x'_{(n,U)}\right) < y_0 + \frac{1}{n} < f(x_0).$$

Since f has the local Darboux property and sets from the local base $\mathscr{B}_{x_0}$ are connected, then for each element (n,U) from the set Σ there exists a point $x_{(n,U)}$ such that

$$x_{(n,U)} \in U \quad \text{and} \quad f\left(x_{(n,U)}\right) = y_0 + \frac{1}{n}.$$

In that way we have constructed a net (Moore-Smith sequence) $\left\{x_{(n,U)}\right\}_{(n,U)\in\Sigma}$ which is convergent to x_0. Hence the set A, where

$$A = \left\{x_{(n,U)} \,:\, (n,U) \in \Sigma\right\},$$

is closed, but its image $f(A)$ is not closed.

Contradiction with assumptions on f completes the proof. □

Corollary 3 *Let X be a locally connected topological space. If a function $f : X \longrightarrow \mathbb{R}$ is closed and strongly connected, then it is a continuous function.*

Corollary 4 *Let X be a locally connected topological space. If a function $f : X \longrightarrow \mathbb{R}$ is closed and locally strongly connected, then it is a continuous function.*

Corollary 5 *Let X be a locally connected topological space. If a function $f : X \longrightarrow \mathbb{R}$ is closed and has the Darboux property, then it is a continuous function.*

The set $T(f,y)$ is defined by

$$T(f,y) = \{x \in X \,:\, y \in C(f,x)\} \quad \text{if} \quad y \in Y \tag{3}$$

and is called *an inverse cluster set* of f at the point y from Y. See [3].

Theorem 9 *Let X be a locally connected topological space and Y be a locally compact topological space. If a function $f : X \longrightarrow Y$ has the Darboux property, $f(X)$ is a closed subset of Y and*

$$T(f, f(x)) = \{x\} \quad \textit{if} \quad x \in X, \tag{4}$$

then f is a continuous function.

Proof. Assume to the contrary that the function f is not continuous at some point x_0.

Let $\mathscr{B}_{x_0}$ be a local base of the space X at the point x_0 which is consisted of connected sets. There exists an open neighbourhood V of the point $f(x_0)$ such that for each set U in $\mathscr{B}_{x_0}$ the set $f(U)$ is not contained in V. Since Y is locally compact, we can assume that the set $\overline{V}$ is compact.

For each U from $\mathscr{B}_{x_0}$ there exists a point x'_U such that

$$x'_U \in U \quad \text{and} \quad f\left(x'_U\right).$$

Since $f(U)$ is a connected subset of Y, $f\left(x'_U\right) \notin U$ and $f(x_0) \in U$, then there exists a point x_U such that

$$x_U \in U \quad \text{and} \quad f(x_U) \in \operatorname{Fr}(V).$$

The class $\mathscr{B}_{x_0}$ is directed by the relation $\supset$, hence $\{x_U\}_{U\in\mathscr{B}_{x_0}}$ is a net convergent to x_0.

Then $\{f\left(x_U\right)\}_{U\in\mathscr{B}_{x_0}}$ is a net in the compact set $f(X)\cap\operatorname{Fr}(V)$. Then there exists a subnet $\{f(x_\lambda)\}_{\lambda\in\Lambda}$ which is convergent to some point y_0; of course, $y_0 \neq f(x_0)$.

The net $\{x_\lambda\}_{\lambda\in\Lambda}$ is a subnet of the net $\{x_U\}_{U\in\mathscr{B}_{x_0}}$, so it is also convergent to the point x_0. Hence, $y_0 \in C(f, x_0)$. Since $f(X)$ is compact, then there exists a point x_1 in X such that $f(x_1) = y_0$.

Therefore,

$$x_0 \in T(f, f(x_1)) \quad \text{and} \quad x_1 \neq x_0,$$

which contradicts to condition (4). □

Corollary 6 *Let X be a locally connected topological space and Y be a locally compact topological space. If a function $f : X \longrightarrow Y$ is strongly connected, $f(X)$ is a closed subset of Y and fulfils condition* (4)*, then f is a continuous function.*

Condition (4) is due to P. Long (see [8]).

The next theorem completes the previous one.

Theorem 10 *If $f : X \longrightarrow Y$ is a continuous and injective function, then it fulfils condition* (4).

Proof. Suppose to the contrary that the function f does not fulfil condition (4). Then there exist two points x_1 and x_2 in X such that

$$x_1 \neq x_2 \quad \text{and} \quad x_2 \in T(f, f(x_1)).$$

This means that there exists a net $\{x_\sigma\}_{\sigma\in\Sigma}$ which is convergent to x_2 and the net $\{f(x_\sigma)\}_{\sigma\in\Sigma}$ is convergent to $f(x_1)$. In such a way we obtained a contradiction to the assumptions of the theorem. □

References

[1] R. Engelking. *General Topology.* PWN, Warszawa 1977.

[2] D. Gillespie. A property of continuity. *Bull. Amer. Math. Soc.* **28**, 245–250, 1922.

[3] T.R. Hamlett, P.E. Long. Inverse cluster sets. *Proc. Amer. Math. Soc.* **53** (2), 470–476, 1975.

[4] J.M. Jędrzejewski. On limit numbers of real functions. *Fund. Math.* **83** 269–281, 1973.

[5] J.M. Jędrzejewski. On limit values of connected functions. *Sci. Issues Jan Długosz Univ. Częstochowa, Mathematics*, **XIV**, 2009. (To appear).

[6] J.M. Jędrzejewski. On different kinds of connectivity of functions, *Sci. Issues Catholic Univ. Ružomberok, Mathematica*, **II**, 2008. (To appear).

[7] J.L. Kelley. *General Topology.* Springer, New York - Heidelberg - Berlin 1955.

[8] P.E. Long. Connected mappings. *Duke Math. J.* **35** (4), 677-682, 1968.