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PART I

MATHEMATICS AND ITS APPLICATIONS

Jan Długosz University in Częstochowa
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ON SOME STABILITY PROPERTIES OF POLYNOMIAL FUNCTIONS

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Abstract. In this paper we present conditions under which a function F with a control function f, in the following sense

$$\|\Delta_y^{n+1} F(x)\| \le \Delta_y^{n+1} f(x), \quad x \in \mathbb{R},$$

can by uniformly approximated by a polynomial function of degree at most n.

1. Introduction

We start with the notation and definitions used in this paper.

Definition 1. Let $(G, +)$ stand for an Abelian group. Let $f : \mathbb{R} \to G$ be a given function and let $y \in \mathbb{R}$ be fixed. Then a difference operator Δ_y is defined by the formula

$$\Delta_y f(x) = f(x+y) - f(x), \quad x \in \mathbb{R},$$

and, for a positive integer n, by

$$\Delta_y^{n+1} f(x) = \Delta_y \Delta_y^n f(x), \quad x \in \mathbb{R}.$$

Definition 2. A map $f : \mathbb{R} \to G$ is called a polynomial function of degree at most n if and only if

$$\Delta_y^{n+1} f(x) = 0$$

for all $x, y \in \mathbb{R}$.

Definition 3. A map $f : \mathbb{R} \to G$ is called a monomial function of order n if and only if

$$\Delta_y^n f(x) = n! f(y)$$

for all $x, y \in \mathbb{R}$.

It is easy to see that a monomial function of order n is a polynomial function of degree at most n.

Definition 4. Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a function. A function f is called convex of order n, or shortly n–convex ($n \in \mathbb{N}$), if and only if

$$\Delta_y^{n+1} f(x) \geq 0$$

for every $x \in I$ and every $y \in (0, +\infty)$ such that $x + (n+1)y \in I$.

A function $f : I \to \mathbb{R}$ is concave of order n, or shortly n–concave, if and only if $-f$ is n–convex. The above notions are due to [3–5].

In [1] we have proved the following

Theorem 1. Let $(S, +)$ be an Abelian semingroup and let $(Y, \| \cdot \|)$ be a k–dimensional real normed linear space. Let further $f : S \to \mathbb{R}$ be a function such that $\Delta_y^n f(x) \geq 0$ for all $x, y \in S$, and $F : S \to Y$ be a mapping such that the inequality

$$\|n!F(y) - \Delta_y^n F(x)\| \leq n!f(y) - \Delta_y^n f(x)$$

holds for all $x, y \in S$.

Then there exists a monomial mapping $M : S \to Y$ of order n such that

$$\|F(x) - M(x)\| \leq k \cdot f(x)$$

for all $x \in S$.

In this paper we consider the functional inequality

$$\|\Delta_y^{n+1} F(x)\| \leq \Delta_y^{n+1} f(x),$$

and we look for the conditions implying the existence of a polynomial function P such that

$$\|F(x) - P(x)\| \leq k \cdot f(x).$$

We shall use the following theorem which was proved in [5]:

Theorem 2. Let $n \in \mathbb{N}$ and let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is n–concave, $g : I \to \mathbb{R}$ is n–convex and $f(x) \leq g(x), x \in I$, then there exists a polynomial w of degree at most n such that $f(x) \leq w(x) \leq g(x), x \in I$.

2. Results

Theorem 3. Let $(Y, \| \cdot \|)$ be a k-dimensional real normed linear space. Let further $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) \geq 0, x \in \mathbb{R}$, and $F : \mathbb{R} \to Y$ be a mapping such that the following inequality

$$\|\Delta_y^{n+1} F(x)\| \leq \Delta_y^{n+1} f(x) \tag{1}$$

holds for all $x, y \in \mathbb{R}$.

Then there exists a polynomial mapping $P:\mathbb{R}\to Y$ such that

$$\|F(x)-P(x)\|\le kf(x),\quad x\in\mathbb{R}.$$

Proof. Assume that $F:\mathbb{R}\to Y$ and $f:\mathbb{R}\to\mathbb{R}$ satisfy (1). Then for every $y^*\in Y^*, \|y^*\|=1$ and for all $x,y\in\mathbb{R}$ we have

$$-\Delta_y^{n+1}f(x)\le\Delta_y^{n+1}y^*\circ F(x)\le\Delta_y^{n+1}f(x).$$

Hence

$$\Delta_y^{n+1}(y^*\circ F+f)(x)\ge 0 \tag{2}$$

and

$$\Delta_y^{n+1}(y^*\circ F-f)(x)\le 0 \tag{3}$$

for every $y^*\in Y^*, \|y^*\|=1$ and for all $x,y\in\mathbb{R}$.

Let $\{e_1,\dots,e_k\}$ be a basis of Y such that $\|e_i\|=1$ for all $i\in\{1,\dots,k\}$. Let further $y_i^*:Y\to\mathbb{R}$ be a projection onto the ith axis, i.e. $y_i^*(y_1e_1+\dots y_ke_k)=y_i$ for $(y_1,\dots,y_k)\in\mathbb{R}^k,\ i\in\{1,\dots,k\}$. Clearly, $y_i^*\in Y^*$ and $\|y_i^*\|=1$ for all $i\in\{1,\dots,k\}$.

For every $i\in\{1,\dots,k\}$, we define functions $p_i:\mathbb{R}\to\mathbb{R}$ and $q_i:\mathbb{R}\to\mathbb{R}$ by the following formulas:

$$p_i(x):=y_i^*\circ F(x)+f(x),\ x\in\mathbb{R} \tag{4}$$

and

$$q_i(x):=y_i^*\circ F(x)-f(x),\ x\in\mathbb{R}. \tag{5}$$

Since $f(x)\ge 0$ for all $x\in\mathbb{R}$, we infer that

$$p_i(x)\ge q_i(x)$$

for every $i\in\{1,\dots,k\}$ and for all $x\in\mathbb{R}$.

From (2) we deduce that for every $i\in\{1,\dots,k\}$ the function p_i is n–convex. From (3) we have that for every $i\in\{1,\dots,k\}$ the function q_i is n–concave.

By virtue of Theorem 2, we infer that for every $i\in\{1,\dots,k\}$ there exists a polynomial function w_i of degree at most n such that

$$q_i(x)\le w_i(x)\le p_i(x),\ x\in\mathbb{R}. \tag{6}$$

Then, by (4), (5) and (6), we obtain

$$|y_i^*\circ F(x)-w_i(x)|\le f(x) \tag{7}$$

for all $i\in\{1,\dots,k\}$ and for all $x\in\mathbb{R}$.

Now, we define a function $P:\mathbb{R}\to Y$ by the formula

$$P(x)=w_1(x)\cdot e_1+\dots+w_k(x)\cdot e_k,\quad x\in\mathbb{R}.$$

The function P is, of course, a polynomial function of degree at most n. We have also, by (7),

$$\begin{aligned}\|F(x)-P(x)\| &= \|\sum_{i=1}^{k}(y_i^*(F(x))-w_i(x))e_i\| \\ &\le \sum_{i=1}^{k}|y_i^*(F(x))-w_i(x)|\cdot\|e_i\| \le k\cdot f(x)\end{aligned}$$

for all $x \in \mathbb{R}$.

Ger [2] considered the operator

$$\delta_y^n f(x) := \Delta^{n+1}_{\frac{y-x}{n+1}} f(x).$$

Then f is n–convex if and only if

$$x \le y \Rightarrow \delta_y^n f(x) \ge 0.$$

Analogically we can prove Theorem 4.

Theorem 4. Let $I \subset \mathbb{R}$ be an open interval and let $(Y, \|\cdot\|)$ be a k–dimensional real normed linear space. Let further $F : I \to Y$ and $f : I \to \mathbb{R}$ be mappings such that the following inequality

$$\|\delta_y^n F(x)\| \le \delta_y^n f(x)$$

holds for all $x, y \in I$.

If $f(x) \ge 0, x \in I$, then there exists a polynomial mapping P of degree at most n such that

$$\|F(x)-P(x)\| \le kf(x),\ x \in I.$$

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A CHARACTERIZATION OF A HOMOGRAPHIC TYPE FUNCTION II

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Abstract. This article is a continuation of the investigations contained in the previous paper [2]. We deal with the following conditional functional equation:

$$f(x)f(y) \neq \frac{1}{\lambda^2} \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)}$$

with $\lambda \neq 0$.

1. Introduction

If $(G, \star)$ is a group or a semigroup and F stands for an arbitrary binary operation in some set H, then a solution of the functional equation

$$f(x \star y) = F(f(x), f(y))$$

is called a homomorphism of structures $(G, \star)$ and (H, F).

Let $J \subset \mathbb{R}$ be a nontrivial interval and $I \subset \mathbb{R}$ be an interval such that $I + I \subset I$. Let further $F : J \times J \longrightarrow J$ be a given map. Functional equations of the form

$$f(x + y) = F(f(x), f(y)), \quad x, y \in I,$$

have nonconstant continuous solutions if and only if there exists an open interval constituting a continuous group with respect to the associative operation F. All such solutions are strictly monotonic (see Aczél [1]). Here we consider a rational function $F : \{(x, y) \in \mathbb{R} : \quad xy \neq \frac{1}{\lambda^2}\} \longrightarrow \mathbb{R}$ of the form

$$F(u, v) = \frac{u + v + 2\lambda uv}{1 - \lambda^2 uv}$$

with $\lambda \neq 0$. This is a rational two-place real-valued function defined on a disconnected subset of the real plane $\mathbb{R}^2$ that with every $\lambda \in \mathbb{R} \setminus \{0\}$ satisfies the equation

$$F(F(x,y),z) = F(x, F(y,z))$$

for all $(x,y,z) \in \mathbb{R}^3$ such that products $xy, yz, F(x,y)z, xF(y,z)$ are not equal to λ^{-2}. Rational functions with such or similar properties are termed associative operations.

A homografic function $\varphi : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ given by the formula

$$\varphi(x) = \frac{x}{\lambda - \lambda x}, \quad x \neq 1,$$

satisfies the functional equation

$$f(x+y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)}$$

for every pair $(x,y) \in \mathbb{R}^2 \setminus D$, where

$$D = \{(x, 1-x) : \quad x \in \mathbb{R}\} \cup \{(x,1) : \quad x \in \mathbb{R}\} \cup \{(1,x) : \quad x \in \mathbb{R}\}.$$

We shall determine all the functions $f : G \longrightarrow \mathbb{R}$, where $(G, \star)$ is a group, that satisfy the functional equation

$$f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)}. \tag{1}$$

By a solution of the functional equation (1) we understand any function $f : G \longrightarrow \mathbb{R}$ that satisfies the equality (1) for every pair $(x,y) \in G^2$ such that $f(x)f(y) \neq \lambda^{-2}$. Thus we deal with the following conditional functional equation:

$$f(x)f(y) \neq \frac{1}{\lambda^2} \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)} \tag{E}$$

for all $x, y \in G$.

The solution of equation (E) in the case $\lambda = 1$ was described in [2].

2. Main result

We proceed with a description of solutions of (E).

Theorem. *Let $(G, \star)$ be a group and $\lambda \in \mathbb{R} \setminus \{0\}$ be fixed. A function $f : G \longrightarrow \mathbb{R}$ yields a nonconstant solution to the functional equation*

$$f(x)f(y) \neq \lambda^{-2} \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)} \tag{E}$$

for all $x, y \in G$ *if and only if either*

$$f(x) := \begin{cases} \frac{1}{\lambda} & \textit{for } x \in H, \\ -\frac{1}{\lambda} & \textit{for } x \in G \setminus H \end{cases}$$

or

$$f(x) := \begin{cases} \frac{A(x)}{\lambda - \lambda A(x)} & \textit{for } x \in \Gamma \\ -\frac{1}{\lambda} & \textit{for } x \in G \setminus \Gamma \end{cases}$$

or

$$f(x) := \begin{cases} \frac{1}{\lambda} & \textit{for } x \in \Gamma \setminus Z \\ 0 & \textit{for } x \in Z \\ -\frac{1}{\lambda} & \textit{for } x \in G \setminus \Gamma, \end{cases}$$

where $(H, \star), (\Gamma, \star)$ *are subgroups of the group* $(G, \star)$, $(Z, \star)$ *is a subgroup of the group* $(\Gamma, \star)$, *and* $A : \Gamma \longrightarrow \mathbb{R}$ *is a homomorphism such that* $1 \notin A(\Gamma)$.

Proof. Assume that f is a nonconstant solution of equation (E), i.e.

$$f(x)f(y) \neq \lambda^{-2} \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)}$$

for all $x, y \in G$. Hence

$$\lambda^2 f(x)f(y) \neq 1 \quad \text{implies} \quad \lambda f(x \star y) = \frac{\lambda f(x) + \lambda f(y) + 2\lambda^2 f(x)f(y)}{1 - \lambda^2 f(x)f(y)}.$$

Thus, it is easy to obserwe that (E) states that the function $g := \lambda f$ satisfies the following functional equation:

$$g(x)g(y) \neq 1 \quad \text{implies} \quad g(x \star y) = \frac{g(x) + g(y) + 2g(x)g(y)}{1 - g(x)g(y)}$$

for all $x, y \in G$. From the theorem proved by the author in [2] we conclude that g is of the form

$$g(x) := \begin{cases} 1 & \text{for } x \in H, \\ -1 & \text{for } x \in G \setminus H \end{cases}$$

or

$$g(x) := \begin{cases} \frac{A(x)}{1 - A(x)} & \text{for } x \in \Gamma \\ -1 & \text{for } x \in G \setminus \Gamma \end{cases}$$

or

$$g(x) := \begin{cases} 1 & \text{for } x \in \Gamma \setminus Z \\ 0 & \text{for } x \in Z \\ -1 & \text{for } x \in G \setminus \Gamma, \end{cases}$$

where $(H, \star), (\Gamma, \star)$ are subgroups of the group $(G, \star)$, $(Z, \star)$ is a subgroup of the group $(\Gamma, \star)$, and $A : \Gamma \longrightarrow \mathbb{R}$ is a homomorphism such that $1 \notin A(\Gamma)$. This means that f is of the form as above.

It is easy to check that each of the functions above yields a solution to the equation (E). Thus the proof has been completed.

The following remark gives the form of constant solutions to equation (E).

Remark. *Let* $(G, \star)$ *be a group. The only constant solutions of equation* (E) *are* $f = -\frac{1}{\lambda}, f = 0$ *and* $f = \frac{1}{\lambda}$.

To check this, assume that $f = c$ fulfils (E). Then

$$c^2 \neq \frac{1}{\lambda^2} \Longrightarrow c = 2c\frac{1 + \lambda c}{1 - \lambda^2 c^2},$$

i.e.

$$c \in \left\{-\frac{1}{\lambda}, \frac{1}{\lambda}\right\} \quad \text{or} \quad c = 0 \quad \text{or} \quad 1 = 2\frac{1 + \lambda c}{1 - \lambda^2 c^2},$$

whence

$$c \in \left\{-\frac{1}{\lambda}, 0, \frac{1}{\lambda}\right\},$$

which was to be shown.

Remark. *Solutions of* (1) *for* $\lambda \in \{-1, 1\}$ *in the class of continuous functions can be found in* [1].

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ON THE QUASI-UNIFORM CONVERGENCE

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Abstract. Arzelá [1] considered the weaker form of uniform convergence which is as good as uniform convergence of sequences of functions in respect to continuity of the limit of a sequence of continuous functions. Some generalization of such convergence can be found in [5]. Similar kinds of convergence of function sequences were considered in [3] and [4]. In our article we generalize those kinds of convergence for functions defined in a topological space with values in a topological space.

In the article we use terminology which is explained in Engelking's monograph "General Topology" [2]. Among others, we use the notion of a star with respect to an open cover. If X is a topological space and α is a cover of this space, then the star $\mathrm{St}(x,\alpha)$ of a point $x \in X$ with respect to the cover α is defined as the union of all the sets from α which contain the point x, i.e.

$$\mathrm{St}(x,\alpha) = \bigcup\{U : x \in U \wedge U \in \alpha\}$$

Definition 1 *Let X, Y be topological spaces and f, f_n, $n \in \mathbb{N}$, be functions defined in X with values in Y. It is said that $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent to f if:*

(1) $(f_n)_{n=1}^{\infty}$ *converges pointwise to* f*;*

(2) *for each open cover* α *of* Y *and each* $n \in \mathbb{N}$ *there exist* $k \in \mathbb{N}$ *and* $n_1, \dots, n_k \geq n$ *such that if* $t \in X$*, then*

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee f_{n_2}(t) \in \mathrm{St}(f(t), \alpha) \vee \dots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha).$$

Theorem 1 *Let* X *be a compact space and* Y *be an arbitrary topological space. If* $f_n \colon X \to Y$*, where* $n \in \mathbb{N}$*,* $f \colon X \to Y$ *are continuous functions and* $(f_n)_{n=1}^{\infty}$ *is pointwise convergent to* f*, then* $(f_n)_{n=1}^{\infty}$ *is quasi-uniform convergent to* f*.*

Proof. Let us take an arbitrary open cover α of Y. For an arbitrary, but fixed, positive integer m denote:

$$A_m = \{t \in X \colon f_m(t) \in \mathrm{St}(f(t), \alpha)\}.$$

Since $(f_n)_{n=1}^{\infty}$ is pointwise convergent, the equality

$$X = \bigcup_{m \geq n} A_m \tag{1}$$

is satisfied. Additionally, the set A_m is open for each $m \geq n$.

Indeed, if $t \in A_m$, then $f_m \in \mathrm{St}(f(t), \alpha)$, i.e. one can find $V \in \alpha$ such that $f(t) \in V$. Continuity of both f_m and f implies the existence of neighborhoods W_1, W_2 of t such that $f(W_1) \subset \mathrm{St}(f(t), \alpha)$ and $f_m(W_2) \subset \mathrm{St}(f(t), \alpha)$.

Now, let $W_0 = W_1 \cap W_2$. Of course, W_0 is a neighborhood of t and for each $t' \in W_0$ we have $f_m(t') \in \mathrm{St}(f(t'), \alpha)$, i.e. $W_0 \subset A_m$. We have shown that the set A_m is open for each $m \geq n$.

By the above, by the fact that X is compact and by (1), one can find positive integers k and $n_1, \dots, n_k \geq n$ such that

$$X = A_{n_1} \cup \dots \cup A_{n_k}.$$

Hence

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee \dots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha)$$

for $t \in X$. Finally, combining this with the pointwise convergence, we get that $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent to f. □

We shall use the symbol $\mathcal{C}(f)$ for the set of all points of continuity of a function f.

Definition 2 *Let $\mathfrak{I}$ be a σ-ideal of subsets of X. A function f from a topological space X to a topological space Y is said to be $\mathfrak{I}$-continuous if $X \setminus \mathcal{C}(f)$ belongs to $\mathfrak{I}$.*

Theorem 2 *Let X be a topological space, Y be a regular one and f_n, f be functions from X to Y for each positive integer n. Let $\mathfrak{I}$ be a σ-ideal of subsets of X such that $X \notin \mathfrak{I}$. If $(f_n)_{n=1}^{\infty}$ is $\mathfrak{I}$-continuous for each $n \in \mathbb{N}$ and $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent to f, then f is $\mathfrak{I}$-continuous as well.*

Proof. Let

$$E = \bigcap_{n=1}^{\infty} \mathcal{C}(f_n), \tag{2}$$

where $\mathcal{C}(f_n)$ denotes the set of points of continuity for f_n. We will show that the set $\mathcal{C}(f)$ of points of continuity for f contains the set E.

Let $x \in E$ and U be an arbitrary neighborhood of $f(x)$. By regularity of the space Y, there exists an open set V such that

$$f(x) \in V \subset \mathrm{cl}(V) \subset U. \tag{3}$$

Consider the family $\alpha = \{U, X \setminus \mathrm{cl}(V)\}$ which forms an open cover of Y. By pointwise convergence of $(f_n)_{n=1}^{\infty}$ to f, there exists n_0 such that

$$f_n(x) \in V \tag{4}$$

for each $n \geq n_0$. By the second condition of quasi-uniform convergence, one can find indexes $n_1 \geq n_0, \ldots, n_k \geq n_0$ such that

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee f_{n_2}(t) \in \mathrm{St}(f(t), \alpha) \vee \ldots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha)$$

for each $t \in X$. By the fact that $f(x) \in V$, $f_{n_i}(x) \in V$ and f_{n_i} is continuous, we infer that there exists a neighborhood W_i of x such that $f(W_i) \subset V$ for each $i = 1, \ldots, k$.

Let $W_0 = \bigcap\limits_{i=1}^{k} W_i$. Thus, W_0 is a neighborhood of x and if $t \in W_0$, then there exists n_j such that $f_{n_j}(t) \in V$ and $f_{n_j}(t) \in \mathrm{St}(f(t), \alpha)$. This means that $f(t) \in U$ for each $t \in W_0$, i.e. the inclusion $f(W_0) \subset U$ holds.

In the consequence, f is continuous at each point $x \in E$. Now, conditions $X \setminus \mathcal{C}(f) \subset E$ and $E \in \mathfrak{I}$ imply that $X \setminus \mathcal{C}(f) \in \mathfrak{I}$, i.e. f is $\mathfrak{I}$-continuous and the proof is complete. $\square$

Definition 3 *Let X, Y be topological spaces and f, f_n, $n \in \mathbb{N}$, be functions defined on X with values in Y. A sequence $(f_n)_{n=1}^{\infty}$ is called* St*-monotonically convergent to f if for every $x \in X$, for every $n \in \mathbb{N}$ and for every open cover of Y the implication*

$$f_n(t) \in \mathrm{St}(f(t), \alpha) \Longrightarrow f_{n+1}(t) \in \mathrm{St}(f(t), \alpha)$$

holds.

Theorem 3 *If X and Y are topological spaces, f_n, f are functions from X into Y, then $(f_n)_{n=1}^{\infty}$ converges uniformly to f if and only if $(f_n)_{n=1}^{\infty}$ is quasi-uniform and* St*-monotonically convergent to f.*

Proof. It is known that uniform convergence implies quasi-uniform convergence. It is not difficult to see that uniform convergence also implies St-monotonic convergence.

Now, assume that $(f_n)_{n=1}^{\infty}$ is quasi-uniformly and St-monotonically convergent to f. Let us choose an arbitrary open cover α of Y. By quasi-uniform convergence, one can find $k \in \mathbb{N}$ and $n_1, \dots, n_k \geq n$ such that

$$f_{n_1}(t) \in \mathrm{St}(f(t), \alpha) \vee f_{n_2}(t) \in \mathrm{St}(f(t), \alpha) \vee \ldots \vee f_{n_k}(t) \in \mathrm{St}(f(t), \alpha) \quad (5)$$

for each $t \in X$. By (5), we infer that for $t \in X$ there exists n_i such that $f_{n_i}(t) \in \mathrm{St}(f(t), \alpha)$, whence by St-monotonic convergence for each $p \in \mathbb{N}$ we have:

$$f_{n_i+p}(t) \in \mathrm{St}(f(t), \alpha).$$

Let $n_0 = \max\{n_1, \dots, n_k\}$. Obviously, if $t \in X$ and $n \geq n_0$, then

$$f_n(t) \in \mathrm{St}(f(t), \alpha),$$

which proves that $(f_n)_{n=1}^{\infty}$ is uniformly convergent to f. □

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UNIFORMLY CONTINUOUS SET-VALUED COMPOSITION OPERATORS IN THE SPACES OF FUNCTIONS OF BOUNDED VARIATION IN THE SENSE OF SCHRAMM

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Abstract. We show that the one-sided regularizations of the generator of any uniformly continuous set–valued Nemytskij operator, acting between the spaces of functions of bounded variation in the sense of Schramm, is an affine function. Results along these lines extend the study [1].

1. Introduction

Let $(X, |\cdot|)$ and $(Y, |\cdot|)$ be two real normed spaces, C be a convex cone in X and $I = [a, b] \subset \mathbb{R}$ $(a, b \in \mathbb{R}, a < b)$ be an interval. Let $cc(Y)$ be the family of all non-empty convex and compact subsets of Y. We consider the Nemytskij operator, i.e. the composition operator defined by $(HF)(t) = h(t, F(t))$, where $F : I \longrightarrow C,\ h : I \times C \longrightarrow cc(Y)$ is a given set-valued function. It is shown that if the operator H maps the space $\Phi BV(I; C)$ of functions of bounded Φ-variation in the sense of Schramm into the space $BS_\Psi(I; cc(Y))$ of set–valued functions of bounded Ψ-variation in the sense of Schramm, and is uniformly continuous, then the one-sided regularizations h^- and h^+ of h with respect to the first variable, are affine with respect to the second variable. In particular,

$$h^-(t,x) = A(t)x + B(t) \quad \text{for } t \in I,\ x \in C,$$

for some function $A : I \longrightarrow \mathcal{L}(C, cc(Y))$ and $B \in BS_\Psi(I; cc(Y))$, where $\mathcal{L}(C, cc(Y))$ stands for the space of all linear mappings acting from C into $cc(Y)$.

2. Preliminaries

We start by recalling some very basic facts as definitions and known results concerning the space of functions of bounded variation in the sense of Schramm.

Let $\mathcal{F}$ be the set of all convex functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that $\phi(0) = \phi(0^+) = 0$ and $\lim\limits_{t\longrightarrow\infty} \phi(t) = \infty$. Then we have

Remark 1. If $\phi \in \mathcal{F}$, then ϕ is continuous and strictly increasing (see [1, 7]).

A sequence $\Phi = (\phi_i)_{i=1}^\infty$ of functions from $\mathcal{F}$ satisfying the following two conditions:

(i) $\phi_{n+1}(t) \le \phi_n(t)$ for all $t > 0$ and $n \in \mathbb{N}$,

(ii) $\sum\limits_{n=1}^{\infty} \phi_n(t)$ diverges for all $x > 0$,

is said to be a Φ–*sequence.*

Let $I = [a,b]$ $(a,b \in \mathbb{R}, a < b)$ be an interval. For a set X we denote by X^I the set of all functions $f : I \to \mathbb{R}$.

If $I_n = [a_n, b_n]$ is a subinterval of the interval I $(n = 1, 2, \dots)$, then we write $f(I_n) := f(b_n) - f(a_n)$.

Definition 1. Let $\Phi = (\phi_n)_{n=1}^\infty$ be a Φ–sequence and $(X, |\cdot|)$ be a real normed space. A function $f \in X^I$ is of *bounded Φ-variation in the sense of Schramm in I* if

$$v_\Phi(f) = v_\Phi(f, I) := \sup \sum_{n=1}^{m} \phi_n\big(|f(I_n)|\big) < \infty, \tag{1}$$

where the supremum is taken over all $m \in \mathbb{N}$ and all non–ordered collections of non–overlapping intervals $I_n = [a_n, b_n] \subset I, n = 1, \dots, m$ ([18]).

It is known that for all $a, b, c \in I$, $a \le c \le b$ we have $v_\Phi(f, [a, c]) \le v_\Phi(f, [a, b])$ (that is v_Φ is increasing with respect to the interval) and $v_\Phi(f, [a, c]) + v_\Phi(f, [c, b]) \le v_\Phi(f, [a, b])$.

In what follows we denote by $V_\Phi(I, X)$ the set of all functions $f \in X^I$ of bounded Φ-variation in the Schramm sense and by $\Phi BV(I, X)$ the linear space of all functions $f \in X^I$ such that $v_\Phi(\lambda f) < \infty$ for some constant $\lambda > 0$.

In the space $\Phi BV(I, X)$ the function $\|\cdot\|_\Phi$ defined by

$$\|f\|_\Phi := |f(a)| + p_\Phi(f), \quad f \in \Phi BV(I, X),$$

where

$$p_\Phi(f) := p_\Phi(f, I) = \inf\Big\{\epsilon > 0 : v_\Phi\big(f/\epsilon\big) \le 1\Big\}, \quad f \in \Phi BV(I, X), \tag{2}$$

is a norm (see for instance [14]).

For $X = \mathbb{R}$, the linear normed space $(\Phi BV(I, \mathbb{R}), \|\cdot\|_\Phi)$ was studied by Schramm [18, Theorem 2.3]. The functional $p_\Phi(\cdot)$ defined by (2) is called *the Luxemburg-Nakano-Orlicz seminorm* [5, 25, 26].

It is worth mentioning that the symbol $\Phi BV(I, C)$ stands for the set of all functions $f \in \Phi BV(I, X)$ such that $f : I \longrightarrow C$ and C is a subset of X.

Let $cc(X)$ be the family of all non-empty convex compact subsets of X, and let D be the *Pompeiu-Hausdorff metric* in $cc(X)$, i.e.

$$D(A, B) := \max\Big\{e(A, B),\ e(B, A)\Big\}, \quad A, B \in cc(X), \tag{3}$$

where

$$e(A, B) = \sup\Big\{d(x, B) : x \in A\Big\}, \qquad d(x, B) = \inf\Big\{d(x, y) : y \in B\Big\}. \tag{4}$$

It is easy to check that the Pompeiu-Hausdorff metric D is invariant with respect to translation, i.e.

$$D(A, B) = D(A + Q, B + Q) \tag{5}$$

(see [4, Lemma 3]) for all $A, B \in cc(X)$ and bounded nonempty subset Q of X.

Definition 2. Let $\Phi = (\phi_n)_{n=1}^\infty$ be a Φ–sequence and $F : I \longrightarrow cc(X)$. We say that F has bounded Φ-variation in the Schramm sense if

$$w_\Phi(F) := \sup \sum_{n=1}^{m} \Phi_n\left(D(F(t_n), F(t_{n-1}))\right) < \infty, \tag{6}$$

where the supremum is taken over all $m \in \mathbb{N}$ and all non–ordered collections of non–overlaping intervals $I_n = [a_n, b_n] \subset I, i = 1, \dots, m$.

From now on, let

$$BS_\Phi(I, cc(X)) := \Big\{ F \in cc(X)^I : \; w_\Phi(\lambda F) < \infty \text{ for some } \lambda > 0 \Big\}. \tag{7}$$

For $F_1, F_2 \in BS_\Phi(I, cc(X))$ put

$$D_\Phi(F_1, F_2) := D\left(F_1(a), F_2(a)\right) + p_\Phi(F_1, F_2), \tag{8}$$

where

$$p_\Phi(F_1, F_2) := \inf \Big\{ \epsilon > 0 : S_\epsilon(F_1, F_2) \le 1 \Big\} \tag{9}$$

and

$$S_\epsilon(F_1, F_2) := \sup \sum_{n=1}^{m} \phi_n \left(\frac{1}{\epsilon} D\big(F_1(t_n) + F_2(t_{n-1}); F_2(t_n) + F_1(t_{n-1})\big) \right), \tag{10}$$

where the supremum is taken over the same collection $([a_n, b_n])_{n=1}^m$ as in Definition 2. Then $(BS_\Phi(I, cc(X)), D_\Phi)$ is a metric space, and it is complete if X is a Banach space [24, Lemma 5.4].

Taking into account [23, Theorem 3.8 (d)] and [24, condition 5.6], we get the following

Lemma 1. Let $\Phi = (\phi_n)_{n=1}^\infty$ be a Φ–sequence and $F_1, F_2 \in BS_\Phi(I, cc(X))$. Then, for $\lambda > 0$,

$$S_\lambda(F_1, F_2) \le 1 \;\text{ if and only if }\; p_\Phi\big(F_1, F_2\big) \le \lambda. \qquad \blacksquare$$

In what follows, let$(X, |\cdot|)$, $(Y, |\cdot|)$ be two real normed spaces and C be a convex cone in X. Given a set-valued function $h : I \times C \longrightarrow cc(Y)$ we set the composition operator $H : C^I \to cc(Y)^I$ generated by h as:

$$(Hf)(t) := h(t, f(t)), \quad f \in C^I, \quad t \in I. \tag{11}$$

Moreover, let us denote by $\mathcal{A}(C, cc(Y))$ the space of all additive functions and by $\mathcal{L}(C, cc(Y))$ the space of all set-valued linear functions, i.e. the space of all set-valued functions $A \in \mathcal{A}(C, cc(Y))$ which are positively homogeneous [1].

Now we quote the following lemma given by Nikodem.

Lemma 2. ([15, Theorem 5.6]). Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be normed spaces and C a convex cone in X. A set-valued function $F : C \longrightarrow cc(Y)$ satisfies the Jensen equation

$$F\Big(\frac{x+y}{2}\Big) = \frac{1}{2}\Big(F(x) + F(y)\Big), \quad x, y \in C, \tag{12}$$

if and only if there exist an additive set-valued function $A : C \longrightarrow cc(Y)$ and a set $B \in cc(Y)$ such that $F(x) = A(x) + B$ for all $x \in C$. ∎

3. The composition operator

Now we will present our main result.

Theorem 1. Let $(X, |\cdot|)$ be a real normed space, $(Y, |\cdot|)$ a real Banach space, C a convex cone in X and suppose that $\Phi = (\phi_n)_{n=1}^{\infty}$ and $\Psi = (\psi_n)_{n=1}^{\infty}$ are Φ–sequences. If the composition operator H generated by a set-valued function $h : I \times C \longrightarrow cc(Y)$ maps $\Phi BV(I, C)$ into $BS_{\Psi}(I, cc(Y))$ and is uniformly continuous, then the left regularization of h, i.e. the function $h^- : I \times C \longrightarrow cc(Y)$ defined by

$$h^-(t, x) := \lim_{s \uparrow t} h(s, x), \quad t \in I, \ x \in C,$$

exists and

$$h^-(t, x) = A(t)x + B(t), \quad t \in I, \quad x \in C,$$

for some $A : I \longrightarrow \mathcal{A}(X, cc(Y))$ and $B : I \longrightarrow cc(Y)$. Moreover, if $0 \in C$, then $B \in BS_{\Psi}(I, cc(Y))$ and the linear set-valued function $A(t)$ is continuous.

Proof. For every $x \in C$, the constant function $I \ni t \longrightarrow x$ belongs to $\Phi BV(I, C)$. Since H maps $\Phi BV(I, C)$ into $BS_{\Psi}(I, cc(Y))$ for every $x \in C$, the function $I \ni t \longrightarrow h(t, x)$ belongs to $BS_{\Psi}(I, cc(Y))$. Now the completeness of $cc(Y)$ with respect to the Pompeiu-Hausdorff metric [24, Lemma 6.12] implies the existence of the left regularization h^- of h.

By the assumption, H is uniformly continuous on $\Phi BV(I, C)$. Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the modulus of continuity of H, that is

$$\omega(\rho) := \sup\Big\{ D_{\Psi}\big(H(f_1), H(f_2)\big) : \|f_1 - f_2\|_{\Phi} \leq \rho; \ f_1, f_2 \in \Phi BV(I, C) \Big\}, \ \rho > 0.$$

Hence we get

$$D_{\Psi}\big(H(f_1), H(f_2)\big) \leq \omega\big(\|f_1 - f_2\|_{\Phi}\big) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I, C). \tag{13}$$

From the definition of the metric D_{Ψ} and (13), we obtain

$$p_{\Psi}\big(H(f_1); H(f_2)\big) \leq \omega\big(\|f_1 - f_2\|_{\Phi}\big) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I, C). \tag{14}$$

From Lemma 1, if $\omega\big(\|f_1 - f_2\|_{\Phi}\big) > 0$, the inequality (14) is equivalent to

$$S_{\omega(\|f_1 - f_2\|_{\Phi})}\Big(H(f_1), H(f_2)\Big) \leq 1, \quad f_1, f_2 \in \Phi BV(I, C). \tag{15}$$

Therefore, for any $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m = b,\ \alpha_i,\ \beta_i \in I$, $i \in \{1, 2, \cdots, m\},\ m \in \mathbb{N},$ the definitions of the operator H and the functional S_ϵ, imply

$$\sum_{i=1}^{\infty} \psi_i\Big(\frac{D\big(h(\beta_i, f_1(\beta_i)) + h(\alpha_i, f_2(\alpha_i)); h(\beta_i, f_2(\beta_i)) + h(\alpha_i, f_1(\alpha_i))\big)}{\omega\big(\|f_1 - f_2\|_\Phi\big)}\Big) \leq 1. \tag{16}$$

For $\alpha, \beta \in \mathbb{R},\ \alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0,1]$ by

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if} \quad t \leq \alpha \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{if} \quad \alpha \leq t \leq \beta \\ 1 & \text{if} \quad \beta \leq t\,. \end{cases} \tag{17}$$

Let us fix $t \in I$. For arbitrary finite sequence $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t$ and $x_1, x_2 \in C,\ x_1 \neq x_2$, the functions $f_1, f_2 : I \longrightarrow X$ defined by

$$f_\ell(\tau) := \frac{1}{2}\big[\eta_{\alpha_i,\beta_i}(\tau)(x_1 - x_2) + x_\ell + x_2\big], \quad \tau \in I,\ \ell = 1, 2, \tag{18}$$

belong to the space $\Phi BV(I, C)$. From (18) we have

$$f_1(\tau) - f_2(\tau) = \frac{x_1 - x_2}{2}\,, \qquad \tau \in I,$$

therefore

$$\|f_1 - f_2\|_\Phi = \Big|\frac{x_1 - x_2}{2}\Big|;$$

moreover

$$f_1(\beta_i) = x_1;\ f_2(\beta_i) = \frac{x_1 + x_2}{2};\ f_1(\alpha_i) = \frac{x_1 + x_2}{2};\ f_2(\alpha_i) = x_2.$$

Using (16), we get

$$\sum_{i=1}^{\infty} \psi_i \left(\frac{D\left(h(\beta_i, x_1) + h(\alpha_i, x_2); h\left(\alpha_i, \frac{x_1+x_2}{2}\right) + h\left(\beta_i, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)} \right) \leq 1. \tag{19}$$

Fix a positive integer m. We have

$$\sum_{i=1}^{m} \psi_i \left(\frac{D\left(h(\beta_i, x_1) + h(\alpha_i, x_2); h\left(\alpha_i, \frac{x_1+x_2}{2}\right) + h\left(\beta_i, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)} \right) \leq 1. \tag{20}$$

From the continuity of ψ_i, passing to the limit in (20) when $\alpha_1 \uparrow t$, we obtain that

$$\sum_{i=1}^{m} \psi_i \left(\frac{D\left(h^-(t,x_1)+h^-(t,x_2); 2h^-\left(t, \dfrac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\dfrac{x_1-x_2}{2}\right|\right)} \right) \leq 1.$$

Hence,

$$\sum_{i=1}^{\infty} \psi_i \left(\frac{D\left(h^-(t,x_1)+h^-(t,x_2); 2h^-\left(t, \frac{x_1+x_2}{2}\right)\right)}{\omega\left(\left|\frac{x_1-x_2}{2}\right|\right)} \right) \leq 1,$$

and, by (ii),

$$D\Big(h^-(t,x_1)+h^-(t,x_2); 2h^-\Big(t, \frac{x_1+x_2}{2}\Big)\Big) = 0.$$

Therefore,

$$h^-\Big(t, \frac{x_1+x_2}{2}\Big) = \frac{h^-(t,x_1)+h^-(t,x_2)}{2}$$

for all $t \in I$ and all $x_1,\ x_2 \in C$.

Thus, for each $t \in I$, the function $h^-(t,\cdot)$ satisfies the Jensen functional equation in C. Consequently, by Lemma 2, for every $t \in I$ there exist an additive set–valued function $A(t) : C \longrightarrow cc(Y)$ and a set $B(t) \in cc(Y)$ such that

$$h^-(t,x) = A(t)x + B(t) \quad \text{for} \quad x \in C,\ t \in I, \tag{21}$$

which proves the first part of our result.

The uniform continuity of the operator $H : \Phi BV(I,C) \longrightarrow BS_\Psi(I, cc(Y))$ implies the continuity of the function $A(t)$ so that $A(t) \in \mathcal{L}(C, cc(Y))$ [15, Theorem 5.3]. Putting $x = 0$ in (21) and taking into account that $A(t)0 = \{0\}$ for $t \in I$, we get

$$h^-(t,0) = B(t), \quad t \in I,$$

which implies that $B \in BS_\Psi(I, cc(Y))$. ∎

Remark 2. The counterpart of Theorem 1 for the right regularization h^+ of h defined by

$$h^+(t,x) := \lim_{s \downarrow t} h(s,x), \quad t \in I,$$

is also true.

Remark 3. Taking $\psi_n(t) = \psi(t)$ $(t \geq 0)$, we obtain the main result of [1].

Remark 4. Denote by S the set of all functions $f \in \Phi BV(I, C)$ such that

$$f(t) = \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(x_1 - x_2) + x + x_2\right],$$

where $\eta_{\alpha,\beta}$ is defined by (17) and $x = x_1$ or $x = x_2$. It follows from the argument used in the proof that Theorem 1 remains valid if the uniform continuity of the operator H is postulated only on the set S.

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SEMIHEREDITARY RINGS AND RELATED TOPICS

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Abstract. A ring A is called right (left) semihereditary if all finitely generated right (left) ideals of A are projective. In this paper we consider non-commutative semihereditary rings and show their connection with non-commutative valuation rings. We also present some criterion for a module to be flat.

1. Introduction

Historically semihereditary rings come from homological algebra and their definition was first appeared in [1].

Definition 1. [1] *A ring A is called right (left) semihereditary if all finitely generated right (left) ideals of A are projective.*

If a ring A is an integral domain, i.e. a commutative ring without divisors of zero, then semihereditary domains coincide with Prüfer domains. Prüfer domains were defined in 1932 by H. Prüfer, and since that time they play a central role in the development of the classical ring theory. Recall that an integral domain is called a Prüfer domain if all its finitely generated ideals are invertible. Since in the case of integral domains any ideal is invertible if and only if it is projective, we obtain that any Prüfer domain is exactly a semihereditary domain. Prüfer domains naturally arise from valuation rings of fields, since for any prime ideal P of a valuation ring A the localization A_P of A is a Prüfer domain. So semihereditary domains can be considered as a global theory for classical valuation rings.

In the non-commutative case there are different generalizations of valuation rings. If we consider invariant valuation rings of division rings which were introduced by Schilling in [6], then we obtain that any invariant valuation ring is a semihereditary ring in the sense of definition 1. So semihereditary rings can be considered as some generalizations of Prüfer domains for non-commutative rings. Another generalization of non-commutative valuation rings were introduced and studied by Dubrovin in [3]. These rings were named Dubrovin valuation rings after him. In this non-commutative valuation theory any Dubrovin valuation ring of a simple Artinian ring Q is exactly a local semihereditary order of Q. So semihereditary orders can be considered as the global theory for Dubrovin valuation rings. Dubrovin valuation rings found a large applications. More information about these rings and semihereditary orders in simple Artinian rings can be found in the book [5].

Semihereditary rings are also interesting from homological point of view, since they belong to the class of rings with weak global dimension ≤ 1.

All rings considered in this paper are assumed to be associative with $1 \neq 0$, and all modules are assumed to be unital. We write $U(A)$ for the group of units of a ring A, and D^* for the multiplicative group of a division ring D. We refer to [4] for general material on theory of rings and modules.

2. Semihereditary rings and valuation rings

For the case of non-commutative rings there are different generalizations for valuation rings. First consider the generalization which was proposed in 1945 by Schilling [6], who extended the concept of a valuation on a field to that on a division ring.

Definition 2. *Let G be a totally ordered group (written additively) with the order relation $\geq$. Add to G a special symbol ∞ such that $x+\infty = \infty + x = \infty$ for all $x \in G$. Let D be a division ring. A valuation on D is a surjective map $v : D \to G \cup \{\infty\}$ which satisfies the following relations*:

1) $v(0) = \infty$,

2) $v(xy) = v(x) + v(y)$;

3) $v(x+y) \geq \min(v(x), v(y))$, *whenever* $x + y \neq 0$,

for any $x, y \in D$.

Then $A = \{x \in D \ : \ v(x) \geq 0\}$ is a ring which is called the (invariant) valuation ring of D with respect to valuation v, and $U = \{u \in D^* \ : \ v(u) = 0\}$ is called the group of valuation units.

In the general case we obtain the following definition.

Definition 3. [6] *A subring A of a division ring D is called an invariant valuation ring of D if there is a totally ordered group G and a valuation $v : D \to G$ of D such that $A = \{x \in D \ : \ v(x) \geq 0\}$.*

The next proposition gives the basic properties of invariant valuation rings.

Proposition 1. *Let A be an invariant valuation ring of a division ring D with respect to a valuation v. Then*

1. *$aA \subseteq bA$ or $bA \subseteq aA$ for any $a, b \in A$.*
2. *Each ideal of A is two-sided.*
3. *A is a right and a left Ore domain. Therefore it has a left and right division ring of fractions.*
4. *Any finitely generated ideal of A is principal.*

As an immediately consequence of this proposition we obtain the following.

Proposition 2. *Any invariant valuation ring of a division ring D is semihereditary and Bézout ring.*[1]

The following theorem gives the equivalent definitions of a invariant valuation ring.

Theorem 1. *Let A be a ring with a division ring of fractions D which is invariant in D. Then the following statements are equivalent:*

1. *A is an invariant valuation ring of some valuation v on D.*
2. *For any element $x \in D^*$ either $x \in A$ or $x^{-1} \in A$.*
3. *The set of principal ideals of A is linearly ordered by inclusion.*
4. *A is a uniserial ring.*[2]

Definition 4. *A subring A of a division ring D is called a total valuation ring if for each $x \in D^*$ we have $x \in A$ or $x^{-1} \in A$.*

Theorem 1 states that any invariant valuation ring is a total valuation ring, but not conversely. Note that in the case of integral domains the notions of invariant valuation rings and total valuation rings are equivalent to the notion of a classical valuation ring of a field. Theorem 1 also states that any invariant totally valuation ring is uniserial. Warfield [7] showed the connection of total valuation rings with semihereditary rings in the case of local rings.

[1] Recall that a ring A is called a right Bézout ring if any its finitely generated ideal is principal.

[2] Recall that a ring A is called uniserial if all ideals of A are linearly ordered with respect to inclusion.

Theorem 2. *For a local ring A the following properties are equivalent:*

(i) *A is uniserial and semihereditary.*

(ii) *A is a total valuation ring.*

The next type of non-commutative valuation rings was introduced and studied by Dubrovin [3].

Definition 5. *Let S be a simple Artinian ring. A subring A of S, with Jacobson radical $J(A)$, is called a Dubrovin valuation ring if*

1) *$A/J(A)$ is a simple Artinian ring*;

2) *for each $s \in S \setminus A$ there are $a_1, a_2 \in A$ such that $sa_1 \in A \setminus J(A)$ and $a_2 s \in A \setminus J(A)$.*

Note that every Dubrovin valuation ring is a total valuation ring if and only if $A/J(A)$ is a division ring. Hence, if S is a field, then Dubrovin valuation rings of S are exactly the usual valuation rings. The class of Dubrovin valuation rings is much wider than the class of total valuation rings. The following theorem gives the basic characterizations of Dubrovin valuation rings.

Theorem 3. [5] *Let A be a subring of a simple Artinian ring Q. Then the following conditions are equivalent:*

(1) *A is a Dubrovin valuation ring of G.*

(2) *A is a local semihereditary order in Q.*

(3) *A is a local Bézout order in Q.*

3. Semihereditary rings and flat modules

While semisimple rings and hereditary rings are defined uniquely by their projective global dimension, for semihereditary rings we have the following statement which gives the equivalent characterization of semihereditary rings.

Theorem 4. [2] *Let A be a ring. The following conditions are equivalent:*

1. *A is a left semihereditary ring.*
2. w.gl.dim$A \leq 1$ *and A is a right coherent ring.*[3]
3. *Every torsion-less right A-module is flat.*

Note that semihereditary rings are not defined uniquely by the flatness property. There are examples of rings with weak global dimension ≤ 1 which are not semihereditary. Note also that for any ring A, w.gl.dim$A \leq 1$ if and only if every ideal of A is flat. The criteria for modules to be flat are very important. In this section we give one of such criteria.

[3]Recall that a ring A is called right coherent if the direct product of an arbitrary family of copies of A is flat as a right A-module.

Theorem 5. *Let* $0 \to X \to P \to M \to 0$ *be an exact sequence of right* A*-modules, where* P *is projective. Then the following statements are equivalent:*

(1) M *is a flat module.*

(2) *For any* $x \in X$ *there is a* $\theta \in \mathrm{Hom}_A(P,X)$ *with* $\theta(x) = x$.

(3) *For any* $x_1, x_2, \dots, x_n \in X$ *there is a* $\theta \in \mathrm{Hom}_A(P,X)$ *with* $\theta(x_i) = x_i$ *for all* i.

Proof.

(1) $\Longrightarrow$ (2). Let P be a projective module, and M a flat module. By the Kaplansky theorem (see e.g. theorem 5.5.1 [4]), P is projective if and only if there is a system of elements $\{p_i \in P\ : i \in I\}$ and a system of homomorphisms $\{\varphi_i\}$, $\varphi_i : P \to A$ such that any element $p \in P$ can be written in the form

$$p = \sum_i p_i(\varphi_i(p)),$$

where only a finite number of elements $\varphi_i(p) \in A$ are not equal to zero.

If $x \in X$, then $x = p_{i_1}a_1 + p_{i_2}a_2 + \dots + p_{i_m}a_m$, where $a_i = \varphi_{i_1}(x) \in A$. Let $\mathcal{I} = Aa_1 + Aa_2 + \dots + Aa_m$. Since M is flat, $x \in X \cap P\mathcal{I} = X\mathcal{I}$, by the flatness test (see e.g. proposition 5.4.11 [4]). Therefore $x = \sum x_j c_j$, where $x_j \in X$ and $c_j \in \mathcal{I}$. Now each $c_j = \sum\limits_i b_{ij}a_i$, so $x = \sum\limits_i x'_i a_i$, where $x'_i = \sum\limits_j x_j b_{ij}$. Define $\theta : P \to X$ by $\theta(p_{i_k}) = x'_k$, while θ sends all the other system elements p_i of P into 0. Then

$$\theta(x) = \theta(\sum_{k=1}^{m} p_{i_k}a_k) = \sum_{k=1}^{m}(\theta(p_{i_k})a_k) = \sum_{k=1}^{m} x'_k a_k = x.$$

(2) $\Longrightarrow$ (1). Let $x \in X \cap P\mathcal{I}$, where $\mathcal{I}$ is a left ideal in A. Then $x = p_{i_1}a_1 + p_{i_2}a_2 + \dots + p_{i_r}a_r$, where $a_i \in A$. Define $\mathcal{I}_x = Aa_1 + Aa_2 + \dots + Aa_r$, which is a finitely generated left ideal in A. It is clear that $\mathcal{I}_x \subseteq \mathcal{I}$, and so $x \in X\mathcal{I}_x \subset X\mathcal{I}$. Let $\theta \in \mathrm{Hom}_A(P,X)$ with $\theta(x) = x$. Then $x = \theta(p_{i_1})a_1 + \theta(p_{i_2})a_2 + \dots + \theta(p_{i_r})a_r \in X\mathcal{I}_x$. Therefore $x \in X \cap P\mathcal{I} \subseteq X\mathcal{I}_x \subseteq X\mathcal{I}$. From the flatness test (see e.g. proposition 5.4.11 [4]) it follows that M is flat.

(2) $\Longrightarrow$ (3). This is proved by induction on n. Let $x_1, x_2, \dots, x_n \in X$. If $n = 1$, then the existence of θ follows from (2). Assume that $n > 1$ and (3) holds for all $k < n$. Let $\theta_n : P \to X$ be a homomorphism such that $\theta_n(x_n) = x_n$. Let $y_i = x_i - \theta_n(x_i)$ for $i = 1, 2, \dots, n-1$. By induction hypothesis, there exists a homomorphism θ' such that $\theta'(y_i) = y_i$ for $i = 1, 2, \dots, n-1$. Now define $\theta = \theta' + \theta_n - \theta'\theta_n \in \mathrm{Hom}_A(P,X)$. Then

$$\theta(x_n) = \theta'(x_n) + \theta_n(x_n) - \theta'\theta_n(x_n) = \theta'(x_n) + x_n - \theta' x_n = x_n,$$

$$\theta(x_i) = \theta'(x_i) + \theta_n(x_i) - \theta'\theta_n(x_i) = \theta'(x_i) + (x_i - y_i) - \theta'(x_i - y_i) =$$
$$= x_i - y_i + \theta'(y_i) = x_i$$

for $i = 1, 2, \dots, n-1$. So θ is a required homomorphism.

(3) $\Longrightarrow$ (2) follows by taking $n = 1$.

From this theorem it immediately follows the theorem which was first proved by Villamayor and was given by Chase in his paper [2].

Theorem 6. [2] *Let $0 \to X \to F \to P \to 0$ be an exact sequence of right A-modules, where F is free with a basis $\{e_i : i \in I\}$. Then the following statements are equivalent:*

(1) *P is a flat module.*

(2) *For any $x \in X$ there is a $\theta \in \mathrm{Hom}_A(F, X)$ with $\theta(x) = x$.*

(3) *For any $x_1, x_2, \dots, x_n \in X$ there is a $\theta \in \mathrm{Hom}_A(F, X)$ with $\theta(x_i) = x_i$ for all i.*

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SOME ALGEBRAIC PROPERTIES OF PREPONDERANTLY CONTINUOUS FUNCTIONS

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Abstract. In the presented paper we study some properties of preponderantly continuous functions and functions satisfying the property A_1. For any family $\mathcal{F}$ of real-valued functions we define $\mathcal{MAX}_{\mathcal{F}} = \{g\colon \max\{f,g\} \in \mathcal{F}$ for all $f \in \mathcal{F}\}$ and $\mathcal{MIN}_{\mathcal{F}} = \{g\colon \min\{f,g\} \in \mathcal{F}$ for all $f \in \mathcal{F}\}$. The aim of the paper is to find $\mathcal{MIN}_{\mathcal{F}}$ for two discussed classes of functions.

1. Preliminaries

Let $\mathbb{R}$, $\mathbb{N}$ be the set of real numbers and natural numbers, respectively. Next, let I denote a closed interval, U any open subset of $\mathbb{R}$ and $\mathrm{Int}(A)$ is the interior of a set $A \subset \mathbb{R}$ in the natural metric. Let λ stand for Lebesgue measure in $\mathbb{R}$. For each measurable set $E \subset \mathbb{R}$ we define the lower and upper density of E at $x_0 \in \mathbb{R}$ by:

$$\underline{d}\,(E,x_0) = \liminf_{\lambda(I)\to 0, x_0\in I} \tfrac{\lambda(I\cap E)}{\lambda(I)} \quad\text{and}\quad \overline{d}\,(E,x_0) = \limsup_{\lambda(I)\to 0, x_0\in I} \tfrac{\lambda(I\cap E)}{\lambda(I)}.$$

If $\underline{d}\,(E,x_0) = \overline{d}\,(E,x_0)$, we denote this common value by $d(E,x_0)$ and call it the density of E at x_0. In a similar way, we also define the one-sided lower and upper density of the set E at the point x_0: $\underline{d}^+\,(E,x_0)$, $\underline{d}^-\,(E,x_0)$, $\overline{d}^+\,(E,x_0)$ and $\overline{d}^-\,(E,x_0)$. It is easy to check that $\underline{d}\,(E,x_0) = \min\{\,\underline{d}^+\,(E,x_0)\,,\underline{d}^-\,(E,x_0)\}$ and $\overline{d}\,(E,x_0) = \max\{\,\overline{d}^+\,(E,x_0)\,,\overline{d}^-\,(E,x_0)\}$. If $\underline{d}^+\,(E,x_0) = \overline{d}^+\,(E,x_0)$ ($\underline{d}^-\,(E,x_0) = \overline{d}^-\,(E,x_0)$), then we denote this common value by $d^+(E,x_0)$ ($d^-(E,x_0)$) and call it the right (the left) density of E at x_0.

There are a few nonequivalent definitions of preponderant density and preponderant continuity [3]. We will use the following.

Definition 1. [1, 3] *A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ if $\underline{d}\,(E, x_0) > \frac{1}{2}$.*

Similarly, we can define the preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ at the right and at the left. Moreover, a point $x_0 \in \mathbb{R}$ is the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ iff it is the point of preponderant density in Denjoy sense of the measurable set E at the right and at the left.

Definition 2. [1, 3] *A function $f\colon U \to \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense at $x_0 \in U$ if there exists a measurable set $E \subset U$ containing x_0 such that $\underline{d}\,(E, x_0) > \frac{1}{2}$ and $f_{|E}$ is continuous at x_0. A function $f\colon U \to \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense if it is preponderantly continuous in Denjoy sense at each point $x_0 \in U$. The class of all functions which are preponderantly continuous in Denjoy sense will be denoted by $\mathcal{PD}$.*

Grande [2] defined a property of real functions called the property A_1. Based on this, we may define a similar property, which extends the notion of preponderant continuity.

Definition 3. [2, 3] *A function $f\colon U \to \mathbb{R}$ is said to have the property A_1 in Denjoy sense at $x_0 \in U$ if there exist measurable sets $E_1 \subset U$ and $E_2 \subset U$ containing x_0 such that x_0 is the point of preponderant density in Denjoy sense of both sets E_1 and E_2, $f_{|E_1}$ is upper semi-continuous at x_0 and $f_{|E_2}$ is lower semi-continuous at x_0. A function $f\colon U \to \mathbb{R}$ has the property A_1 in Denjoy sense if it has the property A_1 in Denjoy sense at each $x_0 \in U$. The class of all functions which have the property A_1 in Denjoy sense will be denoted by $\mathcal{GPD}$.*

Corollary. $\mathcal{PD} \subset \mathcal{GPD}$.

2. Auxiliary lemmas

We will present some known facts and the useful lemma.

Theorem 1. [3, *Corollary 9*] *$\mathcal{GPD} \subset \mathcal{B}_1$ and $\mathcal{PD} \subset \mathcal{B}_1$, where $\mathcal{B}_1$ is the set of Baire class 1 functions.*

Theorem 2. [3, *Theorem 2*]

(i) A measurable function $f\colon U \to \mathbb{R}$ is preponderantly continuous in Denjoy sense at $x_0 \in U$ iff $\lim\limits_{n\to\infty} \underline{d}\Big(\{x \in U\colon |f(x) - f(x_0)| < \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$,

(ii) *A measurable function* $f\colon U \to \mathbb{R}$ *has the property* A_1 *in Denjoy sense at* $x_0 \in U$ *iff* $\lim_{n\to\infty} \underline{d}\Big(\{x \in U\colon f(x) < f(x_0) + \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$ *and* $\lim_{n\to\infty} \underline{d}\Big(\{x \in U\colon f(x) > f(x_0) - \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$.

Theorem 3. [3, *Corollary 6*] *Let* $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, *where* $b_{n+1} < a_n$ *for every* n *and* $x_0 = \lim_{n\to\infty} a_n$. *Then*

1. $\underline{d}^+(E, x_0) = \liminf_{n\to\infty} \dfrac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])}$
2. $\overline{d}^+(E, x_0) = \limsup_{n\to\infty} \dfrac{\lambda([x_0, b_n] \cap E)}{\lambda([x_0, b_n])}$.

Lemma 1. *Let* $\frac{1}{2} < \gamma < 1$, $x \in \mathbb{R}$ *and let* E *be a measurable subset of* $\mathbb{R}$ *such that* $\overline{d}^+(E, x) = c > 0$. *Then there exists a sequence of closed intervals* $\{I_n = [a_n, b_n]\colon n \geq 1\}$ *for which* $x < \ldots < b_{n+1} < a_n < \ldots$, $d^+\Big(\bigcup_{n=1}^{\infty} I_n, x\Big) = \gamma$, *and* $\overline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x\Big) \geq \frac{1}{2}c$.

Proof. Let $c_n = x + \frac{1}{n}$ for $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \dfrac{\lambda([c_{n+1}, c_n])}{\lambda([x, c_{n+1}])} = \lim_{n\to\infty} \dfrac{\frac{1}{n(n+1)}}{\frac{1}{n+1}} = 0$.

Put $U_n^1 = [c_{n+1}, c_{n+1} + \gamma(c_n - c_{n+1})]$ and $U_n^2 = [c_n - \gamma(c_n - c_{n+1}), c_n]$ for $n \geq 1$. Then $\lambda(U_n^1) = \lambda(U_n^2) = \gamma\lambda([c_{n+1}, c_n])$, $[c_{n+1}, c_n] = U_n^1 \cup U_n^2$ and $\lambda(E \cap U_n^1) + \lambda(E \cap U_n^2) \geq \lambda(E \cap [c_{n+1}, c_n])$. It follows that for each $n \geq 1$ we can find a closed interval $J_n \subset [c_{n+1}, c_n]$ such that $\lambda(J_n) = \gamma\lambda([c_{n+1}, c_n])$ and $\lambda(E \cap J_n) \geq \frac{1}{2}\lambda(E \cap [c_{n+1}, c_n])$. Hence $\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap [x, c_k]\Big) = \gamma\lambda([x, c_k])$ for $k \geq 1$.

Let $z \in (x, c_1)$. There is $k \geq 1$ such that $z \in [c_{k+1}, c_k]$. Then

$$\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap [x, z]\Big) = \lambda\Big(\bigcup_{n=k+1}^{\infty} J_n\Big) + \lambda(J_k \cap [c_{k+1}, z]) \leq \gamma\lambda([x, z]) + \lambda([c_{k+1}, c_k]),$$

$$\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap [x, z]\Big) = \lambda\Big(\bigcup_{n=k+1}^{\infty} J_n\Big) + \lambda(J_k \cap [c_{k+1}, z]) \geq \gamma\lambda([x, z]) - \lambda([c_{k+1}, c_k])$$

and

$$\lambda\Big(\bigcup_{n=1}^{\infty} J_n \cap E \cap [x, z]\Big) \geq \lambda\Big(\bigcup_{n=k+1}^{\infty} J_n \cap E\Big) \geq \frac{1}{2}\lambda([x, z] \cap E) - \lambda([c_{n+1}, c_n]).$$

Therefore

$$\gamma - \tfrac{1}{n} = \gamma - \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n+1}} \leq \frac{\lambda\Big(\bigcup\limits_{n=1}^{\infty} J_n \cap [x,z]\Big)}{\lambda([x,z])} \leq \gamma + \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n}} = \gamma + \tfrac{1}{n+1}$$

and

$$\frac{\lambda\Big(\bigcup\limits_{n=1}^{\infty} J_n \cap E \cap [x,z]\Big)}{\lambda([x,z])} \geq \frac{1}{2}\frac{\lambda(E \cap [x,z])}{\lambda([x,z])} - \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n}} = \frac{1}{2}\frac{\lambda(E \cap [x,z])}{\lambda([x,z])} - \tfrac{1}{n}.$$

It follows that $d^+\Big(\bigcup\limits_{n=1}^{\infty} J_n, x\Big) = \gamma$ and $\overline{d}^+\Big(\bigcup\limits_{n=1}^{\infty} J_n \cap E, x\Big) \geq \frac{1}{2}\overline{d}^+(E,x)$.

We have proven that $d^+\left(\bigcup\limits_{n=1}^{\infty} J_n, x\right) = \gamma$ and $\overline{d}^+\left(E \cap \bigcup\limits_{n=1}^{\infty} J_n, x\right) \geq \frac{1}{2}\overline{d}^+(E,x)$, but the elements of the sequence need not be disjoint.

Let $\{I_n \colon n \geq 1\}$ be a sequence of closed intervals such that $I_n \subset \operatorname{Int} J_n$ for all $n \in \mathbb{N}$ and $\overline{d}^+\left(\bigcup\limits_{n=1}^{\infty} (J_n \setminus I_n), x\right) = 0$. Then the sequence $\{I_n \colon n \geq 1\}$ possesses all the required properties. □

3. $\mathcal{MAX}_{\mathcal{F}}$ and $\mathcal{MIN}_{\mathcal{F}}$ for $\mathcal{PD}$ and $\mathcal{GPD}$

Definition 4. *For any family $\mathcal{F}$ of functions from U to $\mathbb{R}$ we define*

$$\mathcal{MIN}_{\mathcal{F}} = \{g \colon U \to \mathbb{R} \colon \forall_{f \in \mathcal{F}} \ \min\{f,g\} \in \mathcal{F}\}.$$

and

$$\mathcal{MAX}_{\mathcal{F}} = \{g \colon U \to \mathbb{R} \colon \forall_{f \in \mathcal{F}} \ \max\{f,g\} \in \mathcal{F}\}.$$

Remark 1. *Observe that* $\max\{f,g\} = -\min\{-f,-g\}$ *and if $\mathcal{F}$ has the property $f \in \mathcal{F} \Rightarrow -f \in \mathcal{F}$, then*

$$\mathcal{MAX}_{\mathcal{F}} = \{g \colon U \to \mathbb{R} \colon -g \in \mathcal{MIN}_{\mathcal{F}}\}.$$

We will find $\mathcal{MAX}_{\mathcal{F}}$ and $\mathcal{MIN}_{\mathcal{F}}$ for $\mathcal{PD}$ and $\mathcal{GPD}$.

Lemma 2. *$\mathcal{MIN}_{\mathcal{PD}} \subset \mathcal{PD}$ and $\mathcal{MIN}_{\mathcal{GPD}} \subset \mathcal{GPD}$.*

Proof. To prove it, it suffices to take any $f \in \mathcal{MIN}_{\mathcal{PD}}$ ($f \in \mathcal{MIN}_{\mathcal{GPD}}$) and for each $x_0 \in U$ define a constant function $g(x) = f(x_0) + 1$. Then $g \in \mathcal{PD} \cap \mathcal{GPD}$ and, since $\min\{f,g\} \in \mathcal{PD}$ ($\min\{f,g\} \in \mathcal{GPD}$), it is easy to verify, applying Theorem 2, that f is preponderantly continuous in Denjoy sense at x_0 (g has the property A_1 in Denjoy sense at x_0). □

Lemma 3. *If $f \in \mathcal{PD}$ ($f \in \mathcal{GPD}$) and g is approximately continuous, then* $\max\{f,g\}, \min\{f,g\} \in \mathcal{PD}$ *(*$\max\{f,g\}, \min\{f,g\} \in \mathcal{GPD}$*).*

Proof. Fix any $x_0 \in U$. Since $f \in \mathcal{PD}$ ($f \in \mathcal{GPD}$), there exists a measurable set E (there exist two measurable sets E_1 and E_2) such that $x_0 \in E$ ($x_0 \in E_1 \cap E_2$), x_0 is a point of Denjoy preponderant density of E (of both sets E_1 and E_2) and $f_{|E}$ is continuous at x_0 ($f_{|E_1}$ is upper semi-continuous at x_0 and $f_{|E_2}$ is lower semi-continuous at x_0). Similarly, since g is approximately continuous at x_0, there exists a measurable set F such that $x_0 \in F$, $\underline{d}(F, x_0) = 1$ and $g_{|F}$ is continuous at x_0. Then $\min\{f,g\}_{|E\cap F}$ and $\max\{f,g\}_{|E\cap F}$ are continuous at x_0 ($\min\{f,g\}_{|E_1\cap F}$, $\max\{f,g\}_{|E_1\cap F}$ are upper semi-continuous at x_0 and $\min\{f,g\}_{|E_2\cap F}$, $\max\{f,g\}_{|E_2\cap F}$ are lower semi-continuous at x_0). Moreover, $\underline{d}(E\cap F, x_0) \geq \underline{d}(E, x_0) - \overline{d}(\mathbb{R}\setminus F, x_0) > \frac{1}{2}$ ($\underline{d}(E_1 \cap F, x_0) > \frac{1}{2}$ and $\overline{d}(E_2\cap F, x_0) > \frac{1}{2}$). It follows that $\min\{f,g\}$ and $\max\{f,g\}$ are preponderantly continuous in Denjoy sense at x_0 ($\min\{f,g\}$ and $\max\{f,g\}$ satisfy the property A_1 in Denjoy sense at x_0). Since x_0 was an arbitrary point, $\min\{f,g\}, \max\{f,g\} \in \mathcal{PD}$ ($\min\{f,g\}, \max\{f,g\} \in \mathcal{GPD}$). □

Lemma 4. *If $g \in \mathcal{PD}$ is not approximately lower semi-continuous at $x_0 \in U$, then there exists $f \in \mathcal{PD}$ such that* $\min\{f,g\} \notin \mathcal{GPD}$*.*

Proof. We may assume that g is not approximately lower semi-continuous at x_0 at the right. Then there exists $\varepsilon > 0$ such that $\overline{d}^+(\{x > x_0\colon f(x) < f(x_0) - \varepsilon\}, x_0) = c > 0$. Applying Lemma 1, we can find a sequence of closed intervals $\{I_n = [a_n, b_n]\colon n \geq 1\}$ such that $x_0 < \ldots < b_{n+1} < a_n < \ldots$, $d^+\big(\bigcup_{n=1}^{\infty} I_n, x_0\big) = \frac{1}{2} + \frac{1}{4}c$ and $\overline{d}\Big(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0\colon f(x) < f(x_0) - \varepsilon\}, x_0\Big) > \frac{1}{2}c$. Pick a sequence of pairwise disjoint closed intervals $\{J_n = [c_n, d_n]\colon n \geq 1\}$ such that $I_n \subset \mathrm{Int}(J_n)$ and $\overline{d}\Big(\bigcup_{n=1}^{\infty}(J_n \setminus I_n), x_0\Big) = 0$. Define a function $f\colon U \to \mathbb{R}$ letting

$$f(y) = \begin{cases} g(x_0) & \text{if } \ y \in \big(U \setminus (x_0, d_1)\big) \cup \bigcup_{n=1}^{\infty} I_n, \\ g(x_0) - 2\varepsilon & \text{if } \ y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on each interval } [c_n, a_n] \text{ and } [b_n, d_n], \ n = 1, 2, \ldots . \end{cases}$$

Obviously, $\min\{f(x_0), g(x_0)\} = g(x_0)$ and $f \in \mathcal{PD}$, because f is continuous at each point except at x_0 and x_0 is a point of preponderant density in Denjoy sense of $\big(E \setminus (x_0, d_1)\big) \cup \bigcup_{n=1}^{\infty} I_n$. Let $E = \{y\colon \min\{f(y), g(y)\} > g(x_0) - \varepsilon\}$.

Then $E \cap \bigcup\limits_{n=1}^{\infty} [d_{n+1}, c_n] = \emptyset$ and

$$\underline{d}(E, x_0) \le \underline{d}^+(E, x_0) \le \underline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x_0\Big) + \overline{d}^+\Big(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\Big) =$$

$$= d^+\Big(\bigcup_{n=1}^{\infty} I_n, x_0\Big) - \overline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 \colon f(x) < f(x_0) - \varepsilon\}, x_0\Big) \le$$

$$\le \frac{1}{2} + \frac{1}{4}c - \frac{1}{2}c = \frac{1}{2} - \frac{1}{4}c < \frac{1}{2}.$$

This implies that $\min\{f, g\}$ does not have the property A_1 in Denjoy sense at x_0 and $\min\{f, g\} \notin \mathcal{PGD}$, which completes the proof. □

Theorem 4. *$\mathcal{MIN}_{\mathcal{PD}} = \mathbf{A}$, where $\mathbf{A}$ is the set of approximately continuous functions.*

Proof. By Lemma 3, we have inclusion $\mathbf{A} \subset \mathcal{MIN}_{\mathcal{PD}}$.

Suppose that g is not approximately continuous at x_0. If g is not approximately lower semi-continuous at x_0, then applying Lemma 4, we obtain that $g \notin \mathcal{MIN}_{\mathcal{PD}}$. Assume that g is not approximately upper semi-continuous at $x_0 \in U$. Without loss of generality we may assume that g is not approximately upper semi-continuous at x_0 at the right. Then we can find $\varepsilon > 0$ such that $\overline{d}^+\big(\{x > x_0 \colon f(x) > f(x_0) + \varepsilon\}, x_0\big) = c > 0$.

As it was shown earlier, we can find $\varepsilon > 0$ and two sequences $\{I_n = [a_n, b_n] \colon n \ge 1\}$, $\{J_n = [c_n, d_n] \colon n \ge 1\}$ of closed intervals such that $x_0 < \ldots < d_{n+1} < c_n < \ldots$, $I_n \subset \mathrm{Int}(J_n)$ for $n \in \mathbb{N}$, $d^+\Big(\bigcup\limits_{n=1}^{\infty} I_n, x_0\Big) = \frac{1}{2} + \frac{1}{4}c$, $\overline{d}^+\Big(\bigcup\limits_{n=1}^{\infty} (J_n \setminus I_n), x_0\Big) = 0$ and $\overline{d}^+\Big(\bigcup\limits_{n=1}^{\infty} I_n \cap \{x > x_0 \colon f(x) > f(x_0) + \varepsilon\}, x_0\Big) > \frac{1}{2}c$. Define $f \colon U \to \mathbb{R}$ letting:

$$f(y) = \begin{cases} g(x_0) + 2 \cdot \varepsilon & \text{if} \quad y \in \big(U \setminus (x_0, d_1)\big) \cup \bigcup\limits_{n=1}^{\infty} I_n, \\ g(x_0) - 2 \cdot \varepsilon & \text{if} \quad y \in \bigcup\limits_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on the intervals } [c_n, a_n] \text{ and } [b_n, d_n], & n = 1, 2, \ldots . \end{cases}$$

It is clear that $f \in \mathcal{PD}$, since it is discontinuous only at x_0 and x_0 is a point of preponderant density in Denjoy sense of $\big(U \setminus (x_0, d_1)\big) \cup \bigcup\limits_{n=1}^{\infty} I_n$. Moreover, $\min\{f(x_0), g(x_0)\} = g(x_0)$. Let $E = \{x \in U \colon |\min\{f(y), g(y)\} - g(x_0)| < \varepsilon\}$.

Then $E \cap \bigcup\limits_{n=1}^{\infty} [d_{n+1}, c_n] = \emptyset$ and

$$\begin{aligned}\underline{d}(E, x_0) \le \underline{d}^+(E, x_0) \le \underline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x_0\Big) + \overline{d}^+\Big(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\Big) = \\ = d^+\Big(\bigcup_{n=1}^{\infty} I_n, x_0\Big) - \overline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n \cap \{x > x_0 \colon f(x) < f(x_0) - \varepsilon\}, x_0\Big) \le \\ \le \frac{1}{2} + \frac{1}{4}c - \frac{1}{2}c = \frac{1}{2} - \frac{1}{4}c < \frac{1}{2}.\end{aligned}$$

Therefore $\min\{f, g\}$ is not Denjoy preponderantly continuous at x_0. It follows that $\min\{f, g\} \notin \mathcal{PD}$. We have proven that if $g \notin \mathbf{A}$, then $g \notin \mathcal{MIN}_{\mathcal{PD}}$. Hence $\mathcal{MIN}_{\mathcal{PD}} \subset \mathbf{A}$, which completes the proof. $\square$

Applying Remark 1, we have:

Corollary.
$$\mathcal{MAX}_{\mathcal{PD}} = \mathbf{A}.$$

Theorem 5. *$\mathcal{MIN}_{\mathcal{GPD}} = \mathcal{GPD} \cap \{f \colon f$ is approximately lower semi-continuous$\}$.*

Proof. Let $g \in \mathcal{MIN}_{\mathcal{GPD}}$. Remark 1 and Lemma 4 imply that $g \in \mathcal{GPD}$ and g is lower semi-continuous.

Let $f, g \colon U \to \mathbb{R}$, $f, g \in \mathcal{GPD}$, $x_0 \in I$ and g be approximately lower semi-continuous at x_0. If $\min\{f(x_0), g(x_0)\} = g(x_0)$, then

$$\{y \in U \colon g(y) < g(x_0) + \varepsilon\} \subset \big\{y \in U \colon \min\{f(y), g(y)\} < g(x_0) + \varepsilon\big\}$$

and if $\min\{f(x_0), g(x_0)\} = f(x_0)$, then

$$\{y \in U \colon f(y) < f(x_0) + \varepsilon\} \subset \big\{y \in U \colon \min\{f(y), g(y)\} < f(x_0) + \varepsilon\big\}$$

for each $\varepsilon > 0$. In both cases x_0 is a point of preponderant density in Denjoy sense of

$$\big\{y \in U \colon \min\{f(y), g(y)\} < \min\{f(x_0), g(x_0)\} + \varepsilon\big\}$$

for each $x_0 \in I$ and each $\varepsilon > 0$, because $f, g \in \mathcal{GPD}$.

On the other hand, the set $\{y \in U \colon g(y) > g(x_0) - \varepsilon\} \cap \{y \colon f(y) > f(x_0) - \varepsilon\}$ is contained in $\big\{y \in U \colon \min\{f(y), g(y)\} > \min\{f(x_0), g(x_0)\} - \varepsilon\big\}$. Since $f \in \mathcal{GPD}$ and g is approximately lower semi-continuous at x_0, we have

$$\underline{d}\big(\{y \colon f(y) > f(x_0) - \varepsilon\}, x_0\big) > \frac{1}{2} \quad \text{and} \quad \underline{d}\big(\{y \colon g(y) > g(x_0) - \varepsilon\}, x_0\big) = 1.$$

Therefore

$$\begin{aligned}&\underline{d}\big(\big\{y\in U\colon\ \min\{f(y),g(y)\}>\min\{f(x_0),g(x_0)\}-\varepsilon\big\},x_0\big)\geq\\ \geq\ &\underline{d}\big(\{y\in U\colon f(y)>f(x_0)-\varepsilon\},x_0\big)-\overline{d}\big(\mathbb{R}\setminus\{y\in U\colon g(y)>g(x_0)-\varepsilon\},x_0\big)\geq\\ &\qquad\qquad\geq\underline{d}\big(\{y\in U\colon f(y)>f(x_0)-\varepsilon\},x_0\big).\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\underline{d}\big(\big\{y\in U\colon\ \min\{f(y),g(y)\}>\min\{f(x_0),g(x_0)\}-\varepsilon\big\},x_0\big)\geq\\ &\qquad\qquad\geq\lim_{\varepsilon\to 0^+}\underline{d}\big(\{y\in U\colon f(y)>f(x_0)-\varepsilon\},x_0\big)>\frac{1}{2}.\end{aligned}$$

It follows that $\min\{f,g\}$ has property A_1 in Denjoy sense at x_0. Since x_0 was an arbitrary point of U, we have $\min\{f,g\}\in\mathcal{GPD}$. Therefore $g\in\mathcal{MIN}_{\mathcal{GPD}}$. This completes the proof. □

Corollary.

$$\mathcal{MAX}_{\mathcal{GPD}}=\mathcal{GPD}\cap\{f\colon f\ \textit{is approximately upper semi-continuous}\}.$$

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THE POISSON INTEGRALS OF FUNCTIONS OF TWO VARIABLES FOR HERMITE AND LAGUERRE EXPANSIONS

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Abstract. In this paper we consider the Poisson integrals of functions of two variables for Hermite and Laguerre expansions in the spaces $L^p(\mathbb{R}^2; \exp(-z_1^2 - z_2^2))$ and $L^p\left(\mathbb{R}_+^2; (z_1 z_2)^\alpha \exp(-z_1 - z_2)\right)$, respectively. We state some estimates of the rate of convergence of the Poisson integrals.

1. Introduction

In [1] Muckenhoupt considered the Poisson integral $A(f)$ of a function $f \in L^p\left(\mathbb{R}_+; z^\alpha \exp(-z)\right)$, $1 \le p \le \infty$, $\alpha > -1$, $\mathbb{R}_+ = [0, \infty)$, defined by

$$A(f)(r,y) = A(f;r,y) = \int\limits_0^\infty K(r,y,z)\, f(z)\, z^\alpha \exp(-z)\, dz,$$

where

$$\begin{aligned} K(r,y,z) &= \sum_{n=0}^\infty \frac{r^n n!}{\Gamma(n+\alpha+1)} L_n^\alpha(y) L_n^\alpha(z) \\ &= \frac{(ryz)^{-\frac{\alpha}{2}}}{1-r} \exp\left(\frac{-r(y+z)}{1-r}\right) I_\alpha\left(\frac{2(ryz)^{\frac{1}{2}}}{1-r}\right), \qquad 0<r<1, \end{aligned}$$

L_n^α is the nth Laguerre polynomial and I_α is the modified Bessel function.

Reference [1] also considered the Poisson integral of a function $f \in L^p(\mathbb{R}; \exp(-z^2))$ for Hermite expansions defined by

$$B(f;r,x) = \int_{-\infty}^\infty P(r,x,z) f(z) \exp\left(-z^2\right) dz, \quad 0<r<1,$$

with the Poisson kernel

$$P(r,x,z)=\sum_{n=0}^{\infty}\frac{r^n H_n(x)H_n(z)}{\sqrt{\pi}\,2^n n!}=\frac{1}{\sqrt{\pi(1-r^2)}}\exp\left(\frac{-r^2x^2+2rxz-r^2z^2}{1-r^2}\right),$$

where H_n is the nth Hermite polynomial. Some approximation properties of these operators are given in [2].

Let us consider the operator $U(f)$:

$$\begin{aligned} U(f)\quad (r,y_1,y_2) &= U(f;r,y_1,y_2)\\ &= \int_0^{\infty}\int_0^{\infty} K(r,y_1,z_1)\,K(r,y_2,z_2)\,f(z_1,z_2)\,(z_1z_2)^{\alpha}\exp(-z_1-z_2)\,dz_1\,dz_2, \end{aligned}$$

where $f\in L^p\left(\mathbb{R}_+^2;(z_1z_2)^{\alpha}\exp(-z_1-z_2)\right),\ 1\le p\le\infty,\ \alpha>-1.$

This paper contains some properties of the above operator and of operator $W(f)$ defined by

$$\begin{aligned} W(f)\quad (r,y_1,y_2) &= W(f;r,y_1,y_2)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(r,y_1,z_1)P(r,y_2,z_2)f(z_1,z_2)\exp\left(-z_1^2-z_2^2\right)\,dz_1\,dz_2, \end{aligned}$$

where $f\in L^p\left(\mathbb{R}^2;\exp(-z_1^2-z_2^2)\right)$, $0<r<1$. The norm of a function f in $L^p\left(X^2;w(z_1,z_2)\right)$ is given by

$$\|f\|_p=\begin{cases}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(t_1,t_2)|^p\,w(t_1,t_2)\,dt_1\,dt_2\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ \sup\limits_{(t_1,t_2)\in\mathbb{R}^2}\text{ess}\ |f(t_1,t_2)|, & p=\infty,\end{cases}$$

where $X=\mathbb{R}_+$ and $w(z_1,z_2)=(z_1z_2)^{\alpha}\exp(-z_1-z_2)$, or $X=\mathbb{R}$ and $w(z_1,z_2)=\exp(-z_1^2-z_2^2)$, respectively.

We state some estimates of the rate of convergence of the integrals $U(f)$ and $W(f)$ using the classical moduli of continuity.

2. Auxiliary results

In this section we shall give some properties of the above operators which we shall apply to the proofs of the main theorems. We begin this section by recalling the following results of Toczek and Wachnicki [2].

Let $\varphi_{n,y}(z)=(z-y)^n,\ n\in\mathbb{N}=\{1,2,...\},\ y,z\in X$, where $X=\mathbb{R}_+$ or $X=\mathbb{R}$, respectively.

Lemma 1. *For each $y \in \mathbb{R}_+$ we have*

$$
\begin{aligned}
A(1;r,y) &= 1,\\
A(\varphi_{1,y};r,y) &= (1-r)(1+\alpha-y),\\
A(\varphi_{2,y};r,y) &= (1-r)\left\{y^2(1-r)+2(\alpha+2)ry-2(\alpha+1)y\right.\\
&\quad \left.+(\alpha+2)(\alpha+1)(1-r)\right\},\\
A(\varphi_{4,y};r,y) &= (1-r)^2\left\{y^4(r-1)^2-4\alpha(r-1)^2y^3-4(4r^2-5r+1)\right.\\
&\quad +6(\alpha+4)(\alpha+3)r^2y^2-12(\alpha+3)(\alpha+2)ry^2\\
&\quad +6(\alpha+2)(\alpha+1)y^2-12(\alpha+3)(\alpha+2)(r-1)y\\
&\quad \left.+(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(r-1)^2\right\}.
\end{aligned}
$$

Lemma 2. *For each $y \in \mathbb{R}$*

$$
\begin{aligned}
B(1;r,y) &= 1,\\
B(\varphi_{1,y};r,y) &= -y(1-r),\\
B(\varphi_{2,y};r,y) &= (1-r)\left\{y^2(1-r)+\frac{1}{2}(r+1)\right\},\\
B(\varphi_{4,y};r,y) &= (1-r)^2\left\{(r-1)^2y^4-3(r^2-1)y^2+\frac{3}{4}(r+1)^2\right\}
\end{aligned}
$$

holds.

From the definitions of U and W we easily obtain:

Lemma 3. *If $f_1, f_2 \in L^p\left(\mathbb{R}_+; z^\alpha \exp(-z)\right)$, $1 \le p \le \infty$, $\alpha > -1$, then*

$$U(f;r,y_1,y_2) = A(f_1;r,y_1)\,A(f_2;r,y_2)$$

for $(y_1,y_2) \in \mathbb{R}_+^2$, $0<r<1$, where $f(z_1,z_2)=f_1(z_1)f_2(z_2)$, $z_1,z_2 \in \mathbb{R}_+$.

Lemma 4. *If $f_1, f_2 \in L^p\left(\mathbb{R}; \exp(-z^2)\right)$, $1 \le p \le \infty$, then*

$$W(f;r,y_1,y_2) = B(f_1;r,y_1)\,B(f_2;r,y_2)$$

for $(y_1,y_2) \in \mathbb{R}^2$, $0<r<1$, where $f(z_1,z_2)=f_1(z_1)f_2(z_2)$, $z_1,z_2 \in \mathbb{R}$.

Applying Lemmas 1 and 2, it is easy to prove the following two lemmas.

Lemma 5. *For every* $(y_1, y_2) \in \mathbb{R}^2_+$ *it follows that*

$$\begin{aligned}
U(1; r, y_1, y_2) &= 1, \\
U(\varphi_{1,y_i}; r, y_1, y_2) &= (1-r)(1+\alpha-y_i), \\
U(\varphi_{2,y_i}; r, y_1, y_2) &= (1-r)\left\{y_i^2(1-r) + 2(\alpha+2)ry_i - 2(\alpha+1)y_i \right. \\
&\quad \left. + (\alpha+2)(\alpha+1)(1-r)\right\}, \\
U(\varphi_{4,y_i}; r, y_1, y_2) &= (1-r)^2\left\{y_i^4(r-1)^2 - 4\alpha(r-1)^2y_i^3 - 4(4r^2-5r+1) \right. \\
&\quad + 6(\alpha+4)(\alpha+3)r^2y_i^2 - 12(\alpha+3)(\alpha+2)ry_i^2 \\
&\quad + 6(\alpha+2)(\alpha+1)y_i^2 - 12(\alpha+3)(\alpha+2)(r-1)y_i \\
&\quad \left. + (\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(r-1)^2\right\}
\end{aligned}$$

for $0 < r < 1$, $i = 1, 2$.

Lemma 6. *For every* $(y_1, y_2) \in \mathbb{R}^2$ *it follows that*

$$\begin{aligned}
W(1; r, y_1, y_2) &= 1, \\
W(\varphi_{1,y_i}; r, y_1, y_2) &= -y_i(1-r), \\
W(\varphi_{2,y_i}; r, y_1, y_2) &= (1-r)\left\{y_i^2(1-r) + \frac{1}{2}(r+1)\right\}, \\
W(\varphi_{4,y_1} + \varphi_{4,y_2}; r, y_1, y_2) &= (1-r)^2\left\{(r-1)^2(y_1^4+y_2^4) \right. \\
&\quad \left. -3(r^2-1)(y_1^2+y_2^2) + \frac{3}{2}(r+1)^2\right\}
\end{aligned}$$

for $0 < r < 1$, $i = 1, 2$.

Using the Hölder inequality and Lemma 5, we obtain

Lemma 7. *For* $(y_1, y_2) \in \mathbb{R}^2_+$ *and* $0 < r < 1$ *we have*

$$\begin{aligned}
U(|\varphi_{1,y_i}|; r, y_1, y_2) &\le (1-r)^{\frac{1}{2}}\left\{y_i^2(1-r) + 2(\alpha+2)ry_i - 2(\alpha+1)y_i \right. \\
&\quad \left. + (\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}}, \quad i = 1, 2.
\end{aligned}$$

Similarly we get

Lemma 8. *For* $0 < r < 1$ *and* $(y_1, y_2) \in \mathbb{R}^2$ *we have*

$$W(|\varphi_{1,y_i}|; r, y_1, y_2) \le (1-r)^{\frac{1}{2}}\left\{y_i^2(1-r) + \frac{1}{2}(r+1)\right\}^{\frac{1}{2}}, \quad i = 1, 2.$$

Applying the Riesz-Thorin theorem, it is easy to prove

Lemma 9. *Let* $f \in L^p\left(\mathbb{R}^2_+; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$, *where* $\alpha > -1$ *and* $1 \le p \le \infty$. *Then* $U(f;r,\cdot,\cdot) \in L^p\left(\mathbb{R}^2_+; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$ *and*

$$\|U(f;r,\cdot,\cdot)\|_p \quad \le \quad \|f\|_p \tag{1}$$

for $0 < r < 1$.

In a similar fashion we can prove the following theorem for operator W.

Lemma 10. *Let* $f \in L^p\left(\mathbb{R}^2; \exp(-z_1^2-z_2^2)\right)$, *where* $1 \le p \le \infty$. *Then* $W(f;r,\cdot,\cdot) \in L^p\left(\mathbb{R}^2; \exp(-z_1^2-z_2^2)\right)$ *and* $\|W(f;r,\cdot,\cdot)\|_p \le \|f\|_p$ *for* $0 < r < 1$.

3. The rate of convergence

In this section we present some estimates of the rate of convergence of the integrals U and W. We state this estimates using the classical modulus of continuity defined by

$$\omega(f;\delta_1,\delta_2) = \sup_{\substack{0<h_1\le\delta_1\\0<h_2\le\delta_2}} \left\{ \sup_{(y_1,y_2)\in X^2} |f(y_1+h_1,y_2+h_2)-f(y_1,y_2)| \right\},$$

$\delta_1, \delta_2 > 0$, where $X = \mathbb{R}_+$ or $X = \mathbb{R}$, respectively.

Theorem 1. *Let* $f \in C(\mathbb{R}^2_+) \cap L^p\left(\mathbb{R}^2_+; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$, $1 \le p \le \infty$ *and* $\alpha > -1$. *Then*

$$|U(f;r,y_1,y_2) - f(y_1,y_2)| \le 6\omega\left(f;\delta_1,\delta_2\right)$$

for $0 < r < 1$ *and* $(y_1,y_2) \in \mathbb{R}^2_+$, *where*

$$\begin{aligned}\delta_i \quad = \quad & (1-r)^{\frac{1}{2}}\left\{y_i^2(1-r) + 2(\alpha+2)ry_i\right.\\ & \left.-2(\alpha+1)y_i + (\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}}, \ i = 1,2.\end{aligned}$$

Proof. First we suppose that $f \in C^1(\mathbb{R}^2_+) \cap L^p\left(\mathbb{R}^2_+; (z_1z_2)^\alpha \exp(-z_1-z_2)\right)$, $1 \le p \le \infty$, $\alpha > -1$. Let $f(z_1,z_2) - f(y_1,y_2) = \lambda_{y_1}(z_1,z_2) + \tau_{y_2}(z_1,z_2)$ for every $(z_1,z_2) \in \mathbb{R}^2_+$, where

$$\lambda_{y_1}(z_1,z_2) = \int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u,z_2)\,du, \qquad \tau_{y_2}(z_1,z_2) = \int_{y_2}^{z_2} \frac{\partial}{\partial v} f(y_1,v)\,dv.$$

Observe that

$$
\begin{aligned}
|\lambda_{y_1}(z_1,z_2)| = \left|\int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u,z_2)\,du\right| &\le \left|\int_{y_1}^{z_1} du\right| \cdot \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right| \\
&= |z_1-y_1| \cdot \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|
\end{aligned}
$$

and

$$
|\tau_{y_2}(z_1,z_2)| \le |z_2-y_2| \cdot \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.
$$

Using Lemma 7, we get

$$
\begin{aligned}
U\left(|\lambda_{y_1}|;r,y_1,y_2\right) \le\ & U\left(|\varphi_{1,y_1}|;r,y_1,y_2\right) \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right| \\
&\le (1-r)^{\frac{1}{2}}\Big\{y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1 \\
&+(\alpha+2)(\alpha+1)(1-r)\Big\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|,
\end{aligned}
$$

$$
\begin{aligned}
U\left(|\tau_{y_2}|;r,y_1,y_2\right) \le\ & (1-r)^{\frac{1}{2}}\Big\{y_2^2(1-r)+2(\alpha+2)ry_2-2(\alpha+1)y_2 \\
&+(\alpha+2)(\alpha+1)(1-r)\Big\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
|U\left(f;r,y_1,y_2\right) & -f(y_1,y_2)| \le U\left(|\lambda_{y_1}|;r,y_1,y_2\right)+U\left(|\tau_{y_2}|;r,y_1,y_2\right) \\
\le\ & (1-r)^{\frac{1}{2}}\left\{y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1\right. \\
& \left.+(\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right| \\
& +(1-r)^{\frac{1}{2}}\left\{y_2^2(1-r)+2(\alpha+2)ry_2-2(\alpha+1)y_2\right. \\
& \left.+(\alpha+2)(\alpha+1)(1-r)\right\}^{\frac{1}{2}} \sup_{(y_1,y_2)\in\mathbb{R}_+^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.
\end{aligned}
$$

Let $f \in C(\mathbb{R}_+^2)\cap L^p\left(\mathbb{R}_+^2;(z_1z_2)^\alpha\exp(-z_1-z_2)\right)$, $1\le p\le\infty$, $\alpha>-1$ and let f_{δ_1,δ_2} be the Steklov function of a function f of two variables given by the formula

$$f_{\delta_1,\delta_2}(y_1,y_2)=\frac{1}{\delta_1\delta_2}\int\limits_0^{\delta_1}\int\limits_0^{\delta_2} f(y_1+u,y_2+v)\,du\,dv \text{ for } (y_1,y_2)\in\mathbb{R}^2_+,\ \delta_1,\delta_2>0.$$

If $f\in C(\mathbb{R}^2_+)\cap L^p\left(\mathbb{R}^2_+;(z_1z_2)^\alpha\exp(-z_1-z_2)\right)$, then

$$f_{\delta_1,\delta_2}\in C^1(\mathbb{R}^2_+)\cap L^p\left(\mathbb{R}^2_+;(z_1z_2)^\alpha\exp(-z_1-z_2)\right)$$

for fixed $\delta_1,\delta_2>0$ and

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|\le\omega(f;\delta_1,\delta_2),$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\left|\frac{\partial}{\partial y_1}f_{\delta_1,\delta_2}(y_1,y_2)\right|\le 2\delta_1^{-1}\omega(f;\delta_1,\delta_2),$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\left|\frac{\partial}{\partial y_2}f_{\delta_1,\delta_2}(y_1,y_2)\right|\le 2\delta_2^{-1}\omega(f;\delta_1,\delta_2)$$

for all $\delta_1,\delta_2>0$.

Moreover, for $f\in L^p\left(\mathbb{R}^2_+;(z_1z_2)^\alpha\exp(-z_1-z_2)\right)$ we can write

$$\begin{aligned}
&|U(f;r,y_1,y_2)-f(y_1,y_2)|\\
&\quad\le |U(f-f_{\delta_1,\delta_2};r,y_1,y_2)|+|U(f_{\delta_1,\delta_2};r,y_1,y_2)-f_{\delta_1,\delta_2}(y_1,y_2)| \qquad (2)\\
&\qquad +|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|,
\end{aligned}$$

where $(y_1,y_2)\in\mathbb{R}^2_+$, $\delta_1,\delta_2>0$.

Now, from the first part of this proof we have

$$\begin{aligned}
&|U(f_{\delta_1,\delta_2};r,y_1,y_2)-f_{\delta_1,\delta_2}(y_1,y_2)|\\
&\quad\le 2\omega(f;\delta_1,\delta_2)\Big\{\delta_1^{-1}(1-r)^{\frac12}\Big[y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1\\
&\qquad +(\alpha+2)(\alpha+1)(1-r)\Big]^{\frac12}+\delta_2^{-1}(1-r)^{\frac12}\big[y_2^2(1-r)+2(\alpha+2)ry_2\\
&\qquad -2(\alpha+1)y_2+(\alpha+2)(\alpha+1)(1-r)\big]^{\frac12}\Big\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&|U(f-f_{\delta_1,\delta_2};r,y_1,y_2)|\le\\
&\qquad\int\limits_0^\infty\int\limits_0^\infty K(r,y_1,z_1)\,K(r,y_2,z_2)\,(z_1z_2)^\alpha\exp(-z_1-z_2)\,dz_1\,dz_2\\
&\times\sup_{(y_1,y_2)\in\mathbb{R}^2_+}|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|\le U(1;r,y_1,y_2)\,\omega(f;\delta_1,\delta_2)\le\omega(f;\delta_1,\delta_2)
\end{aligned}$$

and

$$|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)|\le\omega(f;\delta_1,\delta_2).$$

From (2) we have

$$\begin{aligned}
&|U(f;r,y_1,y_2)-f(y_1,y_2)|\\
&\quad\le\ \Big\{1+\delta_1^{-1}(1-r)^{\frac12}\left[y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1\right.\\
&\qquad\left.+(\alpha+2)(\alpha+1)(1-r)\right]^{\frac12}+\delta_2^{-1}(1-r)^{\frac12}\left[y_2^2(1-r)\right.\\
&\qquad\left.+2(\alpha+2)ry_2-2(\alpha+1)y_2+(\alpha+2)(\alpha+1)(1-r)\right]^{\frac12}\Big\}\,2\omega(f;\delta_1,\delta_2)
\end{aligned}$$

for $0<r<1$, $\delta_1,\delta_2>0$ and all $(y_1,y_2)\in\mathbb{R}_+^2$. Setting

$$\delta_1=(1-r)^{\frac12}\big(y_1^2(1-r)+2(\alpha+2)ry_1-2(\alpha+1)y_1+(\alpha+2)(\alpha+1)(1-r)\big)^{\frac12},$$

$$\delta_2=(1-r)^{\frac12}\big(y_2^2(1-r)+2(\alpha+2)ry_2-2(\alpha+1)y_2+(\alpha+2)(\alpha+1)(1-r)\big)^{\frac12}$$

for fixed $(y_1,y_2)\in\mathbb{R}_+^2$, we get the required inequality.

In a similar fashion we obtain:

Theorem 2. *Let* $f\in C(\mathbb{R}^2)\cap L^p\left(\mathbb{R}^2;\exp(-z_1^2-z_2^2)\right)$, $1\le p\le\infty$. *Then*

$$|W(f;r,y_1,y_2)-f(y_1,y_2)|\le 6\omega\left(f;\delta_1,\delta_2\right)$$

for $0<r<1$ *and* $(y_1,y_2)\in\mathbb{R}^2$, *where*

$$\delta_i\ =\ (1-r)\left\{y_i^2(1-r)+\frac12(r+1)\right\},\ i=1,2.$$

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BETTER VERSUS LONGER SERIES OF HEADS AND TAILS

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Abstract. This article considers a part of games theory by Penney. We intuitionally believe that a shorter series is always a better one. We will prove that this is not always so, and a longer series may happen to be better (see also [1]). In the case of Penney's game, in which players can choose their series, the proved theorems can be a part of a player's game strategy.

Definition 1. Let $k \in \mathbb{N}$ and $k \geq 1$. Each result of the k-fold variation of the $\{H, T\}$ set, which is a result of the k-fold coin toss, is called *a series of heads and tails.* We mark its length as $|\alpha|$.

Definition 2. Let α and β be series of heads and tails. We can say that *the series α is not included in the series β* if it is not the subsequence of the successive elements of the series β.

Definition 3. Let α and β be series of heads and tails. Let the series α be k long and the series β be l long. Let us also assume that the series α is not included in the β one. We repeat a coin toss so long that we get k last results forming the series α or l last results forming the series β. We call this experiment *waiting for one of the two stated series of results* and mark it as $\delta_{\alpha-\beta}$ (see [3], pp. 406–415).

Let us consider a game of two players, G_α and G_β. In the game they conduct the experiment $\delta_{\alpha-\beta}$. If the waiting finishes with the series α – the player G_α wins, and if it finishes with the series β – the player G_β wins. We shall call this game *the Penney game*[1] and mark it as $g_{\alpha-\beta}$.

Let us consider the waiting of $\delta_{\alpha-\beta}$. The sequence ω having its elements from the set $\{H, T\}$ is a result of the experiment $\delta_{\alpha-\beta}$ if it fulfills the following conditions:

- the subsequence of k last results forms the series α or the subsequence of l last results forms the series β, and
- no subsequence of k or l successive results forms the series α or β.

We mark the set of all such sequences (results of the experiment $\delta_{\alpha-\beta}$) as $\Omega_{\alpha-\beta}$.

If the result ω of the experiment $\delta_{\alpha-\beta}$ is an n-element sequence, it is a specific result of an n-fold coin toss. Its probability equals $\left(\frac{1}{2}\right)^n$.
Let $p_{\alpha-\beta}$ be a function of ω,

$$p_{\alpha-\beta}(\omega) = \left(\frac{1}{2}\right)^{|\omega|} \text{ for } \omega \in \Omega_{\alpha-\beta},$$

and $|\omega|$ be the ω sequence length (the number of elements). This function is the distribution of probability in the set $\Omega_{\alpha-\beta}$, and the pair $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$ is a probabilistic model of the waiting $\delta_{\alpha-\beta}$.

Let us state two opposite events in the space $(\Omega_{\alpha-\beta}, p_{\alpha-\beta})$:

$$A = \{\text{the waiting } \delta_{\alpha-\beta} \text{ gives the series } \alpha \text{ at the end}\},$$
$$B = \{\text{the waiting } \delta_{\alpha-\beta} \text{ gives the series } \beta \text{ at the end}\}.$$

Definition 4. If $P(A) = P(B)$, we say that *the series α and β are equally good* and mark them as $\alpha \approx \beta$.

Definition 5. If $P(A) > P(B)$, we say that *the series α is better than the series β* and mark them as $\alpha \gg \beta$.

In the game $g_{\alpha-\beta}$ we conduct the experiment $\delta_{\alpha-\beta}$. If the event A occurs, the player G_α wins. If the experiment ends with the event B, the game winner is the player G_β. Stating the probability of the events A and B, we can also determine the fairness of the Penney game. If the series α and β are equally good, the players have equal chance to win. The game $g_{\alpha-\beta}$ is fair. If one of the series is better than the other, the players chances to win are not equal and the game is not fair.

[1]Proposed by Walter Penney, see [2].

Let $\delta_{\alpha-\beta}$ be waiting for one of two series of heads and tails and k and l be the lengths of series α and β. Let $m \in \{1, 2, 3, \ldots, \min\{k, l\}\}$, $\alpha^{(m)}$, $\beta^{(m)}$ mean sequences of first m elements of series α and β, respectively, and $\alpha_{(m)}$, $\beta_{(m)}$ mean last m elements of the series α and β, respectively. Let us define the sets

$$A_\alpha = \{m : \alpha_{(m)} = \alpha^{(m)}\}, \qquad A_\beta = \{m : \alpha_{(m)} = \beta^{(m)}\},$$

$$B_\beta = \{m : \beta_{(m)} = \beta^{(m)}\}, \qquad B_\alpha = \{m : \beta_{(m)} = \alpha^{(m)}\},$$

and the following sums

$$\alpha : \alpha = \sum_{j \in A_\alpha} 2^j, \qquad \alpha : \beta = \sum_{j \in A_\beta} 2^j,$$

$$\beta : \beta = \sum_{j \in B_\beta} 2^j, \qquad \beta : \alpha = \sum_{j \in B_\alpha} 2^j.$$

Theorem 1. *In the probabilistic space of $\delta_{\alpha-\beta}$ the equation*

$$\frac{P(B)}{P(A)} = \frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha},$$

called the Conway equation, is true[2].

Remark 1. *From the preceding equation, we can tell that if*

$$\mu := \frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha},$$

then

$$\mu > 1 \Leftrightarrow \beta \gg \alpha,$$

$$\mu = 1 \Leftrightarrow \alpha \approx \beta,$$

$$\mu < 1 \Leftrightarrow \alpha \gg \beta.$$

Example 1. Let $\alpha = HTHTHT$ and $\beta = HHTHTH$. Let us notice that $\alpha_{(1)} = T \neq H = \alpha^{(1)}$, so $1 \notin A_\alpha$. Analogously

$$\left.\begin{matrix} \mathbf{HT}HTHT \\ HTHT\mathbf{HT} \end{matrix}\right\} \Rightarrow 2 \in A_\alpha, \qquad \left.\begin{matrix} \mathbf{HTH}THT \\ HTH\mathbf{THT} \end{matrix}\right\} \Rightarrow 3 \notin A_\alpha,$$

$$\left.\begin{matrix} \mathbf{HTHT}HT \\ HT\mathbf{HTHT} \end{matrix}\right\} \Rightarrow 4 \in A_\alpha, \qquad \left.\begin{matrix} \mathbf{HTHTH}T \\ H\mathbf{THTHT} \end{matrix}\right\} \Rightarrow 5 \notin A_\alpha,$$

[2]Discovered by John Horton Conway; the proof of its correctness is shown in [4].

$$\left.\begin{array}{l}\mathbf{HTHTHT}\\ \mathbf{HTHTHT}\end{array}\right\} \Rightarrow \ 6 \in A_\alpha.$$

Therefore

$$A_\alpha = \{2, 4, 6\},$$

so

$$\alpha : \alpha = 2^2 + 2^4 + 2^6 = 84.$$

In the same way, we come to the following:

$$\alpha : \beta = 0, \qquad \beta : \beta = 66, \qquad \beta : \alpha = 42,$$

so

$$\frac{\alpha : \alpha - \alpha : \beta}{\beta : \beta - \beta : \alpha} = \frac{84 - 0}{66 - 42} = \frac{21}{6} > 1.$$

Therefore $HHTHTH \gg HTHTHT$, and this means that $g_{HTHTHT-HHTHTH}$ is not a fair one.

Let $\delta_{\alpha-\beta}$ be waiting for one of the series α or β. Let us assume that $|\alpha| > |\beta|$. Intuitionally we can presume that the series β, being shorter than the series α, is a better one.

Let us consider two series: $\alpha = HHTT...TT$ and $\beta = TT...TT$. The series are such that $|\alpha| = |\beta| + 1 = k + 1$, where $k \geq 2$. In this case

$$\alpha : \alpha = 2^{k+1}, \qquad \alpha : \beta = \sum_{j=1}^{k-1} 2^j,$$

$$\beta : \beta = \sum_{j=1}^{k} 2^j, \qquad \beta : \alpha = 0.$$

From the Conway equation we know that

$$\frac{P(A)}{P(B)} = \frac{\beta : \beta - \beta : \alpha}{\alpha : \alpha - \alpha : \beta} = \frac{\sum_{j=1}^{k} 2^j}{2^{k+1} - \sum_{j=1}^{k-1} 2^j}.$$

Let us notice that $\sum_{j=1}^{n} 2^j$ is a sum of first n elements of the geometrical sequence which has the first element 2 and the quotient 2, so

$$\sum_{j=1}^{n} 2^j = 2\frac{1-2^n}{1-2} = 2^{n+1} - 2. \tag{1}$$

Therefore

$$\frac{\sum\limits_{j=1}^{k} 2^j}{2^{k+1} - \sum\limits_{j=1}^{k-1} 2^j} = \frac{2 \cdot 2^k - 2}{2 \cdot 2^k - 2^k + 2} = \frac{1 - \frac{1}{2^k}}{\frac{1}{2} + \frac{1}{2^k}} > \frac{1}{\frac{1}{2}},$$

and

$$\frac{P(A)}{P(B)} > 2,$$

so $\alpha \gg \beta$ even if the series α is longer than the β one.

If we narrow our consideration to pairs of series that differ by more than one element in length, we can easily see that the shorter series is a better one.

Theorem 2. *Let $\delta_{\alpha-\beta}$ be waiting for one of the α or β series of heads and tails which lengths fulfill the condition $|\alpha| \geq |\beta| + 2$. Then the series β is better than the series α.*

Proof. Let α and β be series of heads and tails and $|\alpha| = k$, $|\beta| = l$. Let $m \geq 2$ be such a number that $k = l + m$. As the series cannot include each other, we have

$$\{k\} \subset A_\alpha \subset \{1, 2, 3, ..., k\}, \qquad A_\beta \subset \{1, 2, 3, ..., l-1\},$$

$$\{k\} \subset B_\beta \subset \{1, 2, 3, ..., l\}, \qquad B_\alpha \subset \{1, 2, 3, ..., l-1\},$$

which leads us to the following approximations:

$$2^k \leq \alpha : \alpha \leq \sum_{j=1}^{k} 2^j, \qquad 0 \leq \alpha : \beta \leq \sum_{j=1}^{l-1} 2^j$$

$$2^l \leq \beta : \beta \leq \sum_{j=1}^{l} 2^j, \qquad 0 \leq \alpha : \beta \leq \sum_{j=1}^{l-1} 2^j.$$

Then

$$\frac{\beta : \beta - \beta : \alpha}{\alpha : \alpha - \alpha : \beta} \leq \frac{\sum\limits_{j=1}^{l} 2^j - 0}{2^k - \sum\limits_{j=1}^{l-1} 2^j}.$$

From (1) we get

$$\frac{\sum_{j=1}^{l} 2^j}{2^k - \sum_{j=1}^{l-1} 2^j} = \frac{2 \cdot 2^l - 2}{2^{m+l} - (2^l - 2)} < \frac{2 \cdot 2^l}{2^m \cdot 2^l - (2^l - 2)} = \frac{2}{2^m - (1 - \frac{2}{2^l})} < \frac{2}{4 - 1},$$

therefore

$$\frac{\beta : \beta - \beta : \alpha}{\alpha : \alpha - \alpha : \beta} < 1.$$

Considering the remark 1, we get $\beta \gg \alpha$.

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DIFFERENT KINDS OF BOUNDARY CONDITIONS FOR TIME-FRACTIONAL HEAT CONDUCTION EQUATION

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Abstract. The time-fractional heat conduction equation with the Caputo derivative of the order $0 < \alpha < 2$ is considered in a bounded domain. For this equation different types of boundary conditions can be given. The Dirichlet boundary condition prescribes the temperature over the surface of the body. In the case of mathematical Neumann boundary condition the boundary values of the normal derivative are set, the physical Neumann boundary condition specifies the boundary values of the heat flux. In the case of the classical heat conduction equation ($\alpha = 1$), these two types of boundary conditions are identical, but for fractional heat conduction they are essentially different. The mathematical Robin boundary condition is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain, while the physical Robin boundary condition prescribes a linear combination of the values of temperature and the values of the heat flux at the surface of a body.

1. Introduction

The conventional theory of heat conduction is based on the classical (local) Fourier law, which relates the heat flux vector $\mathbf{q}$ to the temperature gradient

$$\mathbf{q} = -k \,\mathrm{grad}\, T, \tag{1}$$

where k is the thermal conductivity of a solid. In combination with a law of conservation of energy,

$$\rho C \frac{\partial T}{\partial t} = -\mathrm{div}\, \mathbf{q} \tag{2}$$

with ρ being the mass density, C the specific heat capacity, the Fourier law leads to the parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a\, \Delta T, \tag{3}$$

where a is the thermal diffusivity coefficient.

It should be noted that Eq. (1) is a phenomenological law which states the proportionality of the flux to the gradient of the transported quantity. It is met in several physical phenomena with different names.

For example, it is well known that from mathematical viewpoint the Fourier law (1) in the theory of heat conduction and the Fick law in the theory of diffusion,

$$\mathbf{J} = -k_c\, \mathrm{grad}\, c, \tag{4}$$

where $\mathbf{J}$ is the matter flux, c is the concentration, k_c is the diffusion conductivity, are identical. In combination with the balance equation for mass,

$$\rho \frac{\partial c}{\partial t} = -\mathrm{div}\, \mathbf{J}, \tag{5}$$

the Fick law leads to the classical diffusion equation

$$\frac{\partial c}{\partial t} = a_c\, \Delta c. \tag{6}$$

Here a_c is the diffusivity coefficient.

Similarly, the classical empirical Darcy law, describing the flow of fluid through a porous medium, states proportionality between the fluid mass flux $\mathbf{J}$ and the gradient of the pore pressure p,

$$\mathbf{J} = -k_p\, \mathrm{grad}\, p, \tag{7}$$

and leads to the parabolic diffusion equation for the pressure

$$\frac{\partial p}{\partial t} = a_p\, \Delta p. \tag{8}$$

Though we will consider heat conduction, it obvious that the discussion concerns also diffusion as well as the theory of fluid flow through the porous solid.

Nonclassical theories of heat conduction in which the Fourier law and the standard heat conduction equation are replaced by more general equations, constantly attract the attention of the researchers. For an extensive bibliography on this subject see [1–11] and references therein.

2. Nonlocal generalizations of the Fourier law

For materials with time nonlocality (with memory) the effect at a point $\mathbf{x}$ at time t depends on the histories of causes at a point $\mathbf{x}$ at all past and present times. In the theory proposed by Gurtin and Pipkin [12] the law of heat conduction is given by general time–nonlocal dependence

$$\mathbf{q}(t) = -k \int_0^{\infty} K(u)\,\mathrm{grad}\,T(t-u)\,\mathrm{d}u. \tag{9}$$

Using substitution $\tau = t - u$ leads to the following equation

$$\mathbf{q}(t) = -k \int_{-\infty}^{t} K(t-\tau)\,\mathrm{grad}\,T(\tau)\,\mathrm{d}\tau. \tag{10}$$

Choosing 0 instead of $-\infty$ as a “starting point”, we obtain

$$\mathbf{q}(t) = -k \int_0^{t} K(t-\tau)\,\mathrm{grad}\,T(\tau)\,\mathrm{d}\tau \tag{11}$$

and the heat conduction equation with memory [13]:

$$\frac{\partial T}{\partial t} = a \int_0^{t} K(t-\tau)\,\Delta T(\tau)\,\mathrm{d}\tau. \tag{12}$$

The time-nonlocal dependences between the heat flux vector and the temperature gradient with the “long-tale” power kernel $K(t-\tau)$ were considered in [5, 8, 9] (see also [14])

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^{t} (t-\tau)^{\alpha-1}\,\mathrm{grad}\,T(\tau)\,\mathrm{d}\tau, \qquad 0<\alpha\le 1; \tag{13}$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)}\int_0^{t} (t-\tau)^{\alpha-2}\,\mathrm{grad}\,T(\tau)\,\mathrm{d}\tau, \qquad 1<\alpha\le 2, \tag{14}$$

where $\Gamma(\alpha)$ is the gamma function. Equations (13) and (14) can be interpreted in terms of fractional integrals and derivatives

$$\mathbf{q}(t) = -kD_{RL}^{1-\alpha}\mathrm{grad}\,T(t), \qquad 0<\alpha\le 1; \tag{15}$$

$$\mathbf{q}(t) = -kI^{\alpha-1}\mathrm{grad}\,T(t), \qquad 1 < \alpha \le 2, \tag{16}$$

where $I^{\alpha} f(t)$ and $D^{\alpha}_{RL} f(t)$ are the Riemann–Liouville fractional integral and derivative of the order α, respectively [15–18]:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad \alpha > 0, \tag{17}$$

$$D^{\alpha}_{RL} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \right], \quad n-1 < \alpha < n. \tag{18}$$

The constitutive equations (15) and (16) yield the time-fractional heat conduction equation

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\Delta T, \qquad 0 < \alpha \le 2, \tag{19}$$

with the Caputo fractional derivative of order $0 < \alpha \le 2$

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \quad n-1 < \alpha < n. \tag{20}$$

3. Boundary conditions

The Dirichlet boundary condition (the boundary condition of the first kind) specifies the temperature over the surface of the body under consideration

$$T\big|_S = g_0(\mathbf{x}_S, t). \tag{21}$$

For fractional heat conduction equations, two types of Neumann boundary condition (the boundary condition of the second kind) can be considered: the mathematical condition with the prescribed boundary value of the normal derivative

$$\frac{\partial T}{\partial n}\bigg|_S = G_0(\mathbf{x}_S, t) \tag{22}$$

and the physical condition with the prescribed boundary value of the heat flux

$$\begin{aligned} D^{1-\alpha}_{RL} \frac{\partial T}{\partial n}\bigg|_S &= G_0(\mathbf{x}_S, t), \quad 0 < \alpha \le 1, \\ I^{\alpha-1} \frac{\partial T}{\partial n}\bigg|_S &= G_0(\mathbf{x}_S, t), \quad 1 < \alpha \le 2. \end{aligned} \tag{23}$$

In the case of the classical heat conduction equation ($\alpha = 1$), these two types of boundary condition are identical, but for fractional heat conduction they are essentially different. Similarly, the mathematical Robin boundary condition (the boundary condition of the third kind) is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain

$$\left(c_1 T + c_2 \frac{\partial T}{\partial n} \right) \bigg|_S = H_0(\mathbf{x}_S, t) \tag{24}$$

with some nonzero constants c_1 and c_2, while the physical Robin boundary condition specifies a linear combination of the values of temperature and the values of the heat flux at the boundary of the domain. The condition of convective heat exchange between a body and the environment with the temperature T_e leads to

$$\begin{aligned} \left(hT + k D_{RL}^{1-\alpha} \frac{\partial T}{\partial n} \right) \bigg|_S &= hT_e(\mathbf{x}_S, t), \quad 0 < \alpha \leq 1, \\ \left(hT + k I^{\alpha-1} \frac{\partial T}{\partial n} \right) \bigg|_S &= hT_e(\mathbf{x}_S, t), \quad 1 < \alpha \leq 2, \end{aligned} \tag{25}$$

where h is the convective heat transfer coefficient.

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AUTOMORPHISMS OF WITT RINGS OF FINITE FIELDS

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Abstract. The problem of general description of the group of automorphisms of any Witt ring W seems to be very difficult to solve. However, there are many types of Witt rings, which automorphism are described precisely (e.g. [1], [2], [4], [5], [6], [7], [8]). In our paper we characterize automorphisms of abstract Witt rings (cf. [3]) isomorphic to powers of Witt rings of quadratic forms with coeffitients in finite fields with characteristic different from 2.

1. Introduction

A Witt ring of quadratic forms is one of the centaral notion in bilinear algebra. Investigation of automorphisms of Witt rings is a natural problem, when we consider such objects. The special property of automorphisms of a Witt ring W is the fact that every automorphism of W has to preserve the dimension of forms or, in other words, it has to send one-dimensional forms to one-dimensional forms. In the abstract Witt ring theory an automorphism of a Witt ring $W = (R, G)$ is such an automorphism σ of the ring R that $\sigma(G) = G$. Introducing abstract Witt rings in [3], Marshall showed that every Witt ring of quadratic forms over a field F fulfills axioms of abstract Witt rings contained in his book [3], where as a distinguished group G one can take the group F^*/F^{*2} of square classes of the feld F. We say that an abstract Witt ring $W = (R, G)$ is *realized by a field* F if there exist a Witt ring of quadratic forms over the field F and such a ring isomorphism between W and $W(F)$, which maps the group G into the group F^*/F^{*2}. In our paper we consider Witt rings which are powers of Witt rings realized by finite fields with characteristic different from 2 and we investigate their groups of automorphisms.

In proofs of our results we use the notion of quaternionic structure (G, Q, q) associated to a Witt ring $W = (R, G)$. Here q is a surjection $q\colon G \times G \to \mathrm{Q}$ and four suitable axioms hold (for details see [3, Chapter 2]). By the theory presented in [3], categories of Witt rings and quaternionic structures are naturally equivalent. Therefore, the group of automorphisms of quaternionic structure is isomorphic to the group of automorphisms of associated Witt ring. By Marshall (cf. [3]), $\sigma \in \mathrm{Aut}(G)$ is an automorphism of quaternionic structure (G, Q, q) if q fulfills the following conditions:

1. $\sigma(-1) = -1$,
2. $q(a, b) = 0 \Leftrightarrow q(\sigma(a), \sigma(b)) = 0$.

Hence, we can investigate group automorphisms $\sigma \in \mathrm{Aut}(G)$ instead of automorphisms of a Witt ring $W = (R, G)$.

Let $W = (R, G)$ be a Witt ring realized by a finite field with characteristic different from 2 and let (G, Q, q) be the quaternionic structure associated to W. There are two cases.

1) $W \cong \mathbb{Z}/2\mathbb{Z}[C_2] \cong W(\mathbb{F}_5)$, where $\mathbb{Z}/2\mathbb{Z}[C_2]$ denotes the group ring of 2-element cyclic group C_2 with coefficients in the ring $\mathbb{Z}/2\mathbb{Z}$. Then the group G is isomorphic to $\mathbb{F}_5^*/\mathbb{F}_5^{*2} = \{1, x\}$ and the distinguished element is $-1 = 1$.
2) $W \cong \mathbb{Z}/4\mathbb{Z} \cong W(\mathbb{F}_3)$. Then the group G is isomorphic to $\mathbb{F}_3^*/\mathbb{F}_3^{*2} = \{1, -1\}$ $(-1 \neq 1)$.

In both cases the set $\mathrm{Q} = \{0\}$, hence $q(a, b) = 0$ for all $a, b \in G$.

2. Witt rings of finite fields and their powers

Theorem 1. *Let $W = (R, G)$ be a Witt ring realized by a finite field with characteristic different from 2. Then every automorphism of the group G^k, $k \in \mathbb{N}$, mapping the distinguished element -1 into itself induces an automorphism of a Witt ring W^k.*

Proof.
Since $q(a, b) = 0$ for all $a, b \in G$, it follows that quaternionic map q does not influence on the form and the number of automorphisms of quaternionic structure (G, Q, q). This means that every automorphism of the group G such that $\sigma(-1) = -1$ is an automorphism of (G, Q, q).

Finally, by the construction of powers of Witt rings and quaternionic structures (see [3, Chapter 5, §4]), we conclude that every automorphism of the group G^k, $k \in \mathbb{N}$, mapping the distinguished element $-1 \in G^k$ into itself is an automorphism of quaternionic structure (G^k, Q^k, q^k), consequently it induces an automorphism of a Witt ring W^k.

Corollary 1. *Let $W = (R, G)$ be a Witt ring realized by the Galois field $\mathbb{F}_5$. Then the group of automorphisms of the Witt ring $W' = (W^k, G^k)$ is isomorphic to $GL(k, \mathbb{F}_2)$.*

Proof. Let $W' = (W^k, G^k)$. Then $1 = -1$ in G and consequently in G^k, hence any automorphism of the group G^k is an automorphism of the quaternionic structure (G^k, Q^k, q^k). In fact, the group G^k is a vector space over the two-element Galois field $\mathbb{F}_2$ and $\dim_{\mathbb{F}_2} G^k = k$. Let $\mathcal{B} = \{b_1, \dots, b_k\}$ be a basis of the vector space of G^k over $\mathbb{F}_2$ and let $\sigma \in \mathrm{Aut}(G, \mathrm{Q}, q)$. Then a system of vectors $\{\sigma(b_1), \dots, \sigma(b_k)\}$ is another basis of the vector space G^k over $\mathbb{F}_2$. Every $\sigma(b_i)$, $1 \leq i \leq k$ can be represented as a combination of vectors from basis $\mathcal{B}$ as follows: $\sigma(b_i) = \alpha_{i1} b_1 + \ldots + \alpha_{ik} b_k$. Define a map $\Phi\colon \mathrm{Aut}(G^k, \mathrm{Q}^k, q^k) \to \mathrm{GL}(k, \mathbb{F}_2)$ by the following formula:

$$\Phi(\sigma) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2k} \\ \dots & \dots & \dots & \dots \\ \alpha_{k1} & \alpha_{k2} & \dots & \alpha_{kk} \end{bmatrix}.$$

The map Φ is a group isomorphism (this follows from uniqueness of representation of vectors $\sigma(b_i)$ in the basis $\mathcal{B}$).

Corollary 2. *Let $W = (R, G)$ be a Witt ring realized by the Galois field $\mathbb{F}_3$. Then the group of automorphisms of the Witt ring $W' = (W^k, G^k)$ is isomorphic to the affine group $\mathrm{Aff}(k-1, \mathbb{F}_2)$.*

Proof. Let $W' = (W^k, G^k)$. In this case we have $1 \neq -1$ in G. Consider the map Φ defined in the previous proof. We want to find all $\sigma \in \mathrm{Aut}(G^k)$ such that $\sigma(-1) = -1$, thus we fix the last vector in the basis $\mathcal{B}$ as $b_k = -1$. For all the automorphisms which preserve that vector, we get the following form of the map Φ:

$$\Phi(\sigma) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2k} \\ \dots & \dots & \dots & \dots \\ \alpha_{k-1,1} & \alpha_{k-1,2} & \dots & \alpha_{k-1,k} \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

So in this case we have block matrices of the form $\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}$. Let us examine the form of the product of this kind of matrices:

$$\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} B & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} AB & bA + a \\ 0 & 1 \end{bmatrix}.$$

On the other hand, affine transformations have the form: $f(x) = xA + a$ and $g(x) = xB + b$, hence the composition of them has the form: $f(g(x)) = f(xB + b) = (xB + b)A + a = AB + (bA + a)$.

Therefore, the map given by $f(\Phi) = \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}$ is a group isomorphism between $\mathrm{Aut}(G, \mathrm{Q}, q)$ and $\mathrm{Aff}(k-1, \mathbb{F}_2)$.

Summarizing the results of the paper, we can write out the groups of automorphisms of Witt rings of two types:

$\mathrm{Aut}((\mathbb{Z}/2\mathbb{Z}[C_2])^k) \cong \mathrm{Aut}((W(\mathbb{F}_5))^k) \cong GL(k, \mathbb{F}_2)$,

$\mathrm{Aut}((\mathbb{Z}/4\mathbb{Z})^k) \cong \mathrm{Aut}((W(\mathbb{F}_3))^k) \cong \mathrm{Aff}(k-1, \mathbb{F}_2) \cong \mathbb{F}_2^{k-1} \ltimes GL(k-1, \mathbb{F}_2)$.

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ON 4-DIMENSIONAL CUBE AND SUDOKU

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Abstract. The number puzzle SUDOKU (Number Place in the U.S.) has recently gained great popularity. We point out a relationship between SUDOKU and 4-dimensional Latin cubes. Namely, we assign the cells 4-tuples of numbers – coordinates in 4-dimensional space and then consider the game plan as a 4-dimensional cube. We also mention some variants of SUDOKU.

Recently the SUDOKU puzzle became popular in Europe. This game came into existence about 30 years ago in the U.S. The name of the game – a number puzzle – is derived from Japanese words SU (number) and DOKU (single), and the game became popular in Japan in the year 1986. SUDOKU was introduced in Europe in 2004, in the well know British journal The Times. As the signs for numbers are common to many languages, the game spread very quickly and became popular in many other countries. In the U.S. the SUDOKU game is usually called *Number Place*. For more information we advise to consult the Internet.

The *game plan* of the SUDOKU puzzle is a grid consisting of 9×9 squares, called *cells*. The grid is further divided into 3×3 *subgrids* or *regions*. Some cells contain numbers ranging from 1 to 9, known as *givens*. They are usually inscribed into cells which are symmetrical with respect to the center of the game plan (see Figure 1). The aim of the game is to inscribe a number from 1 to 9 into each of the empty cells, so that each row, column and region contains only one instance of each numeral. To simplify our formulas, we shall use numbers from the set $M = \{0, 1, 2, \ldots, 8\}$ instead of $\{1, 2, 3, \ldots, 9\}$.

From the point of view of mathematics, there arise many questions concerning SUDOKU. Several papers have considered the questions connected with solving the game. These are question of the type: For which configurations of the givens does the game possess a unique solution? How many solutions does the game have? What is the best time complexity of an algorithm solving SUDOKU? Here, however, we shall consider a different viewpoint. We shall label the individual cells by quadruples of numbers – coordinates, and represent the game plan by a 4-dimensional cube (also called a *hypercube*). The goal of the paper is to show some connections between the SUDOKU puzzle, Latin squares and their 4-dimensional analogies. The reader can get himself acquainted with 4-dimensional Latin cubes (Latin hypercubes) which have been getting an ever greater attention of mathematicians recently. Hence the text offers a small excursion into the 4-dimensional space and it can also inspire scientific research for both teachers and students. The involved reader can design different variations of the SUDOKU puzzle. These modifications can be adjusted with respect to the age and capabilities of the solver.

Figure 1 shows a SUDOKU puzzle (the numbers inscribed are from the set $M = \{0, 1, 2, \dots, 8\}$) and Figure 2 shows its solution in the ternary, i.e. base-3 numeral system. The ternary numbers range from 00 (decimal 0) to 22 (decimal 8) and we consider them as ordered pairs of numerals $[i, j]$, $0 \le i, j \le 2$. Each ordered pair occurs exactly once in each row, column and region. We recommend the reader to divide the solution from Figure 2 into two tables – one containing the first numerals and the other one containing the second numerals of the ternary numbers. In both tables, each row, column and region will contain exactly three occurrences of the numbers $0, 1, 2$. This property might give the reader a new perspective on the SUDOKU puzzle or even help him solving it.

4			6	7				0
	8	0					3	
6		7			3	1		2
			7		5	3		
8	3			0			2	7
		2	1		8			
3		8	2			6		5
	1					4	0	
7				6	4			1

FIGURE 1

11	02	10	20	21	01	22	12	00
01	22	00	12	11	02	21	10	20
20	12	21	00	22	10	01	11	02
00	20	11	21	02	12	10	01	22
22	10	01	11	00	20	12	02	21
12	21	02	01	10	22	00	20	11
10	11	22	02	01	00	20	21	12
02	01	20	22	12	21	11	00	10
21	00	12	10	20	11	02	22	01

FIGURE 2

Because we assume that most of the readers have never before encountered the notion of a hypercube (i.e. a 4-dimensional cube), which we shall use later in this text, we shall first simplify the SUDOKU puzzle into its "3-dimensional version" and demonstrate its relation to the usual 3-dimensional cube. Figure 3 shows a cube consisting of $3 \times 3 \times 3$ cells. Letters A, B, C denote three of its nine layers. Each layer consists of 3×3 cells. To obtain the second and the third triple of layers, we have to "cut" the cube using two planes parallel to the base and the side face of the cube, respectively. The numbers from the set $M = \{0, 1, 2, \ldots, 8\}$ are inscribed into the cells in such a way that the numbers in each of the nine layers are pairwise different.

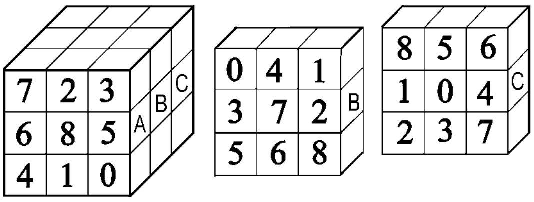

FIGURE 3

Let us assign a triple of coordinates to each cell in the natural way, as shown in Figure 4. (The commas between the coordinates are omitted in the picture.) We can now formalize the notion of a layer: a *layer* is a nine-tuple of cells which have the same coordinate in one of the three positions. Each layer is determined by one of its elements (cells) and two of the three directions given by the edges of the cube.

$a(111)$	$a(112)$	$a(113)$	$a(211)$	$a(212)$	$a(213)$	$a(311)$	$a(312)$	$a(313)$
$a(121)$	$a(122)$	$a(123)$	$a(221)$	$a(222)$	$a(223)$	$a(321)$	$a(322)$	$a(323)$
$a(131)$	$a(132)$	$a(133)$	$a(231)$	$a(232)$	$a(233)$	$a(331)$	$a(332)$	$a(333)$

FIGURE 4

Exercise: Figure 5 shows a cube with numbers inscribed into one third of its cells. Inscribe a number from the set $M = \{0, 1, 2, \ldots, 8\}$ into each cell so that each number occurs in each layer exactly once.

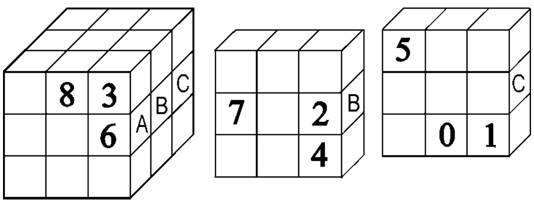

FIGURE 5

After this "3-dimensional trip", let us return to SUDOKU.

When the SUDOKU puzzle is solved, i.e. when the 9×9 grid correctly completed, then each row and column of the grid contains a permutation of the set M. In mathematics, a grid with this property is called a *Latin square*. Formally: a *Latin square* of order n is an $n \times n$ matrix (table) $\mathbf{R}_n = |r(k,l);\ 1 \leq k,l \leq n|$ comprising n^2 numbers $r(k,l) \in \{0,1,2,\dots,n-1\}$ with the property that each row and column is a permutation of the set $\{0,1,\dots,n-1\}$. The SUDOKU puzzle contains an additional constraint that each region is a permutation of the set M.

We assign to each cell of the game plan a quadruple of numbers – its coordinates – as in Figure 6. The first pair of the quadruple determines the region, and the second pair the position in the particular region.

A *hypercube* of order n is a 4-dimensional matrix

$$\mathbf{A}_n = |a(i,j,k,l);\ 1 \leq i,j,k,l \leq n|$$

comprising n^4 elements $a(i,j,k,l)$.

$a(1111)$	$a(1112)$	$a(1113)$	$a(1211)$	$a(1212)$	$a(1213)$	$a(1311)$	$a(1312)$	$a(1313)$
$a(1121)$	$a(1122)$	$a(1123)$	$a(1221)$	$a(1222)$	$a(1223)$	$a(1321)$	$a(1322)$	$a(1323)$
$a(1131)$	$a(1132)$	$a(1133)$	$a(1231)$	$a(1232)$	$a(1233)$	$a(1331)$	$a(1332)$	$a(1333)$
$a(2111)$	$a(2112)$	$a(2113)$	$a(2211)$	$a(2212)$	$a(2213)$	$a(2311)$	$a(2312)$	$a(2313)$
$a(2121)$	$a(2122)$	$a(2123)$	$a(2221)$	$a(2222)$	$a(2223)$	$a(2321)$	$a(2322)$	$a(2323)$
$a(2131)$	$a(2132)$	$a(2133)$	$a(2231)$	$a(2232)$	$a(2233)$	$a(2331)$	$a(2332)$	$a(2333)$
$a(3111)$	$a(3112)$	$a(3113)$	$a(3211)$	$a(3212)$	$a(3213)$	$a(3311)$	$a(3312)$	$a(3313)$
$a(3121)$	$a(3122)$	$a(3123)$	$a(3221)$	$a(3222)$	$a(3223)$	$a(3321)$	$a(3322)$	$a(3323)$
$a(3131)$	$a(3132)$	$a(3133)$	$a(3231)$	$a(3232)$	$a(3233)$	$a(3331)$	$a(3332)$	$a(3333)$

FIGURE 6

A *row* of a hypercube $\mathbf{A}_n$ of order n is an n-tuple of elements whose coordinates differ on exactly one position. A *layer* of a hypercube is an n^2-tuple of elements, whose coordinates differ on exactly two positions. A layer is determined by one of its elements and two directions. There are four *directions* in a hypercube; they are given by its edges. Each pair of directions defines a *lay*. (Note: We use the notions of *direction* and *lay* in a similar sense as in analytic geometry. There, direction (lay) denotes a one-dimensional (two-dimensional) vector space and together with a point it defines a straight line (plane).) Each direction is contained in three lays. Two different layers of a hypercube have the same lay if and only if they are disjoint. An order n hypercube contains $6n^2$ layers belonging to six lays.)

Let us return to the table in Figure 6. We know now that it shows the coordinates of the cells of an order 3 hypercube. The 3×3 blocks bounded by thick lines represent the 9 layers of the hypercube which have the same lay.

For example, the element $a(1,1,1,1)$ is contained in four rows which comprise the following elements:

$$\begin{array}{ll} \text{1-direction:} & (a(1,1,1,1), a(1,1,1,2), a(1,1,1,3)), \\ \text{2-direction:} & (a(1,1,1,1), a(1,1,2,1), a(1,1,3,1)), \\ \text{3-direction:} & (a(1,1,1,1), a(1,2,1,1), a(1,3,1,1)), \\ \text{4-direction:} & (a(1,1,1,1), a(2,1,1,1), a(3,1,1,1)). \end{array}$$

These rows are contained in four different directions which we have denoted by numbers 1,2,3,4. Six layers of the hypercube whose lays are pairwise different and which contain the element $a(1,1,1,1)$ are given by quadruples of corner cells (the symbol *(a-b)-lay* denotes the lay which is given by the directions a and b):

$$\begin{array}{lllll} \text{(1-2)-lay:} & a(1,1,1,1), & a(1,1,1,3), & a(1,1,3,1), & a(1,1,3,3), \\ \text{(1-3)-lay:} & a(1,1,1,1), & a(1,1,1,3), & a(1,3,1,1), & a(1,3,1,3), \\ \text{(1-4)-lay:} & a(1,1,1,1), & a(1,1,1,3), & a(3,1,1,1), & a(3,1,1,3), \\ \text{(2-3)-lay:} & a(1,1,1,1), & a(1,1,3,1), & a(1,3,1,1), & a(1,3,3,1), \\ \text{(2-4)-lay:} & a(1,1,1,1), & a(1,1,3,1), & a(3,1,1,1), & a(3,1,3,1), \\ \text{(3-4)-lay:} & a(1,1,1,1), & a(1,3,1,1), & a(3,1,1,1), & a(3,3,1,1). \end{array}$$

Using this terminology, we can formulate the rules of the SUDOKU puzzle in the following way: Numbers from the set M are inscribed into some cells of an order 3 hypercube. Inscribe a number from M into each empty cell so that each layer with lay (1-2) and (1-3) and (2-4) contains all numbers from the set M.

We would obtain different versions of SUDOKU if we required having all numbers from M in layers with different lays.

Next we show how to fill in such a table in a certain special case. First though, we have to define two notions: orthogonal Latin squares and a Latin hypercube.

Two Latin squares $\mathbf{R}_n = |r(k,l)|$ and $\mathbf{S}_n = |s(k,l)|$ of order n are said to be *orthogonal* if all the ordered pairs $[r(k,l), s(k,l)]$ are pairwise different. It has been known since Leonhard Euler's time that pairs of orthogonal Latin squares can be constructed for all odd n using the following formulas:

$$r(k,l) = (k + l + a) \mod n, \qquad s(k,l) = (k - l + b) \mod n \tag{1}$$

for all $1 \le k, l \le n$, where a, b are arbitrary integers. The elements of the table in Figure 7 are pairs $[r(k,l), s(k,l)]$ of elements of two orthogonal Latin squares $\mathbf{R}_9$ a $\mathbf{S}_9$ of order 9 which have been obtained using the above equations with $a = b = 0$.

2, 0	3, 8	4, 7	5, 6	6, 5	7, 4	8, 3	0, 2	1, 1
3, 1	4, 0	5, 8	6, 7	7, 6	8, 5	0, 4	1, 3	2, 2
4, 2	5, 1	6, 0	7, 8	8, 7	0, 6	1, 5	2, 4	3, 3
5, 3	6, 2	7, 1	8, 0	0, 8	1, 7	2, 6	3, 5	4, 4
6, 4	7, 3	8, 2	0, 1	1, 0	2, 8	3, 7	4, 6	5, 5
7, 5	8, 4	0, 3	1, 2	2, 1	3, 0	4, 8	5, 7	6, 6
8, 6	0, 5	1, 4	2, 3	3, 2	4, 1	5, 0	6, 8	7, 7
0, 7	1, 6	2, 5	3, 4	4, 3	5, 2	9, 1	7, 0	8, 8
1, 8	2, 7	3, 6	4, 5	5, 4	6, 3	7, 2	8, 1	0, 0

FIGURE 7

The French De la Hire knew already 300 years ago that a *magic square* can be constructed using a pair of orthogonal Latin squares. (A *magic square* of order n is an $n \times n$ matrix $\mathbf{M}_n = |m(k,l);\ 1 \leq k,l \leq n|$ consisting of n^2 consecutive positive integers $m(k,l)$ such that the sums of elements in each row, column and on both diagonals are the same.)

Setting $a = 3$ and $b = 4$ and using the formula

$$m(k,l) = 9 \cdot r(k,l) + s(k,l) + 1,$$

we obtain a magic square $\mathbf{M}_9 = |m(k,l); 1 \leq k,l \leq 9|$ of order 9 whose elements are from the set $\{0,1,2,\dots,9^2-1\}$. The choice of the parameters a,b ensures that not only the sums in the rows and columns, but also on the diagonals are the same.

When we generalize the notion of a Latin square to four dimensions, we obtain a Latin hypercube. A *Latin hypercube* of order n is a hypercube

$$\mathbf{T}_n = |t(i,j,k,l);\ 1 \leq i,j,k,l \leq n|,$$

whose elements are from the set $\{0,1,2,\dots,n-1\}$ and each row and diagonal contains a permutation of this set.

Consider two Latin hypercubes $\mathbf{T}_n = |t(i,j,k,l)|$ and $\mathbf{U}_n = |u(i,j,k,l)|$ of order n given by the following formulas:

$$\begin{aligned} t(i,j,k,l) &= r(i,(r(j,r(k,l))) = (i+j+k+l) \mod n, \\ u(i,j,k,l) &= s(i,(s(j,s(k,l))) = (i-j+k-l) \mod n; \end{aligned}$$

with $\mathbf{R}_n$ and $\mathbf{S}_n$ being a pair of orthogonal Latin squares given by formulas (1). The 9×9 grid from Figure 8, which is a SUDOKU solution, has been constructed using the formula

$$v(i,j,k,l) = t(i,j,k,l) \cdot n + u(i,j,k,l). \tag{2}$$

The careful reader may notice that not only all the layers with the three lays (1-2), (1-3) and (2-4) contain all the elements of M but that the same is true for all the layers with a fourth lay. We have obtained this property thank to a suitable choice of the pair of Latin hypercubes. For a different choice of the pair of Latin squares (obtained for example by a different choice of a and b) we obtain different Latin hypercubes of order n and hence a different hypercube – a different game plan completion. Moreover, the construction based on (2) is such that the sums of the numbers in all the rows of the hypercube are the same for every choice of the parameters a, b. Applying suitable exchanges of rows and columns, we can obtain different grids (SUDOKU solutions). (You can find more details on Latin squares and hypercubes in Internet or [3].)

3	1	8	7	5	0	2	6	4
7	5	0	2	6	4	3	1	8
2	6	4	3	1	8	7	5	0
1	8	3	5	0	7	6	4	2
5	0	7	6	4	2	1	8	3
6	4	2	1	8	3	5	0	7
8	3	1	0	7	5	4	2	6
0	7	5	4	2	6	8	3	1
4	2	6	8	3	1	0	7	5

FIGURE 8

The above text poses more questions than it gives answers. Since the formulas are true for every odd n, the reader may think of game plans of different sizes.

The reader can also design numerous variations of the SUDOKU puzzle, inspired for example by the following notes:

1. Assume that a hypercube of order 3 contains in some of its cells numbers from the set $\{0, 1, 2\}$. The aim of the game is to fill in the missing numbers so that each row contains each of the numbers $0, 1, 2$ exactly once. Hence, the solution of the puzzle is a Latin hypercube of order 3. (A construction of such hypercubes is given in [3].)

2. Instead of the 9×9 grid we can consider a $m^2 \times m^2$ grid consisting of m^2 regions containing $m \times m$ cells. Some cells contain numbers from the set $N = \{1, 2, 3, \dots, m^2\}$. The aim of the game is to fill in the missing numbers so that each row, column and region contains a permutation of the set N. (The formulas (1) and (2) are valid for all odd n.) In our experience the version with $m = 2$ is suitable for young kids from 6 to 10 years of age.

5	4	3	2	6	1
2	1	6	5	3	4
4	3	5	1	2	6
1	6	2	4	5	3
6	2	1	3	4	5
3	5	4	6	1	2

FIGURE 9

3. We can obtain a different SUDOKU puzzle if we drop the requirement that the regions are squares. For example, the 6×6 grid in Figure 9 is divided into rectangular regions with 3×2 cells.

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ON APPROXIMATION OF SUBQUADRATIC FUNCTIONS BY QUADRATIC FUNCTIONS

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Abstract. In this paper we establish an approximation of subquadratic functions, which satisfy the condition

$$\exists\ \epsilon > 0 \quad \forall\ x \in X \quad |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon,$$

by quadratic functions.

1. Introduction

Let X be a group and let $\mathbb{R}$ denotes the set of all reals. A function $\varphi\colon X \to \mathbb{R}$ is said to be subquadratic iff it satisfies the inequality

$$\varphi(x+y) + \varphi(x-y) \leq 2\varphi(x) + 2\varphi(y) \tag{1}$$

for all $x, y \in X$. If the sign " $\leq$ " in the inequality above is replaced by " $=$ ", then we say that φ is a quadratic one.

Section 2 of this paper contains some basic properties of subquadratic functions which play a crucial role in our proofs of the main theorems of this paper.

At the beginning of the third part of this paper, we will consider the problem of approximation of a subquadratic function $\varphi\colon X \to \mathbb{R}$, which satisfies the following condition

$$\exists\ \epsilon > 0 \quad \forall\ x \in X \quad |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon, \tag{2}$$

by a quadratic function $\omega\colon X \to \mathbb{R}$.

At the end of this section, we will present some conditions for subquadratic functions defined on a topological group having additional properties or on $\mathbb{R}^N$, under which we will establish an approximation of functions of this type by continuous quadratic functions.

2. Some basic properties

At the beginning of this section we remind one of basic properties of subquadratic functions which is proved in [1], [2].

Lemma 1. [1], [2] *Let $X = (X, +)$ be a group and let $\varphi : X \to \mathbb{R}$ be a subquadratic function. Then*

$$\varphi(0) \geq 0$$

and

$$\varphi(kx) \leq k^2 \varphi(x), \qquad x \in X,$$

for each positive integer k.

In [2] it was proved that if for some positive integer $k > 1$ a subquadratic function $\varphi : X \to \mathbb{R}$ satisfies equality

$$\varphi(kx) = k^2 \varphi(x), \quad x \in X, \tag{3}$$

in the case when the domain of the function considered is a linear space, then it has to be a quadratic one. Essentially, the same argumentation yields the validity of the next lemma if the domain is a group and the condition (3) is replaced by another.

Lemma 2. *Let $X = (X, +)$ be a group and let $\varphi : X \to \mathbb{R}$ be a subquadratic function. If for every $x \in X$ there exists some positive integer $k > 1$ such that*

$$\varphi(k_x x) \geq {k_x}^2 \varphi(x),$$

then $\varphi(2x) \geq 4\varphi(x)$ for every $x \in X$.

Lemma 3. *Let $X = (X, +)$ be an Abelian group. If a subquadratic function $\varphi \colon X \to \mathbb{R}$ satisfies the condition*

$$\varphi(2x) \geq 4\varphi(x)$$

for all $x \in X$, then it is a quadratic function.

Proof. Let $x, y \in X$. Then

$$\varphi(x+y) + \varphi(x-y) \leq 2\varphi(x) + 2\varphi(y) = \frac{1}{2}\left[4\varphi(x) + 4\varphi(y)\right] \leq$$

$$\leq \frac{1}{2}\left[\varphi(2x) + \varphi(2y)\right] = \frac{1}{2}\left[\varphi\left((x+y)+(x-y)\right) + \varphi\left((x+y)-(x-y)\right)\right] \leq$$

$$\leq \varphi(x+y) + \varphi(x-y).$$

Thus φ is a quadratic function. $\square$

Now, using Lemmas 2 and 3, the Theorem 1 from [2] has the following form:

Theorem 1. *Let $X = (X, +)$ be an Abelian group and let $\varphi : X \to \mathbb{R}$ be a subquadratic function. If for every $x \in X$ there exists some positive integer $k > 1$ such that*

$$\varphi(k_x x) \geq {k_x}^2 \varphi(x),$$

then φ is a quadratic function.

By a topological group we mean a group endowed with a topology such that the group operation as well as taking inverses are continuous functions.

We adopt the following definition.

Definition 1. *We say that 2-divisible topological group X has the property $(\frac{1}{2})$ if and only if for every neighbourhood V of zero there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset V$.*

At the end of this section we present Theorem 2 which was proved in [3].

Theorem 2. [3] *Let X be a uniquely 2-divisible topological Abelian group having the property $(\frac{1}{2})$, which is generated by any neighbourhood of zero in X. Assume that a subquadratic function $\varphi : X \to \mathbb{R}$ satisfies the following conditions:*

(i) $\varphi(0) \leq 0$;

(ii) *φ is locally bounded from below at a point of X;*

(iii) *φ is upper semicontinuous at zero.*

Then φ is continuous everywhere in X.

3. The main result

The next lemma plays a crucial role in our proofs. This lemma is valid for an arbitrary function defined on a semigroup with values in a normed space.

Lemma 4. *Let X be an arbitrary semigroup, Y a normed space and let $f\colon X \to Y$ be an arbitrary function. If there exists $\epsilon > 0$ such that the inequality*

$$\|f(2x) - 4f(x)\| \leq 3\epsilon \tag{4}$$

holds for every $x \in X$, then the inequality

$$\|4^{-n}f(2^n x) - f(x)\| \leq (1 - 4^{-n})\epsilon \tag{5}$$

holds for every $x \in X$ and $n \in \mathbb{N}$.

Proof. Let $x \in X$. By induction, we show that for every $n \in \mathbb{N}$ the inequality (5) holds. For $n = 1$, by (4), we have

$$4\left\|\frac{1}{4}f(2x) - f(x)\right\| \leq 3\epsilon.$$

Thus

$$\left\|\frac{1}{4}f(2x) - f(x)\right\| \leq \frac{3}{4}\epsilon = \left(1 - \frac{1}{4}\right)\epsilon.$$

Now assume that (5) holds for $n \in \mathbb{N}$. Then we have

$$\left\|4^{-n-1}f(2^{n+1}x) - f(x)\right\| \leq$$

$$\leq \left\|\frac{1}{4^{n+1}}f(2^{n+1}x) - \frac{1}{4^n}f(2^n x)\right\| + \left\|\frac{1}{4^n}f(2^n x) - f(x)\right\| =$$

$$= \frac{1}{4^n}\left\|\frac{1}{4}f(2(2^n x)) - f(2^n x)\right\| + \left\|\frac{1}{4^n}f(2^n x) - f(x)\right\| \leq$$

$$\leq \frac{1}{4^n}\frac{3}{4}\epsilon + \left(1 - \frac{1}{4^n}\right)\epsilon = \left(1 - \frac{1}{4^{n+1}}\right)\epsilon$$

for $n + 1$, which completes the induction. $\square$

Applying Lemma 4, we will prove the following theorem.

Theorem 3. *Let X be an arbitrary Abelian group and let $\varphi\colon X \to \mathbb{R}$ be a subquadratic function. If there exists $\epsilon > 0$ such that the inequality*

$$|\varphi(2x) - 4\varphi(x)| \le 3\epsilon \tag{6}$$

holds for every $x \in X$, then there exists a quadratic function $\omega\colon X \to \mathbb{R}$ such that

$$0 \le \varphi(x) - \omega(x) \le \epsilon$$

for every $x \in X$.

Proof. It follows by Lemma 1 that for arbitrary $x \in X$ and $n \in \mathbb{N}$ we have

$$\varphi(2^n x) \le 4^n \varphi(x). \tag{7}$$

Whence

$$\frac{\varphi(2^n x)}{4^n} \le \varphi(x), \qquad x \in X, \quad n \in \mathbb{N}. \tag{8}$$

Let us fix $x \in X$. We will consider the sequence $\left\{\frac{\varphi(2^n x)}{4^n}\right\}_{n\in\mathbb{N}}$. For arbitrary $m, n \in \mathbb{N}$, by Lemma 4, we have

$$\left|\frac{1}{4^{n+m}}\varphi(2^{n+m}x) - \frac{1}{4^n}\varphi(2^n x)\right| = \frac{1}{4^n}\left|\frac{1}{4^m}\varphi(2^{n+m}x) - \varphi(2^n x)\right| =$$

$$= \frac{1}{4^n}\left|\frac{1}{4^m}\varphi(2^m 2^n x) - \varphi(2^n x)\right| \le \frac{1}{4^n}\left(1 - \frac{1}{4^m}\right)\epsilon < \frac{1}{4^n}\epsilon,$$

which means that for every $x \in X$ the sequence $\left\{\frac{\varphi(2^n x)}{4^n}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence and consequently converges. Let

$$\omega(x) := \lim_{n\to\infty} \frac{\varphi(2^n x)}{4^n}, \qquad x \in X.$$

On letting $n \to \infty$ in (8), we obtain

$$\omega(x) \le \varphi(x), \quad x \in X. \tag{9}$$

Since $\omega(2x) = 4\omega(x)$ for every $x \in X$, by Theorem 1, ω is a quadratic function.

Again, by Lemma 4, we have

$$\left|\frac{\varphi(2^n x)}{4^n} - \varphi(x)\right| \le \left(1 - \frac{1}{4^n}\right)\epsilon, \quad x \in X. \tag{10}$$

Whence, on letting $n \to \infty$, we obtain

$$|\omega(x) - \varphi(x)| \leq \epsilon, \quad x \in X. \tag{11}$$

By (9) and (11), we have

$$0 \leq \varphi(x) - \omega(x) \leq \epsilon$$

for every $x \in X$. This ends the proof. □

As a consequence of Theorem 3, we have the following corollary.

Corollary 1. *Let X be a uniquely 2-divisible Abelian topological group having the property $\left(\frac{1}{2}\right)$, which is generated by any neighbourhood of zero in X. Assume that a subquadratic function $\varphi \colon X \to \mathbb{R}$ satisfies the following conditions:*

(i) $\exists \;\; \epsilon > 0 \;\; \forall \;\; x \in X \;\; |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon$;

(ii) φ is upper semicontinuous at zero in X;

(iii) φ is locally bounded from below at a point of X.

Then the function ω, which appears in thesis of Theorem 3, is continuous.

Proof. Due to the upper semicontinuoity of φ at zero, the function ω is also upper semicontinuous at zero [5, p. 131].

If φ is locally bounded from below at some point $x_0 \in X$, then by the inequality

$$\varphi(x) - \epsilon \leq \omega(x) \leq \varphi(x) + \epsilon, \qquad x \in X,$$

the function ω is also locally bounded from below at the point $x_0 \in X$. Since the quadratic function ω satisfies the condition $\omega(0) = 0$, then according to Theorem 2, it is continuous. This completes the proof. □

Now, let $X = \mathbb{R}^N$. In this case we have the following theorem.

Theorem 4. *Let $A \subset \mathbb{R}^N$ be a set of positive inner Lebesgue measure or of the second category with the Baire property. If a subquadratic function $\varphi \colon \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:*

(i) $\exists \;\; \epsilon > 0 \;\; \forall \;\; x \in X \;\; |\varphi(2x) - 4\varphi(x)| \leq 3\epsilon$;

(ii) φ is bounded on A,

then there exists a continuous quadratic function $\omega \colon \mathbb{R}^N \to \mathbb{R}$ such that

$$0 \leq \varphi(x) - \omega(x) \leq \epsilon$$

for every $x \in \mathbb{R}^N$.

Proof. Due to Theorem 3, there exists a quadratic function $\omega\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$0 \le \varphi(x) - \omega(x) \le \epsilon, \quad x \in \mathbb{R}^N. \tag{12}$$

Therefore, we have

$$\varphi(x) - \epsilon \le \omega(x) \le \varphi(x), \quad x \in \mathbb{R}^N. \tag{13}$$

Since φ is bounded on the set A, there exist real constants m, M such that

$$m \le \varphi(x) \le M, \quad x \in A. \tag{14}$$

Inequalities (13) and (14) imply

$$m - \epsilon \le \omega(x) \le M, \quad x \in A. \tag{15}$$

Finally, ω is a bounded function on the set A. A quadratic function is a polynomial function of degree 2. Due to Theorem 3 from [4, p. 386], ω is continuous in $\mathbb{R}^N$. □

We end our paper with the following corollary.

Corollary 2. *Let $A \subset \mathbb{R}$ be a set of positive inner Lebesgue measure or of the second category with the Baire property. If a subquadratic function $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(i) $\exists \;\; \epsilon > 0 \;\; \forall \;\; x \in X \quad |\varphi(2x) - 4\varphi(x)| \le 3\epsilon$;

(ii) φ is bounded on A,

then

$$\varphi(x) \ge cx^2$$

for every $x \in \mathbb{R}$, where c is a real constant .

Proof. Due to Theorem 4, there exists a continuous quadratic function $\omega\colon \mathbb{R} \to \mathbb{R}$ such that

$$\omega(x) \le \varphi(x), \quad x \in \mathbb{R}. \tag{16}$$

Since $\omega\colon \mathbb{R} \to \mathbb{R}$ is a continuous quadratic function, then it takes the form

$$\omega(x) = cx^2, \quad x \in \mathbb{R}, \tag{17}$$

where $c \in \mathbb{R}$ is a constant. By (16) and (17), we obtain the thesis of the Corollary 2. □

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SOLUTIONS OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

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Abstract. Let $(G,+)$ be a uniquely 2-divisible Abelian group. In the present paper we will consider the solutions of functional equation
$[f(x+y)]^2-[f(x-y)]^2+f(2x+2y)+f(2x-2y)=f(2x)[f(2y)+2g(2y)],\ x,y\in G$, where f and g are complex-valued functions defined on G.

1. Introduction

We know many trigonometric identities. To us, important will be the following:

$$\left[\sin\left(\frac{x+y}{2}\right)\right]^2-\left[\sin\left(\frac{x-y}{2}\right)\right]^2=\sin(x)\sin(y),\quad x,y\in\mathbb{R}, \tag{1}$$

$$\sin(x+y)+\sin(x-y)=2\sin(x)\cos(y),\quad x,y\in\mathbb{R}, \tag{2}$$

$$\sinh(x-y)=\sinh(x)\cosh(y)-\cosh(x)\sinh(y),\quad x,y\in\mathbb{R}. \tag{3}$$

Let $(G,+)$ be a uniquely 2-divisible Abelian group and $f,g\colon G\to\mathbb{C}$. Equation (1) translates into the well known *sine functional equation* [1, 8]

$$\left[f\left(\frac{x+y}{2}\right)\right]^2-\left[f\left(\frac{x-y}{2}\right)\right]^2=f(x)f(y)\quad\text{for all}\quad x,y\in G, \tag{4}$$

and (2) gives rise to the familiar *cosine functional equation* [1, 6, 7]

$$f(x+y)+f(x-y)=2f(x)g(y) \quad \text{for all} \quad x,y\in G, \tag{5}$$

and (3) leads to the *Aczel-Dhombres functional equation* [1]

$$f(x-y)=f(x)g(y)-g(x)f(y) \quad \text{for all} \quad x,y\in G. \tag{6}$$

From now on, f_o and f_e stand for the odd and the even part of a function f.

Theorem 1 (Aczél and Dhombres [1]). *Let* $(G,+)$ *be a uniquely 2-divisible Abelian group. Then* $f,g:G\to\mathbb{C}$ *satisfy equation* (6) *if and only if*

(i) $f=0$ *and* g *is arbitrary; or*

(ii) *there exists an additive function* $A:G\to\mathbb{C}$ *and a constant* $\alpha\in\mathbb{C}$ *such that* $f(x)=A(x),\ g(x)=\alpha A(x)+1, x\in G$*; or*

(iii) *there exists an exponential function* $m:G\to\mathbb{C}$ *and constants* $\beta,\gamma\in\mathbb{C}$ *such that* $f(x)=\beta m_o(x),\ g(x)=\gamma m_o(x)+m_e(x),\ x\in G.$

From the system of equations

$$\begin{cases} f(x+y)=f(x)+f(y), \\ f(xy)=f(x)f(y), \end{cases}$$

we get *the Dhombres functional equation* (see [2])

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y)$$

for functions f mapping a given ring into another one. A different system of the functional equations has been studied by Ger [3, 4, 5]. Here we consider the sum of equations (4) and (5).

2. Main results

We replace x by $2x$ and y by $2y$ in (4) and (5). Summing up these functional equations side by side, for all $x,y\in G$, we get

$$[f(x+y)]^2-[f(x-y)]^2+f(2x+2y)+f(2x-2y)=f(2x)[f(2y)+2g(2y)]. \tag{7}$$

Remark 1. *Put* $x=y=0$ *in* (7)*, so that we have* $f(0)=0\vee f(0)=2-2g(0)$.

Lemma 1. *Let* $(G,+)$ *be a uniquely 2-divisible Abelian group and let functions* $f,g:G\to\mathbb{C}$ *satisfy equation* (7)*. In this case*

(i) *if* $f=0$*, then* g *is arbitrary;*

(ii) *if* $g=0$*, then* $f=0$ *or* $f=2$.

Proof. Ad (ii). For $g = 0$, putting $y = x$ in (7), we get

$$f(x) = [f(0)]^2 - f(0) = f(0)[f(0) - 1] = \gamma, \quad x \in G. \tag{8}$$

From equation (7) we obtain

$$\gamma^2 - \gamma^2 + 2\gamma = \gamma^2,$$

whence $\gamma = 0 \vee \gamma = 2$. By (8), we conclude that $f = 0 \vee f = 2$. □

Lemma 2. *Let $(G, +)$ be a uniquely 2-divisible Abelian group and let nonzero functions $f, g : G \to \mathbb{C}$ be functions defined by*

$$f(x) = aA(x) + f(0), \; g(x) = bA(x) + g(0), \quad x \in G, \tag{9}$$

with some additive function $A : G \to \mathbb{C}$ and $a, b \in \mathbb{C}$ satisfy equation (7). *In this case we have the following possibilities:*

(i) *If $f(0) = 0$, then $f(x) = aA(x),\; g(x) = 1,\; x \in G$.*

(ii) *If $f(0) \neq 0$, then $f(x) = f(0),\; g(x) = 1 - \frac{1}{2}f(0),\; x \in G$.*

Proof. Applying (9) to (7), we have

$$\begin{aligned}&[aA(x+y)+f(0)]^2 - [aA(x-y)+f(0)]^2 + aA(2x+2y) + 2f(0) + aA(2x-2y)\\ &\quad = [aA(2x) + f(0)][(a+2b)A(2y) + f(0) + 2g(0)],\; x, y \in G.\end{aligned}$$

From the properties of additive function A for all $x, y \in G$, we infer that

$$[2 - f(0) - 2g(0)][2aA(x) + f(0)] + 2f(0)[a - 2b]A(y) = 8abA(x)A(y). \tag{10}$$

Case 1. Assume that $f(0) = 0$. Then equation (10) has a form

$$aA(x)[1 - g(0) - 2bA(y)] = 0, \quad x, y \in G. \tag{11}$$

If $a = 0 \vee A = 0 \vee (a \neq 0 \wedge A \neq 0 \wedge b \neq 0)$, then we get $f = 0$, a contradiction. Hence we have only one possibility $(a \neq 0 \wedge A \neq 0 \wedge b = 0)$. Consequently, equation (11) gives $g(0) = 1$. From (9) we obtain (i).

Case 2. Let $f(0) \neq 0$. By Remark 1 and equation (10), we get the relation

$$A(y)[f(0)(a - 2b) - 4abA(x)] = 0, \quad x, y \in G.$$

If $A = 0$, then (ii). Assume that $A \neq 0$. From above we have

$$f(0)(a - 2b) = 4abA(x), \quad x \in G.$$

Therefore $(a = 0 \Rightarrow b = 0) \vee (b = 0 \Rightarrow a = 0)$, the case (ii). If $a \neq 0 \wedge b \neq 0$, then $A = 0$, a contradiction. □

Now, we formulate some properties of the exponential function without a proof.

Lemma 3. *Let $(G,+)$ be a uniquely 2-divisible Abelian group. Then a nonzero exponential function $m : G \to \mathbb{C}$ has the following properties*

(i) $m_e(x+y) + m_e(x-y) = 2m_e(x)m_e(y), \quad x, y \in G;$

(ii) $[m_o(x+y)]^2 - [m_o(x-y)]^2 = m_o(2x)m_o(2y), \quad x, y \in G;$

(iii) $m_o(x+y) + m_o(x-y) = 2m_o(x)m_e(y), \quad x, y \in G;$

(iv) $m_o(2x) = 2m_o(x)m_e(x), \quad x \in G;$

(v) $[m_e(x+y)]^2 - [m_e(x-y)]^2 = m_o(2x)m_o(2y), \quad x, y \in G;$

(vi) $m_o(x+y) - m_o(x-y) = 2m_e(x)m_o(y), \quad x, y \in G.$

Lemma 4. *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let nonzero functions $f, g : G \to \mathbb{C}$ be functions defined by*

$$f(x) = am_o(x) + bm_e(x),\ g(x) = cm_o(x) + dm_e(x),\ x \in G, \tag{12}$$

with some exponential function $m : G \to \mathbb{C}$ and $a, b, c, d \in \mathbb{C}$ satisfying equation (7). Then we have the following possibilities:

(i) $f(x) = am_o(x),\ g(x) = m_e(x),\ x \in G;$ *or*

(ii) $f(x) = b \neq 0,\ g(x) = 1 - \frac{1}{2}b,\ x \in G;$ *or*

(iii) $f(x) = bm_o(x) + bm_e(x), g(x) = \frac{1}{2}bm_o(x) + (1 - \frac{1}{2}b)m_e(x), x \in G;$ *or*

(iv) $f(x) = -bm_o(x) + bm_e(x), g(x) = -\frac{1}{2}bm_o(x) + (1 - \frac{1}{2}b)m_e(x), x \in G.$

Proof. Inserting functions (12) into equation (7), for all $x, y \in G$, we obtain

$$[am_o(x+y) + bm_e(x+y)]^2 - [am_o(x-y) + bm_e(x-y)]^2 + a[m_o(2x+2y)$$
$$+m_o(2x-2y)] + b[m_e(2x+2y) + m_e(2x-2y)]$$
$$= [am_o(2x) + bm_e(2x)][(a+2c)m_o(2y) + (b+2d)m_e(2y)].$$

From above and Lemma 3, we get

$$[b^2 - 2ac]m_o(x)m_o(y) + b[a-2c]m_e(x)m_o(y)$$

$$+a[2-b-2d]m_o(x)m_e(y) + b[2-b-2d]m_e(x)m_e(y) = 0,\ x, y \in G. \tag{13}$$

Directly from the definition (12), we see that $f(0) = b$ and $g(0) = d$. Moreover, from Remark 1 we infer that $b = 0$ or $b = 2(1-d)$.

Now we shall distinguish two cases regarding the value of function f at zero.

Case 1. Let $f(0) = b = 0$. Then, by (13), we conclude that

$$am_o(x)[-cm_o(y) + (1 - d)m_e(y)] = 0, \ x, y \in G.$$

If $a = 0$ or $m_o = 0$, then also $f = 0$. Hence

$$-cm_o(y) + (1 - d)m_e(y) = 0, \ y \in G. \tag{14}$$

Putting $y = 0$ in (14) and using $m_o(0) = 0$, we have $d = 1$. Jointly with (14), for all $y \in G$, this implies that $-cm_o(y) = 0$, whence $c = 0$, which ends the proof of (i).

Case 2. Assume that $f(0)=b\neq 0$. Set $b=2(1-d)$ in (13). Then, we get

$$m_o(y)[(b^2 - 2ac)m_o(x) + b(a - 2c)m_e(x)] = 0, \ x, y \in G. \tag{15}$$

Subcase 2.1. Let $m_o = 0$. By equation (12), we conclude that $f = bm_e$, $g = dm_e$. Replacing y by $-y$ in (7), we arrive at

$$[f(x-y)]^2-[f(x+y)]^2+f(2x-2y)+f(2x+2y)=f(2x)[f(2y)+2g(2y)]. \tag{16}$$

Subtracting (7) and (16), we get

$$[f(x + y)]^2 = [f(x - y)]^2, \quad x, y \in G.$$

Putting here $y = x$ and replacing x by $\frac{x}{2}$, we obtain $f^2 = b^2$. The case $f = -b$ is impossible. In other words, we have (ii): $f = b, \ m_e = 1, \ g = d = \frac{2-b}{2} = 1 - \frac{1}{2}b$.

Subcase 2.2. Suppose $m_o \neq 0$. Then (15) yields

$$(b^2 - 2ac)m_o(x) + b(a - 2c)m_e(x) = 0, \quad x \in G. \tag{17}$$

Putting $x = 0$, we have $a = 2c$. From (17), for all $x \in G$, we get $(b^2 - 4c^2)m_o(x) = 0$, i.e. $b^2 = 4c^2$. If $a = 2c \wedge b = 2c$, then we have the case (iii). However, $a = 2c \wedge b = -2c$ yields (iv). $\square$

Theorem 2. *Let $(G, +)$ be a uniquely 2-divisible Abelian group. Then functions $f, g : G \to \mathbb{C}$ satisfy equation (7) if and only if*

(i) $f = 0$ *and g is arbitrary; or*

(ii) $f(x) = \alpha \neq 0, \ g(x) = 1 - \frac{1}{2}\alpha, \ x \in G$*; or*

(iii) *there exists an additive function $A{:}G\to\mathbb{C}$ such that $f=A, g=1$; or*

(iv) *there exists an exponential function $m : G \to \mathbb{C}$ and some constant $\beta \in \mathbb{C}$ such that $f=\beta m_o, \ g=m_e$; or*

(v) *there exists an exponential function* $m\colon G \to \mathbb{C}$ *such that* $f(x) = f(0)m_o(x) + f(0)m_e(x)$, $g(x) = \frac{f(0)}{2}m_o(x) + \left(1 - \frac{f(0)}{2}\right) m_e(x)$, $x \in G$; *or*

(vi) *there exists an exponential function* $m : G \to \mathbb{C}$ *such that* $f(x) = -f(0)m_o(x) + f(0)m_e(x)$, $g(x) = -\frac{f(0)}{2}m_o(x) + \left(1 - \frac{f(0)}{2}\right) m_e(x)$, $x \in G$.

Proof. From Lemma 1 we obtain (i) and (ii) for $\alpha = 2$. Assume that $f \neq 0$ and $g \neq 0$. Putting $x = 0$ in (7), we get

$$[f(y)]^2 - [f(-y)]^2 + f(2y) + f(-2y) = f(0)[f(2y) + 2g(2y)], \quad y \in G.$$

Let $2C := f(0)$. Thus, from above

$$f(2y) + f(-2y) - 2C[f(2y) + 2g(2y)] = [f(-y)]^2 - [f(y)]^2, \quad y \in G. \tag{18}$$

Interchanging the roles of x and y in (7), we obtain

$$\begin{aligned}&[f(y+x)]^2 - [f(y-x)]^2 + f(2y+2x) + f(2y-2x)\\ &\quad = f(2y)f(2x) + 2f(2y)g(2x), x, y \in G.\end{aligned} \tag{19}$$

Subtracting (7) and (19), we get

$$\begin{aligned}&[f(y-x)]^2 - [f(x-y)]^2 + f(2x-2y) - f(2y-2x)\\ &\quad = 2f(2x)g(2y) - 2f(2y)g(2x), x, y \in G.\end{aligned} \tag{20}$$

Applying (18) for y equal $x - y$, we recive

$$\begin{aligned}&f(2x-2y) + f(-2x+2y) - 2C[f(2x-2y) + 2g(2x-2y)]\\ &\quad = [f(-x+y)]^2 - [f(x-y)]^2, x, y \in G.\end{aligned} \tag{21}$$

By (20) and (21), we get the relation

$$(1 - C)f(2x - 2y) - 2Cg(2x - 2y) = f(2x)g(2y) - f(2y)g(2x), \quad x, y \in G.$$

Replacing x by $\frac{x}{2}$ and y by $\frac{y}{2}$, we obtain

$$(1 - C)f(x - y) - 2Cg(x - y) = f(x)g(y) - f(y)g(x), \quad x, y \in G. \tag{22}$$

Case 1. Let $f(0) = 0 \Rightarrow C = 0$. Thus, from (22) we get

$$f(x - y) = f(x)g(y) - f(y)g(x), \quad x, y \in G,$$

By Theorem 1 (ii), we infer that $f(x) = A(x), g(x) = \alpha A(x) + 1$ for some additive function A and some constant α. In view of Lemma 2 for $a = 1$, $b = \alpha$, $f(0) = 0$, $g(0) = 1$, we deduce that

$$f(x) = A(x), \quad g(x) = 1, \quad x \in G.$$

This is the case (iii) of our theorem. By Theorem 1 (iii), we get

$$f(x) = \beta m_o(x), \quad g(x) = \gamma m_o(x) + m_e(x), \quad x \in G.$$

For $a = \beta,\ b = 0,\ c = \gamma,\ d = 1$ in Lemma 4 (i) we have the case (iv), i.e.

$$f(x) = \beta m_o(x), \quad g(x) = m_e(x), \quad x \in G.$$

Case 2. Assume that $f(0) \neq 0$. Then $g(0) = 1 - C$, and (22) gives

$$g(0)f(x-y) - f(0)g(x-y) = f(x)g(y) - f(y)g(x), \quad x, y \in G. \tag{23}$$

Subcase 2.1. If $g(0) = 0$, then $f(0) = 2$. By (23), we infer that

$$g(x-y) = g(x)\frac{f(y)}{2} - g(y)\frac{f(x)}{2}, \quad x, y \in G. \tag{24}$$

Theorem 1 (ii) yields $g(x) = A(x)$, $\frac{f(x)}{2} = \alpha A(x)+1$ for some additive function A and some constant α. Thus

$$f(x) = 2\alpha A(x) + 2, \quad g(x) = A(x), \quad x \in G.$$

By Lemma 2 for $a = 2\alpha$, $b = 1$, $f(0) = 2$, $g(0) = 0$, we get $f = 2$, $g = 0$. This is the case (ii). Theorem 1 (iii) leads us to

$$f(x) = 2\gamma m_o(x) + 2m_e(x), \quad g(x) = \beta m_o(x), \quad x \in G.$$

From Lemma 4 (ii) for $a = 2\gamma$, $b = 2, c = \beta, d = 0$, we get (ii) of the theorem. The case (iii) for $f(0) = 2$ gives (v), and (iv) gives (vi).

Subcase 2.2. Let $f(0) \neq 0$ and $g(0) \neq 0$. Thus, from (23) for

$$F(x) := g(0)f(x) - f(0)g(x), \quad G(x) := \frac{g(x)}{g(0)}, \quad x \in G,$$

we conclude that

$$F(x-y) = F(x)G(y) - F(y)G(x), \quad x, y \in G. \tag{25}$$

Again, by Theorem 1 (ii), we obtain

$$g(x) = g(0)\alpha A(x) + g(0), \quad f(x) = \frac{1 + f(0)g(0)\alpha}{g(0)}A(x) + f(0), \quad x \in G.$$

By Lemma 2 (ii) for $a = \frac{1+f(0)g(0)\alpha}{g(0)}$, $b = g(0)\alpha$, we get (ii) of the theorem. Further, Theorem 1 (iii) yields

$$F(x) := g(0)f(x) - f(0)g(x) = \beta m_o(x), \quad x \in G,$$

$$G(x) := \frac{g(x)}{g(0)} = \gamma m_o(x) + m_e(x), \quad x \in G,$$

or, equivalently,

$$g(x) = g(0)\gamma m_o(x) + g(0)m_e(x), \quad x \in G,$$

$$f(x) = \frac{\beta + \gamma f(0)g(0)}{g(0)} m_o(x) + f(0)m_e(x), \quad x \in G.$$

Now, using Lemma 4 for $a = \frac{\beta+\gamma f(0)g(0)}{g(0)}$, $b = f(0)$, $c = \gamma g(0)$, $d = g(0)$, the case (ii) gives (ii) of our theorem, however (iii) yields (v), and (iv) gives (vi). □

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A CORRECTION METHOD FOR PROJECTS OF SYSTEM SAFETY FROM THE RISK MINIMIZATION CRITERION

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Abstract. In this paper we give an analytical method of solution to the problem of correction of the technical system parameters in order to reduce the planned common risk of technical system safety violation.

In the article [1] we considered the possibility and assumptions under which the common risk (criterion) $y = f(x)$ of technical system safety violation is expressed as a sum of particular risks $u_i(x)$ on a certain threats for safety violation, i.e.

$$y = f(x) = u_1(x) + u_2(x) + \ldots + u_n(x). \tag{1}$$

Let $u_i(x)$ be a polynomial

$$u_i(x) = C_i \cdot \prod_{j=1}^{m} x_j^{a_{ij}}, \quad C_i > 0, \quad i = 1, 2, \ldots, n, \tag{2}$$

and the vector $x = (x_1, x_2, \ldots, x_m)$ of some parameters x_j be positive. The matrix $A = (a_{ij})$ is called an exponent matrix. Suppose that the matrix $A = (a_{ij}) = \begin{pmatrix} B \\ H \end{pmatrix}$, where the basis B is an $m \times m$ matrix ($\mid B \mid \neq 0$) and the submatrix H contains d rows of matrix A, which do not belong to the basis B. The difficulty level is characterized by this number d: $d = n - m$. The coefficients a_{ij} and C_i are got by methods of linear regression analysis [2].

A selected set (vector) x will be called *a project* of system safety.

Suppose that the project x_* ensures the minimum value $y_* = f(x_*)$ of the common risk y (see [3]); then the project x_* will be called a *perfect* project.

Numerical method for obtaining the values y_* and x_* was proposed in [4,5].

Assume that the perfect project x_* is not accepted on account of costs. Therefore, a maximum acceptable risk $\bar{y}$ such that $y_* < \bar{y}$ is given, and the requirement $y = f(x) < \bar{y}$ should be fulfilled (for more details we refer the reader to [5, 6]).

Let $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots , x_m^{(1)})$ of some parameters $x_1^{(1)}$ be a project satisfying $y = f(x) \leq \bar{y}$ for $x = x^{(1)}$.

The value $y^{(1)} = f(x^{(1)})$ will be called the *planned* risk of technical system safety violation. By the above, we have $y_* \leq y^{(1)} \leq \bar{y}$.

On the basis of project $x^{(1)}$ and x_*, recomendations on possible improvement of the planned parameters $x_j^{(1)}$, $j = 1, 2, \dots , m$ should be given.

In other words, we should propose a new project $\tilde{x}$ such that the value $\tilde{y} = f(\tilde{x})$ of the common risk is less than the planned risk. Furthermore, it satisfies the inequality $y_* \leq \tilde{y} < y^{(1)}$.

In order to solve this problem we shall need some results of the Theorem which is proved below. According to this Theorem, the function $y = f(x)$ of (1) satisfies the inequality

$$f(x_*) < f((x^{(1)})^{\lambda} \cdot x_*^{1-\lambda}) < f(x^{(1)}). \tag{3}$$

Here $(x^{(1)})^{\lambda} \cdot x_*^{1-\lambda} = \tilde{x}$ is a new project with parameters

$$\tilde{x}_j = (x_j^{(1)})^{\lambda} \cdot x_{j*}^{1-\lambda} \tag{4}$$

for each λ from the interval $0 < \lambda < 1$.

The number $\bar{y}$ is the value of the common risk for the *improved* project.

Substituting the different values $\lambda \in (0; 1)$ in (4), we obtain the set of new projects $\tilde{x}$, each of which satisfies $f(\tilde{x}) \leq \bar{y}$.

The common risk $\tilde{y}$ for the new project $\tilde{x}$ is less than or equal to $y^{(1)}$: $f(\tilde{x}) \leq f(x^{(1)})$. For this reason, it is advisable to change the project $x^{(1)}$ by the project $\tilde{x}$.

Example. Let the common risk be given as a function

$$y = f(x) = \frac{1}{100} \cdot \left(2x_1^2 \cdot x_2 + 3x_1 \cdot x_2^2 + 4\frac{1}{x_1} \cdot \frac{1}{x_2} \right)$$

of the parameters x_1, x_2. Let $\bar{y}$=0.109 be the maximum acceptable value of this risk.

Minimizing the function $y = f(x)$ according to [3], we get the minimum value y_*=0.085 of the criterium and the perfect project x_* with parameters $x_1* = 1.08$ and $x_2* = 0.72$.

Assume that the project x_* is not accepted on account of costs. Suppose that the project $x^{(1)}$ =[1.04;1.04] is brought up for discussion. Substituting $x^{(1)}$ into (1) yields the value of the common risk: $y^{(1)} = f(x^{(1)}) = 0.093$, which satisfies the inequality $y^{(1)} < \bar{y} = 0.109$.

It is necessary to propose an improved project $\tilde{x}$.

Solution. Take a number $\lambda \in (0;1)$, for example, $\lambda = 0.4$ and define the new project $\tilde{x} = [\tilde{x_1}, \tilde{x_2}]$, where $\tilde{x_1} = 1.04^{0.4} \cdot 1.08^{0.6} = 1.07$, $\tilde{x_2} = 1.04^{0.4} \cdot 0.72^{0.6} = 0.84$. It can be easily checked that the corresponding value $\tilde{y} = f(\tilde{x}) = 0.086$. If the new project satisfies us for the economic reason, then $\tilde{x}$ can be accepted because $\tilde{y} < y^{(1)}$.

Now these calculations will be substantiated for $d = 1$.

Theorem. Let $A = (a_{ij})$ be the matrix corresponding to the criterion $y = f(x)$ of (1). Further $A_0 = (A, \mathbf{1})$, where $\mathbf{1}$ is a column of ones, is a square and nonsingular matrix. Moreover a unique solution $\delta = (\delta_1, \delta_2, \ldots, \delta_n) = (\underbrace{0, 0, \ldots, 0}_{m}, 1) \cdot A_0^{-1}$ of the equality $\delta \cdot A_0 = (\underbrace{0, 0, \ldots, 0}_{m}, 1)$ is positive: $\delta > 0$. Suppose that $f(x) \leq \overline{y}$ and $y_* \leq \overline{y}$; then the following statements are true:

1. The vector x_* with the components

$$x_{j*} = \prod_{i=1}^{n} \left(\frac{\delta_i \cdot y_*}{C_i} \right)^{k_{ji}}, \quad j = 1, 2, \ldots, m, \tag{5}$$

is one of the solutions of the inequality $f(x) \leq \bar{y}$, where k_{ji} are the elements of the inverse matrix $B^{-1} = (k_{ji})$. Similarly, the vector

$$x_j = \prod_{i=1}^{n} \left(\frac{\delta_i \cdot \bar{y}}{C_i} \right)^{k_{ji}}, \quad j = 1, 2, \ldots, m, \tag{6}$$

is the solution of the same inequality $f(x) \leq \bar{y}$.

2. Suppose that $\alpha^{(1)} = (\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_n^{(1)})$ and $\alpha^{(2)} = (\alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_n^{(2)})$ are the solutions of the so called *generative inequality*

$$\alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \geq r, \tag{7}$$

where $\alpha_i > 0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, and $r = \frac{\prod_{i=1}^{n} C_i^{\delta_i}}{\bar{y}}$.

Substituting these solutions into (6) in place of δ_i, we obtain two solutions: $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}) > 0$ and $x^{(2)} = (x_1^{(2)}, x_2^{(2)}, \ldots, x_m^{(2)}) > 0$ of the inequality $f(x) \leq \bar{y}$.

Then for any number $\lambda \in [0;1]$ the vector $\tilde{x} = (x^{(1)})^{\lambda} \cdot (x^{(2)})^{1-\lambda}$ with the components $\bar{x}_j = (x_j^{(1)})^{\lambda} \cdot (x_j^{(2)})^{1-\lambda}$ is the solution of the same inequality $f(x) \leq \bar{y}$, i.e.

$$(f(x^{(1)}) \leq \bar{y},\ f(x^{(2)}) \leq \bar{y}) \Rightarrow (f(\tilde{x}) \leq \bar{y}). \tag{8}$$

Moreover,

$$\left(x_j \in \left[x_j^{(1)}, x_j^{(2)}\right], j = 1, 2, \ldots, m\right) \Longrightarrow \left(y = \sum_{i=1}^{n} C_i \prod_{j=1}^{m} x_j^{a_{ij}} \in [y_*, \overline{y}]\right) \tag{9}$$

Proof.

1. According to [3], the vector x_* with components from (5) guarantees that $f(x)$ of (1) has a minimum at x_*: $f(x_*) = y_*$. Under the conditions of the Theorem, it follows that x_* is the solution of $f(x) \leq \bar{y}$, i.e. $f(x_*) = y_* \leq \bar{y}$.

Let us consider inequality (7) and the vector $\alpha = \delta$ with components $\alpha_i = \delta_i$ for which the minimum $y_* = \prod_{i=1}^{n} (\frac{C_i}{\delta_i})^{\delta_i}$ is occurred (see [3]).

We claim that the inequality $\alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \leq \delta_1^{\delta_1} \cdot \delta_2^{\delta_2} \cdots \delta_n^{\delta_n}$ is fulfilled for fixed numbers $\delta_i > 0$, where $\delta_1 + \delta_2 + \ldots + \delta_n = 1$ and arbitrary numbers $\alpha_i > 0$, where $\alpha_1 + \alpha_2 + \ldots \alpha_n = 1$. Indeed, if the function $g(\alpha)$ has the form $g(\alpha) = \alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n}$, then there exists the *Lagrange* function $L = \ln g(\alpha) + \lambda(1 - (\alpha_1 + \alpha_2 + \ldots \alpha_n))$. Consider the equations $L'_{\alpha_i} = (\frac{\delta_i}{\alpha_i} - \lambda) = 0$, so $\frac{\delta_i}{\alpha_i} = \lambda$, $i = 1, 2, \ldots, n$. From these equations we obtain $\delta_i = \lambda \alpha_i$, $\sum_{i=1}^{n} \delta_i = \lambda \cdot \sum_{i=1}^{n} \alpha_i$, and $\lambda = 1$.

Thus, the function $g(\alpha)$ has an extremum at the point with components $\alpha_i = \delta_i$. Since $L''_{\alpha_i^2} = -\frac{\delta_i}{\alpha_i^2} < 0$ and $L''_{\alpha_i \alpha_j} = 0$, (i≠j), we see that the matrix $(L''_{\alpha_i \alpha_j})$ of second partial derivatives for the *Lagrange* function is negative definite. This means that the function $g(\alpha)$ takes the maximum at the point with components $\alpha_i = \delta_i$. Hence $g(\alpha) = \alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \leq \delta_1^{\delta_1} \cdot \delta_2^{\delta_2} \cdots \delta_n^{\delta_n}$. An equality sign is achieved if and only if $\alpha_i = \delta_i$. Now the inequality yields $(\alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \geq q) \Rightarrow (\delta_1^{\delta_1} \cdot \delta_2^{\delta_2} \cdots \delta_n^{\delta_n} \geq q)$. This relationship and Theorem 1 from [3] show that the vector α $(= \delta)$ with components $\alpha_i = \delta_i$ is one of the solutions of (7) corresponding to the solution (6) of the inequality $f(x) \leq \bar{y}$.

2. Suppose that $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}) > 0$ is the solution of $f(x) \leq \bar{y}$. The parameters $x_j^{(1)}$ are received under change of δ_i by $\alpha_i^{(1)}$ in (6), where $\alpha^{(1)}$ is one of the solutions of the generative inequality (7).

Then, by the results of [3], there exist $x_{m+1}^{(1)} = x_{m+2}^{(1)} = \ldots = x_{m+n-1}^{(1)} = 1$, $x_{m+n}^{(1)} = \prod_{i=1}^{n} \left(\frac{\alpha_i^{(1)} \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu} \geq 1$ such that the vector $\ln(x^{(1)})' = (\ln x_1^{(1)}, \ln x_2^{(1)},$

$\dots, \ln x^{(1)}_{m+n})$ is the solution of the equation $A_1 \cdot \ln x' = \ln b(\alpha)$, where the matrix $A_1 = (A.I_n)$ and $b(\alpha) = (\ln(\frac{\alpha_1 \cdot \bar{y}}{C_1}, \ln(\frac{\alpha_2 \cdot \bar{y}}{C_2}, \dots, \ln(\frac{\alpha_n \cdot \bar{y}}{C_n})$.

Similarly, if $x^{(2)} > 0$ is the solution of $f(x) \leq \bar{y}$ with components $x^{(2)}_j$, $j = 1, 2, \dots, m$, which are received under substitution of δ_i by $\alpha^{(2)}_i$ in (6), where $\alpha^{(2)}$ is one of the solutions of the generative inequality (7), then there exist $x^{(2)}_{m+1} = x^{(2)}_{m+2} = \dots = x^{(2)}_{m+n-1} = 1$, $x^{(2)} = \prod_{i=1}^{n} \left(\frac{\alpha^{(2)}_i \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu} \geq 1$ such that the vector $\ln(x^{(2)})' = (\ln x^{(2)}_1, \ln x^{(2)}_2, \dots, \ln x^{(2)}_{m+n})$ is the solution of the equation $A_1 \cdot \ln x' = \ln b(\alpha)$.

Suppose that $\lambda \in [0; 1]$. Since $A_1(\lambda \ln(x^{(1)})' + (1 - \lambda)\ln(x^{(2)})') = \lambda A_1 \cdot \ln(x^{(1)})' + (1 - \lambda)A_1 \cdot \ln(x^{(2)})' = \lambda \ln b(\alpha) + (1 - \lambda)\ln b(\alpha) = \ln b(\alpha)$, we see that the vector $\ln(\tilde{x})' = \lambda \ln(x^{(1)})' + (1 - \lambda)\ln(x^{(2)})'$, where $\tilde{x}' = (x^{(1)})'^{\lambda} \cdot (x^{(2)})'^{(1-\lambda)}$, is the solution of the inequality $A_1 \cdot \ln x' = \ln b(\alpha)$.

The elements $\tilde{x}'_j = (x^{(1)}_j)'^{\lambda} \cdot (x^{(2)}_j)'^{(1-\lambda)}$ are components of the vector $\tilde{x}'$. Thus, from the formulas for $(x^{(1)}_j)'$ and $(x^{(2)}_j)'$, it follows that

$$\tilde{x}'_j = \tilde{x}_j = (\prod_{i=1}^{n} \left(\frac{\alpha^{(1)}_i \cdot \bar{y}}{C_i}\right)^{k_{ij}})^{\lambda} \cdot (\prod_{i=1}^{n} \left(\frac{\alpha^{(2)}_i \cdot \bar{y}}{C_i}\right)^{k_{ij}})^{1-\lambda}, \quad j = 1, 2, \dots, m,$$

$$\tilde{x}'_{m+1} = \tilde{x}'_{m+2} = \dots = \tilde{x}'_{m+n-1} = 1,$$

$$\tilde{x}'_{m+n} = (\prod_{i=1}^{n} \left(\frac{\alpha^{(1)}_i \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu})^{\lambda} \cdot (\prod_{i=1}^{n} \left(\frac{\alpha^{(2)}_i \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu})^{1-\lambda} \geq 1.$$

Here μ is equal to the sum of components of the row matrix $(-a_n \cdot B^{-1}, 1)$, where a_n is the last row of the exponent matrix A. These expressions and the results of [3] allow us to reach the conclusion that the vector $\tilde{x}'$ with components $\tilde{x}'_j = \tilde{x}_j = (x^{(1)}_j)^{\lambda} \cdot (x^{(2)}_j)^{1-\lambda}$, $j = 1, 2, \dots, m$ is the solution of the inequality $f(x) \leq \bar{y}$. Hence the relationship (8) holds.

Since λ is any number in [0;1], then $\min(x^{(1)}_j, x^{(2)}_j) \leq \tilde{x}_j \leq \max(x^{(1)}_j, x^{(2)}_j)$, i.e. $\tilde{x}_j \in [x^{(1)}_j, x^{(2)}_j]$.

Therefore, the inequality $f(x) \leq \bar{y}$ is fulfilled for all vectors $x = \tilde{x}$ with components $x_j = [x^{(1)}_j, x^{(2)}_j]$. Moreover, since $y_* = f(x_*)$ and $y_* \leq \bar{y}$, then $\left(x_j \in \left[x^{(1)}_j, x^{(2)}_j\right], j = 1, 2, \dots, m\right) \Longrightarrow \left(y = \sum_{i=1}^{n} C_i \prod_{j=1}^{m} x_j^{a_{ij}} \in [y_*, \overline{y}]\right)$.

Thus, we have proved the relationship (9).

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PART II

COMPUTER SCIENCE

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SAT-BASED CRYPTANALYSIS OF MODIFIED VERSIONS OF FEISTEL NETWORK

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Abstract. It is well known that Feistel Network (FN) is the foundation of many symmetric ciphers used in practice. In this paper we present some remarks and experimental results on SAT based cryptanalysis of several modified versions of FN. We investigate different cryptographic functions used in FN schema for better understanding their properties from a security point of view. In our work we study the notions widely used in many ciphers: the *xor* function, bits rotations, permutations and $S-boxes$.

1. Introduction

Boolean SATisfiability problem is the well known and celebrated NP-complete problem [2]. The Boolean encoding of some system models and checking satisfiability of obtained formulas sometimes gives the answer to the question about important system's properties [1, 8]. So far, there is no known algorithm that solves efficiently all the instances of SAT. It is generally believed that no such effective algorithm can already exist. On the other hand, in many instances a lot of Boolean formulas can be solved surprisingly efficiently, even very large formulas appearing naturally in description of various industrial systems as well as in decision and optimization problems [1, 2, 7]. There are many competing algorithms searching for a satisfying valuation for a given Boolean formula. A lot of them are highly optimised versions of the DPLL procedure of [4] and [5]. Usually SAT-solvers take input formulas in the conjunctive normal form (CNF). It is a conjunction of clauses, where a clause is a disjunction of literals, and a literal is a propositional variable or the complement of a propositional variable.

In this paper we use SAT for investigation of security properties of several modified versions of FN, that is some easy but very important cipher used as a basis for many strong symmetric ciphers applied in practice (see, for example, [9]). We show how using several different cryptographic functions as a main function F in FN can change security properties and computational complexity of FN's SAT based cryptanalysis. We show this on the well known functions used in many other symmetric ciphers: the xor function, bits rotation, permutation and $S-box$. Carrying out current research we want to check how SAT cryptanalysis works in the simple cases discussed in order to have the ability to select some other ciphers used in practice for future study. Their cryptanalysis may be promising.

The methodology is similar to that used in [3] and [8], and this paper presents an investigation additional to our previous paper [6].

The rest of this paper is organized as follows. In the second section, we introduce all the basic information on the FN cipher to the extent necessary for explaining our Boolean encoding method. The third section gives a process of a direct Boolean encoding of FN and the main functions which are considered. In the fourth section, we present some experimental results which have been obtained. Some conclusions and remarks concerning the future work are given in the last section.

2. Feistel Network

This section presents the basic information on FN which is needed for understanding the Boolean encoding of investigated ciphers. It is well known that FN is a symmetric-key block algorithm widely used as a design principle of many symmetric ciphers, including the famous Data Encryption Standard (DES). FN has the advantage that its encryption and decryption procedures are almost identical, requiring only a reversal of the key schedule. FN is an iterated algorithm which is executed many times on the same input. Due to a simple structure and easy hardware implementation, Feistel-like networks are widely used as a component of various cipher designs. Some famous, strong and used in practice FN are the following: MISTY1, Skipjack, Blowfish, RC5, Twofish (see, for example, [9]).

Consider a given bit block M that represents a plaintext. Let F denote the round main function of FN and $K_1, \ldots, K_n$ denote a sequence of keys obtained in some way from the main key K for the rounds $1, \ldots, n$, respectively. We use the symbol $\otimes$ for denoting the exclusive-OR (xor) operation.

The basic operations of FN are specified as follows:

1. break the plaintext block M into two equal length parts denoted by (L_0, R_0);

2. for each round $i = 0, 1, \ldots, n$ compute:
   a) $L_{i+1} = R_i$,
   b) $R_{i+1} = L_i \otimes F(R_i, K_i)$.

Then the ciphertext sequence is (R_{n+1}, L_{n+1}).

The structure of FN allows for an easy method of decryption. For explanation of the decryption procedure of FN, let us recall the basic properties of operation $\otimes$:

1. $x \otimes x = 0$,

2. $x \otimes 0 = x$,

3. $x \otimes (y \otimes z) = (x \otimes y) \otimes z$.

A given ciphertext (R_{n+1}, L_{n+1}) is decrypted by computing

$$R_i = L_{i+1}, \qquad L_i = R_{i+1} \otimes F(L_{i+1}, K_i)$$

for $i = n, n-1, \ldots, 0$.

It is easy to see that (L_0, R_0) is the plaintext again. Observe that we have the following equations:

$$R_{i+1} \otimes F(L_{i+1}, K_i) \;=\; (L_i \otimes F(R_i, K_i)) \otimes F(L_i, K_i)$$

$$= L_i \otimes (F(R_i, K_i) \otimes F(L_i, K_i)) \;=\; L_i \otimes 0 \;=\; L_i.$$

It should be noted that the power of the cipher depends on the choice of the function F. In practice, many different solutions are used in this case. In the next sections we investigate four of them: the *xor* function, bits rotation, permutation, and $S\!-\!box$.

3. Boolean encoding for cryptanalysis

This section presents a direct Boolean encoding of FN versions with different functions F. As it was mentioned above, FN constitutes the basic structure for many well respected symmetric ciphers. Hence, its Boolean encoding will be helpful in the SAT-based cryptanalysis which we want to pursue in the future.

For our explanation we consider FN with a 64-bit block of a plaintext and a 32-bit key. Let $p_1, \ldots, p_{64}$, $k_1, \ldots, k_{32}$ and $c_1, \ldots, c_{64}$ be the propositional variables representing a plaintext, a key, and a ciphertext, respectively. Observe that following the Feistel algorithm for the first, left half of a plaintext we have:

$$\bigwedge_{i=1}^{32}(c_i \Leftrightarrow p_{i+32}).$$

It is easy to see that for the second, right half of a plaintext we have:

$$\bigwedge_{i=1}^{32}(c_i \Leftrightarrow (p_i \otimes F(k_i, p_{i+32})).$$

Hence the encoding formula for one round of FN is as follows:

$$\Phi^1_{Feistel} : \bigwedge_{i=1}^{32}(c_i \Leftrightarrow p_{i+32}) \ \wedge \ \bigwedge_{i=1}^{32}(c_i \Leftrightarrow (p_i \otimes F(k_i, p_{i+32})).$$

In the case of t rounds of FN we have the following: Let $(p^1_1, \dots, p^1_{64})$, $(k_1, \dots, k_{32})$ be a plaintext and a key vectors of variables, respectively. By $(p^j_1, \dots, p^j_{64})$ and $(c^i_1, \dots, c^i_{64})$ we describe the vectors of variables representing input of the jth round for $j = 2, \dots, t$ and output of the ith round for $i = 1, \dots, t-1$. We denote by $(c^t_1, \dots, c^t_{64})$ the variables of a cipher vector after the tth round.

The formula which encodes the whole tth round of a Feistel Network is as follows:

$$\Phi^t_{Feistel} : \bigwedge_{i=1}^{32}\bigwedge_{s=1}^{t}(c^s_i \Leftrightarrow p^s_{i+32}) \ \wedge \ \bigwedge_{i=1}^{32}\bigwedge_{s=1}^{t}[c^s_{i+32} \Leftrightarrow (p^s_i \ \otimes \ F(k_i, p^s_{i+32}))] \ \wedge$$

$$\wedge \bigwedge_{i=1}^{64}\bigwedge_{s=1}^{t-1}(p^{s+1}_i \ \Leftrightarrow \ c^s_i).$$

Observe that the last part of $\Phi^t_{Feistel}$ states that the outputs from the sth round are the inputs of the $(s+1)$th round.

As we can see, the obtained formula is a conjunction of ordinary, or rather simple, equivalences. This is important from the viewpoint of translating into CNF. The second advantage of this description is that we can automatically generate the formula for many investigated rounds.

It is well known that the security of FN cipher depends on the function F. In our investigation, as a simple instantiation of the function F, we firstly use the function *xor*, denoted as before without any other changes of the used bits. In this case we obtain the following formula that encodes the function F in the considered cipher:

$$F(k_i, p^s_{i+32}) \;\Leftrightarrow\; (k_i \otimes p^s_{i+32}).$$

The second considered approach is adding the rotation of key's bits into F. If we use one right rotation for one round of FN, we arrive at the following formula:

$$F(k_i, p^s_{i+32}) \;\Leftrightarrow\; (k_{i\oplus s} \otimes p^s_{i+32}),$$

where $\oplus$ denotes + modulo 32.

The third example presents some permutation in the function F. In this work we consider the PC permutation – one of permutations used in the DES cipher [9]. In this case we get the following formula:

$$F(k_i, p^s_{i+32}) \;\Leftrightarrow\; (k_i \otimes p^s_{PC(i)\oplus 32}).$$

The last, most powerfull, considered modification of the function F is adding the $S-box$ into F. Like before, we use the first $S-box$ used in DES cipher [9]. Firstly, we increase the length of the block half from 32 to 48 by repeating the proper bits like in the DES algorithm. Observe that each of $S{-}boxes$ of this type is the matrix with four rows and sixteen columns, where in each row we have one different permutation of numbers belonging to Z_{16}. These numbers are denoted in binary form as four-tuples of bits. Following this, we can consider each $S-box$ as a function of the type $S_{box} :\ \{0,1\}^6 \to \{0,1\}^4$.

For simplicity, let us denote by $\overline{x}$ the vector $(x_1, \ldots, x_6)$ and by $S^k_{box}(\overline{x})$ the kth coordinate of the value $S_{box}(\overline{x})$ for $k = 1, 2, 3, 4$.

We can encode the $S-box$ as the following Boolean formula:

$$\Phi_{S_{box}} : \bigwedge_{\overline{x}\in\{0,1\}^6} (\bigwedge_{i=1}^{6} (\neg)^{1-x_i} r_i \;\Rightarrow\; \bigwedge_{j=1}^{4} (\neg)^{1-S^j_{box}(\overline{x})} q_j),$$

where $(r_1, \ldots, r_6)$ is the input vector of the $S-box$ and $(q_1, \ldots, q_4)$ is the output one. Additionally, by $(\neg)^0 r$ and $(\neg)^1 r$ we mean r and $\neg r$, respectively. Using this, we can encode the $S-box$ used as 256 simple implication. This number is equal to the size of the S–box matrix. Due to the strongly irregular and random character of S–$boxes$, we are sure that this is the simplest method of their Boolean encoding. Having them, we can encode any given number of rounds of modified FN as a Boolean propositional formula. The next step of our investigation is applying our cryptanalysis procedure to the obtained formulas.

Rounds	Variables	Clauses	Encoding Time (s.)	Encoding Memory (MB)	Solving Time (s.)	Solving Memory (MB)
4	672	9952	0.008	8	0.012	8
8	1248	19808	0.016	8	0.181	9
12	1824	29664	0.021	9	90.74	25
16	2400	39520	0.028	9	11370.5	231

Table 1: Experimental results with $S-box$.

4. Cryptanalysis procedure and experimental results

The cryptanalysis procedure used in our investigation proceeds as follows:

1) encode a single round of the cipher considered as a Boolean propositional formula;
2) generate automatically the formula encoding an iterated desired number of rounds;
3) convert the obtained formula into the CNF form;
4) (randomly) choose a plaintext and the key vector as the $0, 1$ valuation of the variables representing them in the formula;
5) insert the chosen valuation into the formula;
6) calculate the corresponding ciphertext using an appropriate key and insert it into the formula;
7) run some SAT-solver to find a satisfying valuation, including a valuation of the key variables.

To test how the SAT based cryptanalysis works for the functions mentioned above we have used the previously outlined procedure for four different number of rounds of a modified version of the FN: 4, 8, 12, and 16. The obtained results show that the SAT based cryptanalysis is similar to other methods of cryptanalysis. The SAT based cryptanalysis proceeds along very well with simple functions such as xor, rotations, and permutations. The results obtained for these features show a strictly linear relationship to receive the key from the known plaintext ciphertext pair with increasing the number of iterations of the algorithm.

In the case of adding the $S-box$ into the function F we obtain results that show an exponential relation to receiving the key with increasing the number of iterations of the algorithm. We can see these results in Table 1.

In this Table we show the number of variables and clauses of the encoding formula obtained for proper iterations of FN, time of generating the formula and time of searching key satisfiable valuation (break the key).

The computer used to perform the experiments was equipped with the processor Intel Pentium D (3000 MHz), 2 GB main memory, the operating system Linux, and the SAT-solver MiniSat.

5. Conclusion and the future work

In this paper we have shown how the SAT based cryptanalysis works for breaking some modified versions of Feistel Network. We have investigated four different main functions of FN using the *xor* function, bits rotation, permutation, and *S–box*. The obtained results show that this type of cryptanalysis proceeds well with the ciphers with the *xor* function, bits rotation, and permutation. Using the $S-box$ in cipher algorithm increases computational complexity of this type cryptanalysis into an exponential one. The next step of our work will consist in choosing some of the ciphers used in practice that have functions simply from the SAT based cryptanalysis and trying to break them. We hope that this investigation will be helpfull in choosing such ciphers. We are sure that the Kazumi cipher will be a good example for our work. Clearly, the success of our method depends on finding a cipher which can be broken.

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A PICTURE THAT IS SOMETHING MORE THAN JUST A PICTURE

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Abstract. Our paper presents the ways of hiding information with the usage of a digital picture. A branch of science that deals with hiding messages in the wider media is called steganography. Due to the rapid expansion of Internet and the associated increase in data exchange, this field appears to be a subject of interest. In this paper we present our own algorithms of hiding data in pictures and their implementations.

1. Introduction

1.1. Introduction to steganography

This article concerns steganography – the field related to cryptography. While cryptography hides the content of the information through its encryption, steganography tries to hide the fact of its occurrence.

There is the following division of steganographic systems: Pure Steganography, Private Key Steganography, and Public Key Steganography.

Kerckhoffs principle says that the cryptosystem should be secure even if all the details of its operation (besides the key) are known. Modification of the least significant bit is a classic representative of methods of replacing (substitution). The last bit (e.g. pixel component values) is replaced by a bit (or bits) from the message.

1.2. Steganography in pictures

Steganography for pictures, as well as for other media, uses a carrier (in this case a picture) to forward a confidential message. Just as in other cases, in steganography it is so important to prepare the carrier (media) that it does not arouse any suspicion of outsiders.

In addition to normal hiding, digital images allow for the usage of specific properties of the graphic format in which the image is saved, in order to provide confidential information. One of the techniques that allow us to hide information in a picture is to modify the least significant bit (Least Significant Bit) [4, 5]. As its name indicates, the technique involves the modified least significant bits of a numerical value describing the intensity of the color at the selected location of a picture.

Generally, with the LSB method in an image, we can hide any information stored in the form of consecutive bits. It is not important, whether it is a text message, the encrypted text message or other image, or any other type of binary file. The only limitation is the capacity of the media. In the example picture of 800x600 pixels in size, assuming the modification of only one bit of blue and one bit of green, we can send a secret message on a maximum of 120 000 bytes.

2. Proposed algorithm for steganography in pictures

2.1. Algorithm

In this chapter we present the algorithm for generating an image with the hidden information in it (a text). There are two approaches to this problem: the first, when a key and a text are hidden together in a picture, and the second, when a key is outside the picture. In the last case, the key can be delivered to the addressee through an encrypted connection, hidden in another image or delivered through another way. The key to our algorithm consists of a pair of numbers. The first number is a generator of multiplicative group Z_p and the second is a prime number p. The group generator is found by using the following theorem.

Theorem 1. *Let p–1 have a decomposition into prime factors p–1$= p_1p_2...p_k$, then $g \in Z_p$ is a generator of multiplicative group Z_p if and only if $g^{p-1/p_i} \neq 1$ for each $1 \leq i \leq k$.*

Theorem 2. *The number of generators of the group Z_p is $\phi(\phi(p))$, where ϕ is the Euler function.*

In the algorithm 1 an input is an array of size of k consisting of primes that are factors of $p - 1$. The algorithm returns the generator of a multiplicative group Z_p.

Algorithm 1: Algorithm of creating the generator

Ingresso: $d[k]$ - array of prime divisors $p-1$
Uscita: g - generator

1 Choose randomly g from the range $< 2, p-2 >$;
2 **per** $i \leftarrow 0$ **a** $k-1$ **fai**
  se $(g^{p-1/d[i]} = 1)$ **allora**
    go to step 1;
3 return g

Suppose that we have generated a prime number p greater than the number of bits of a bitmap and a generator of the group Z_p. Now we show an example of an algorithm to hide a text in the image. If the number of pixels in our bitmap is n, we can put $3n$ bits of text in this picture. This means that we put at most $3n/8$ bytes of text in the bitmap. In our algorithm, we will put 2 pixels in each pixel of the bitmap. The realization will take two approaches here: the first, when the information will be placed in two fixed RGB colors (green, blue), and the second, when the information will be placed in any two of the three colors (in the sequence: red color with green, red with blue and green with blue). Let us assume then that we have k bytes of text ($k < n/4$), which are included in the bitmap.

Algorithm 2: Algorithm of hiding text in an image

Ingresso: $t[k]$ - array of text, $b[n]$ - array of values of bitmap pixels
Uscita: $b[n]$ - array of values of bitmap pixels

1 **per** $i \leftarrow 0$ **a** $k-1$ **fai**
2   t[i] divide into four blocks of two bits;
3   compute the successive four values:
  $g^{4i} \bmod p, g^{4i+1} \bmod p, g^{4i+2} \bmod p, g^{4i+3} \bmod p$;
4   put the next block of the text in places of the $b[n]$ array determined by instruction 3 (from instruction 2).
5 return $b[n]$;

Note: when some of values calculated in point 3 is bigger than $n-1$ of the index of the last bitmap pixel, we calculate the next value.

Generating the places on a bitmap, in which the bits of hidden text will be placed, is achieved by a special case of linear pseudo-random number generator. Generators of this type have been proposed by American mathematician Derrick Lehmer [1]. In the generator, pseudo-linear consecutive numbers are calculated based on a recursive pattern: $X_{n+1} = aX_n \bmod N$ ($X_0 < N$ and $a < N$).

Theorem 3. *If $N = p$ is prime, the linear generator has a maximum period equal to p if and only if a is a primitive root of p.*

In our case the places, where the bits of hidden text are put, are analogous to that of the generator for $a = g$ and $N = p$.

2.2. Implementation

In order to present the method and effectiveness of the proposed solutions, we presented an example implementation of one of the variants of the proposed algorithm. A hidden message is the Latin text 'Lorem ipsum ...' This text was taken from publicly available sources that the reader can find at `http://lipsum.com`.

Carrier of secret information is a photograph of 768x1024 pixels and saved as 24-bit bitmap in BMP format. Comparison of the appearance of the image before encoding and after hiding the information is presented in Figure 1.

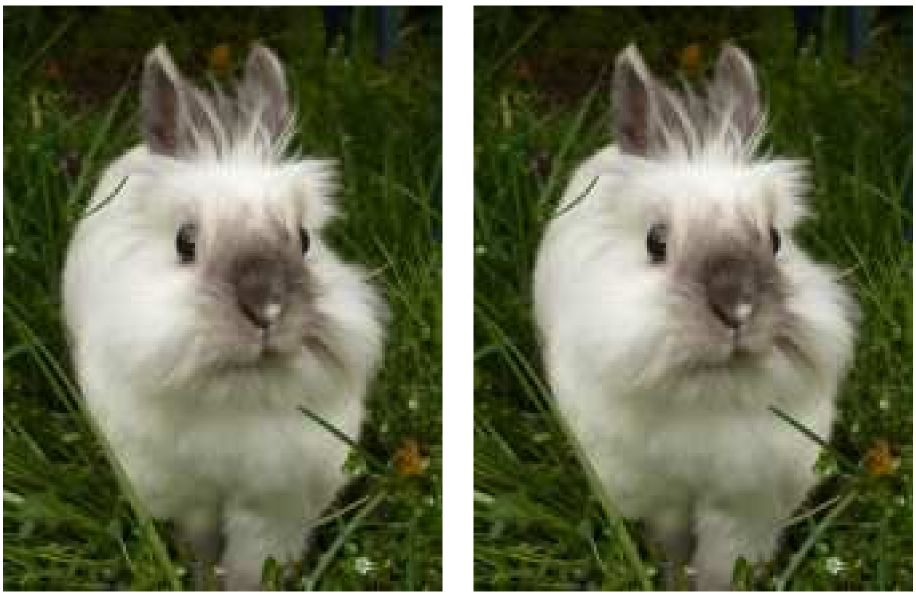

Figure 1: The original carrier (without modification) is shown on the left, while the image-bearer of hidden information counting 1000 words is presented on the right.

As we can see, comparing 'with the naked eye' the two pictures has a small chance of detecting any suspicious elements in any of them. The hidden message in this photo consists of 1,000 words, which use 6717 characters (bytes). Modification of the least significant bit of color is virtually undetectable to the human eye. It should be noted that in the real situation the original image should be removed in order to hinder the direct comparison. In a situation, where an attacker has the original image, he can make a simple 'byte by byte' comparison of both images, which immediately reveals that the picture shown in Figure 1 on the right has undergone some editing.

3. Steganoanalysis – χ^2 test

Frequently used attack is a test based on chi-square test. It assumes that in an ordinary image the distribution of least significant bits of the value 1 is not the random normal distribution. Introduction of the message may change the distribution of ones in such a way that it is close to a normal distribution.

Preparation for this test begins from counting the number of occurrences of all possible values of certain color (this is to create a histogram). Then, the values are grouped in pairs of different values in the last bit (each pair has a value of even and odd). In a typical photograph, each pair of the numbers of odd and even values should deviate from equilibrium. In the case of pictures with a hidden message, this number will move towards equality (if the image is analyzed with noise only). The most important advantage is that the attacker does not need to have an original picture.

Chi-square test is based on verifying the hypothesis that the last bits of the image have normal distribution. If there is no such distribution, the picture probably does not contain any hidden information. If not, the picture may contain a hidden message.

Implementation of chi-squared test involves drawing up statistics:

$$\chi^2 = \sum_i \frac{(o_i - e_i)^2}{e_i},$$

where o is the observed number of ones on the last bit, while e is an expected number estimated on the basis of the normal distribution [2], [3], [4].

Checking the picture with the message in Figure 1 (right) by using the test does not qualify the picture as a suspected one, the same the original picture in Figure 1 (left). However, filling in this picture the last bits by the random value (noise) results in qualifying such images as suspected.

Details of the chi-squared test auxiliary values are given in Table 1.

	Blue color	Green color
Value of χ^2	312.548	365.336
Number of degrees of freedom	123	124

Table 1: Values of chi-squared test for accepting the hypothesis counted for a picture with a hidden message.

4. Summary

In this paper, our carrier was a 24-bit bitmap. It should be noted that high-quality scanners offer 30- or 36-bit sample depth. With 30-bit RGB color is derived after 10 bits per component, and a 36-bit, each component is a 12-bit. In these cases, 3 bits (for 10-bit component) can be placed in each component, and in the second case – 5 bits of concealed information.

Another interesting issue is the fact that different areas of the picture are more or less susceptible to steganoanalysis. In areas where every pixel has a color different from its neighbors, the changes caused by placing hidden bits in them are difficult to detect.

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GLOBAL MINIMUM SEARCH USING DMC ALGORITHM WITH CONTINUOUS WEIGHTS

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Abstract. In this study we presented an algorithm for an unconstrained optimization of a continuous objective function, inspired by the Diffusion Monte Carlo method using a weight-based implementation. In this algorithm a cloud of replicas explores the solution space. Replicas are moved and evaluated after each step. Each replica carries an additional parameter (weight) which reflects the quality of its local solution. This parameter is updated after each step. Most inefficient replicas, i.e. replicas with the lowest weights, are occasionally replaced with their highest weight counterparts. In our study we present the basic implementation of the algorithm and compare its performance with other approaches, including the previously used implementation of DMC algorithm with a fluctuating population.

1. Introduction

Finding a global minimum of a nontrivial multidimensional function is a challenging problem in many areas of science and engineering [1, 2]. Many of these problems belong to the class of NP-hard problems, which make them extremely difficult to solve – except for a relatively small and simple cases.

There is a large number of algorithms for solving various types of global minimum problems (GOP), unfortunately there is no generic algorithm which can be applied to a wide selection of GOP. Most of the algorithms rely on the specific characteristics of the optimized function, although there are also more general methods, e.g. genetic algorithms [3], and other evolutionary approaches [4].

In our study we present a different approach to the global minimum problem, inspired by the Diffusion Monte Carlo method [5, 6] used in quantum physics and chemistry. We have already used a different implementation of this algorithm in other studies [7–10] with a promising results.

In this paper we present a weight-based implementation of the DMC optimization scheme and compare it with the algorithm used in the previous study.

In the next sections we discuss the details of the algorithm, show the efficiency of both approaches on a set of simple problems, and discuss the strong and weak points of both schemes.

2. Methodology

The DMC algorithm is often used in computational physics and chemistry to solve numerically a time dependent Schrödinger equation by a random walk of a cloud of replicas of a quantum system. Based on the weights distribution of replicas, the approximate wave-function of the system can be obtained. Two implementations of the algorithms are used. One, suggested by Anderson [5], involves the modifications of the population size (kill/clone process). Another approach, used by Suhm and Watts [6], uses continuous weighting method.

In this study we applied the Suhm and Watts implementation of the DMC algorithm. The following procedure was used in our simulations:

Initialize population. The initial population of replicas is randomly generated. Each replica represents a possible solution (i.e. the vector of objective function variables). The size of the population N_{rep} is an empirical parameter and depends on the problem. The additional parameter (weight) is assigned to each replica. The usual value of the initial weight is $\frac{1}{N_{rep}}$.

Move replicas. Each replica is moved randomly with displacement Δx generated from the Gaussian distribution with $\mu = 0$ and σ depending on the problem:

$$x_{n+1} = x_n + \Delta x. \tag{1}$$

Calculate objective function values and modify weights. The objective function value is calculated for each replica. The weight (w_i) of each replica is then modified according to Eq. 2, where f_i is the objective function value of the replica i, $\bar{f}$ is the mean value calculated over the total population, τ is the empirical parameter, and n is the step number,

$$w_{i,n+1} = w_{i,n} \exp\left[-(f_i - \bar{f})\tau\right]. \tag{2}$$

After the modification, weights of all replicas are renormalized to avoid numerical errors (underflows or overflows).

Exchange replicas. During the simulation, some replicas explore regions of the solution space with high objective function values. To avoid the unnecessary computations, these replicas are occasionally removed. Each time the replica is removed from the population, the replica with the largest weight is cloned and the weight is divided between both copies. This procedure eliminates worst solutions while keeping the population size constant.

Check for stopping criteria. In our study we use a fixed number of steps, although other criteria can also be used.

In our study we use two test functions, namely Ackley's problem in N-dimensions [11] and Griewangk's problem [12]. Ackley's problem is a multimodal, non-separable, differentiable and scalable function defined as:

$$\begin{aligned} F(\vec{x}) = &-20 \cdot \exp\left(-0.2\sqrt{\frac{1}{n}\cdot\sum_{i=1}^{n} x_i^2}\right) \\ &- \exp\left(\frac{1}{n}\cdot\sum_{i=1}^{n}\cos(2\pi x_i)\right) + 20 + \exp(1), \end{aligned} \tag{3}$$

where $\vec{x} = \{x_1, x_2, \ldots, x_N\}$, and $x_i \in (-32.768, 32.768)$. It has a known optimal solution for $\vec{x} = \{0, 0, \ldots, 0\}$, and $F(\vec{x}) = 0$.

Griewangk's problem is also a multimodal, non-separable, differentiable and scalable function defined as:

$$F(\vec{x}) = 1 + \sum_{i=1}^{n}\frac{x_i^2}{4000} - \prod_{i=1}^{n}\cos\left(\frac{x_i}{\sqrt{i}}\right), \tag{4}$$

where $\vec{x} = \{x_1, x_2, \ldots, x_N\}$, and $x_i \in (-600, 600)$. It has a known optimal solution for $\vec{x} = \{0, 0, \ldots, 0\}$, and $F(\vec{x}) = 0$.

In our study we performed two sets of tests for each function, using the number of dimensions $N_{dim} = 5$ and $N_{dim} = 20$, respectively. For each test we performed DMC simulations using the randomly generated initial population of $N_{rep} = 100$ replicas, and the fixed number of steps, $N_{steps} = 1000$. The empirical parameter τ was set as 0.5 in all the DMC runs, and the value of σ was equal to 0.5 for Ackley's problem and 5.0 for Griewangk's one (to account for its larger solution space). After each step 10% of the population was replaced.

The values of the simulation parameters were based on the educated guess based on results from [8], although we are aware that these values may not be optimal for our test functions.

For a reference, we used results from a blind search, simple random walk, and the DMC approach based on variable population size – DMC-VP (the details of this method can be found in [10]). In the first method, N_{rep} random solution candidates in each of the N_{steps} steps are generated and evaluated. In the second one, replicas are moved randomly according to the Gaussian distribution with a given σ value, but without the modification of the population. In order to better compare both DMC approaches, we used the same simulation parameters in both DMC runs.

The computational cost of all approaches is similar, therefore their performance can be directly compared.

All experiments were repeated three times to remove possible artifacts. Each time, the different initial population was used for random walk and DMC runs. The averaged values from these experiments were used for comparison of the algorithms efficiency.

3. Results and discussion

Results of the simulations are shown in Table 1. The values f_{init} are the best solutions from the initial (randomly generated) populations. In the case of the blind search, each sampling is independent and therefore the initial solution is defined as the result of first N_{rep} samples. The values of f_{best} are the best solutions found during the simulation. The value f_{best}/f_{init} gives the factor, by which the initial solution was improved during the simulation. All the values in Table 1 are averaged over three independent runs.

Table 1: Simulation results.

Test function:	Ackley's			Griewangk's		
$N_{dim} = 5$						
Method	f_{init}	f_{best}	$\frac{f_{best}}{f_{init}}$	f_{init}	f_{best}	$\frac{f_{best}}{f_{init}}$
Blind search	18.09	6.99	0.386	31.01	2.10	0.068
Random walk		11.27	0.623		18.39	0.593
DMC-VP		0.59	0.033		0.19	0.006
DMC-CW		**0.66**	**0.036**		**0.12**	**0.004**
$N_{dim} = 20$						
Method	f_{init}	f_{best}	$\frac{f_{best}}{f_{init}}$	f_{init}	f_{best}	$\frac{f_{best}}{f_{init}}$
Blind search	20.40	18.78	0.921	315.22	161.19	0.511
Random walk		19.44	0.953		292.42	0.928
DMC-VP		4.66	0.228		1.31	0.004
DMC-CW		**3.92**	**0.192**		**1.19**	**0.004**

The results obtained from both DMC-VP and DMC-CW simulation clearly outperform other approaches. In the case of Griewangk's problem the improvement of the initial solution is very good for both $N_{dim} = 5$ and $N_{dim} = 20$ cases, while for blind search and random walk the efficiency is not only much worse, but it also deteriorates for a larger problem size. There is no significant difference between DMC-VP and DMC-CW results, both algorithm give similar results.

Ackley's problem seems to be more challenging for all algorithms. The improvement for DMC is not as good as in the previous case, especially for $N_{dim} = 20$, where the factor of the solution improvement is only 0.2. Nevertheless, the efficiency of both DMC algorithms is much better than other methods, e.g. for $N_{dim} = 20$ both random walk and blind search were able to reduce the initial solution by less than 10%. Both DMC-VP and DMC-CW runs give similar results, although DMC-CW is slightly better for $N_{dim} = 20$.

From the efficiency point of view, both DMC algorithms give similar results. However, the DMC-CW approach has several advantages over the DMC-VP. The fixed size of the population is easier to implement and handle in the computer storage (static vs. dynamic). The DMC-VP population size can drastically change (population explosion or annihilation) if incorrect parameters are used. The DMC-CW does not use random number generator in the modification phase, reducing the possible error from the low quality random numbers (although random numbers are still used for moving replicas).

Unfortunately, there are also several disadvantages of the DMC-CW scheme: The range of weights increases very fast and must be normalized to avoid overflows and underflows. Some replacement strategy must be used to remove inefficient replicas without inhibiting the exploration process.

4. Conclusions

In this study we have presented a new global optimization approach based on the Diffusion Monte Carlo method with continuous weighting (DMC-CW). We have shown that performance of this algorithm is similar to the DMC implementation with variable population size, while current approach is easier to implement and is more stable numerically. Both DMC algorithms outperform algorithms based on the blind search and simple random walks.

Both DMC schemes used in the current study are conceptually simple, they are easy to implement on a multiprocessor machine. They require only the value of the objective function. Therefore, they are good candidates for a general global optimization scheme, although they require the large number of function evaluation, so their usage is limited to inexpensive functions, which can be quickly and cheaply calculated.

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FINDING THE SHORTEST PATH BETWEEN VERTICES IN A GRAPH HANOI

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Abstract. Three algorithms for finding the shortest path between two vertices with arbitrary labels of any fractal graph Hanoi $S(k, n)$ and the exact estimation of the minimal distance between these vertices for the case $k \geq 3$ and $n < k$ are proposed.

1. Introduction

The problem "Multi-peg Tower of Hanoi" has many variations and generalizations in different directions [1].

In this paper we consider the following generalization: initial and terminal configurations are any arbitrary (no regular) legal distributions of n disks among k pegs; our goal is to get from a given arbitrary initial state to one of the different states by the shortest path between legal configurations [2]. The shortest sequences of moves leading from a given initial configuration to a given terminal configuration is equal to the shortest path between two vertices V_i , V_j of the special graph (graph Hanoi) with special labels [3].

This problem for $k = 3$ was investigated by Andreas Hinz in [3]. If the initial configuration is any arbitrary (no regular) node and terminal configuration is perfect (regular) configuration, the shortest sequences of moves leading from a given initial configuration to a given configuration is equal, on the average, to $(2/3) * (2^n - 1)$.

We consider these problem for the case $k \geq 3$.

The main result of the proposed article is an algorithm for finding the shortest path (A, B) between arbitrary vertices with labels A and B of a fractal graph Hanoi $H(k, n)$ for the case $n < k$ and $k \geq 3$.

Some components in the labels A and B may coincide. This means that the disks in the appropriate configuration A are installed in the same manner as required in a given configuration B. Such components are painted (colored). As building a path (A, B), a growing number of components will be colored.

Bearing in mind the need to minimize the number of moves, note that when $k > n$, each disk can be moved to its desired location at most in two moves. Therefore, each component of the label A in the path (A, B) can be changed at most two times.

2. The exact estimation of the minimal distance between vertices of $S(k, n)$ for $k \geq 3$ and $n < k$

Theorem. For a graph $H(k, n)$ with $k \geq 3$ and $n < k$, the minimal distance between the vertices with arbitrary labels A and B in the path (A, B) is equal to

$$d(A, B) = 2(n - n_0) - n_1, \tag{1}$$

where n is the number of components in A, n_0 the number of invariable components, n_1 the number of components which are variable one-off.

Proof. The proof is carried out by induction with respect to n.

1. For $n = 1$ we have one variable component and two cases:

1a) Our component is variable one-off.

It is clear that $d(A, B) = 1$. On the other hand, $n_1 = 1$, $n_0 = 0$ and from our formula we have $d(A, B) = 2 * 1 - 1 = 1$.

1b) Our component does not change.
Then $d(A, B) = 0$. On the other hand, $n_1 = 0$, $n_0 = 1$ and from our formula we have $d(A, B) = 2 * (1 - 1) - 0 = 0$.

2. Our induction hypothesis for the parameter $n = m$ is

$$d(A^m, B^m) = 2(m - m_0^m) - m_1^m$$

3. For $n = m + 1$ we have three cases:

3a) The new component does not change.
Then $m_0^{m+1} = m_0^m + 1$, $m_1^{m+1} = m_1^m$, and $d(A^{m+1}, B^{m+1}) = d(A^m, B^m) + 0 = d(A^m, B^m) = 2(m - m_0^m) - m_1^m = 2(m - m_0^m + 1 - 1) - m_1^m = = 2((m + 1) - (m_0^m + 1)) - m_1^m = 2((m + 1) - m_0^{m+1}) - m_1^{m+1}$.

3b) The new component is variable one-off.
Then $m_1^{m+1} = m_1^m + 1$, $m_0^{m+1} = m_0^m$, and $d(A^{m+1}, B^{m+1}) = d(A^m, B^m) + 1 = 2(m - m_0^m) - m_1^m + 1 = 2(m - m_0^m) - m_1^m + 1 + 2 - 2 = 2((m + 1) - m_0^m) -(m_1^m + 1) = 2((m + 1) - m_0^{m+1}) - m_1^{m+1}$.

3c) The new component is variable twice.

Then $m_1^{m+1} = m_1^m$, $m_0^{m+1} = m_0^m$, and $d(A^{m+1}, B^{m+1}) = d(A^m, B^m) + 2 =$ $= 2(m - m_0^m) - m_1^m + 2 = 2((m+1) - m_0^m) - m_1^m = 2((m+1) - m_0^{m+1}) - m_1^{m+1}$.

Our statement is true for all possible cases.

Remark. This result allows us to prove the minimal feature of the path (A, B) built by our algorithm.

3. Sets of invariable components of the label A being variable one-off or invariable

The first step in the process of finding the shortest path (A, B) is to build the set of invariable components in the label A.

Algorithm 1.

Source data: $A = a_1 a_2 \dots a_n$, $B = b_1 b_2 \dots b_n$, where $[a_i], [b_i] \in \{1, \dots, k\}$, $k > n$.

Output data: Z – the set of invariable components in the label A.

1. The set of colored components $Z = \emptyset$, the set of analyzed components $M = \emptyset$. $i := 1$.

2. If $i = n + 1$, then go to step 5. Otherwise, we go (from left to right) and compare a_i with b_j (only no painted).

If $a_i \in M$ or $a_i \in Z$, then $i := i + 1$ and go to step 2.

If $[a_i] \neq [b_i]$, then $M := \{a_i\} \cup M$, $i := i + 1$ and go to step 2.

If $[a_i] = [b_i]$, then $j = i$, and go to step 3.

3. We compare b_i with b_j (only no painted), where $j < i$. If $[b_i] = [b_j]$, then $i := i + 1$ and go to step 2.

If $[b_j] \neq [b_{j-1}]$ and $[b_j] \neq [b_{j-2}]$ and $\dots$ and $[b_j] \neq [b_1]$, we compare a_i with a_j (only no painted), where $j < i$.

If $[a_j] \neq [a_{j-1}]$ and $[a_j] \neq [a_{j-2}]$ and $\dots$ and $[a_j] \neq [a_1]$, then $Z := \{a_j\} \cup Z$, $M := \{a_j\} \cup M$ and go to step 4.

If $[b_j] = [b_{j-1}]$ or $[b_j] = [b_{j-2}]$ or $\dots$ or $[b_j] = [b_1]$, or $[a_j] = [a_{j-1}]$ or $[a_j] = [a_{j-2}]$ or $\dots$ or $[a_j] = [a_1]$, then $M := \{a_j\} \cup M$, $i := i + 1$ and go to step 2.

4. We have s:=j+1. If $s > n$, then $i := i + 1$ and go to step 2.
Otherwise, we compare $[a_j]$ with $[a_s]$, where $s = j + 1, j + 2, \dots, n$.
If $[a_j] \neq [a_s]$, go to step 4.
If s is such that $[a_j] = [a_s]$, where $j < s \leq n$, we compare $[a_s]$ with $[b_s]$.
If $[a_s] \neq [b_s]$, then $M := \{a_s\} \cup M$, $i := i + 1$ and go to step 2.
If $[a_s] = [b_s]$, then $j = s$ and go to step 3.

5. Stop. All the elements of the set Z are painted components.

Each component of the label A in the path (A, B) can be changed at most two times. Elements of Z are invariable components in the label A.

The following algorithm (A2) creates the set $V1$ of components of A which are variable one-off. The algorithm A2 is based on verification of properties to be satisfied by elements from $V1$.

Properties of components to create the set $V1$:

1) $\exists a_i([a_i] = p) \wedge \forall j(j < i)([a_j] \neq p) \Rightarrow (a_i \in V1)$

2) $\exists b_i([b_i] = q) \wedge \forall j(j < i)([b_j] \neq q) \Rightarrow (a_i \in V1)$

3) $\exists a_i([a_i] \neq [b_i]) \wedge (b_{i-1} \in Z) \Rightarrow (a_i \in V1)$

4) $\exists a_i([a_i] = p = [b_i]) \wedge \exists j(j < i)([a_j] = p)(b_{j-1} \in V1) \Rightarrow (a_i \in V1)$

5) $\exists a_i([a_i] = p) \wedge \exists j(j < i)([a_j] = p) \wedge ([b_i] = q \neq p)([b_j] = q) \Rightarrow (a_i \notin V1)$

6) $\exists a_i([a_i] = p) \wedge \exists j(j < i)([a_j] = p) \wedge ([b_i] = p)([b_j] \neq p) \Rightarrow (a_i \notin V1)$

7) $\exists a_i([a_i] = p = [b_i]) \wedge \exists j(j < i)([a_j] \neq p)(b_j] = p) \Rightarrow (a_i \notin V1)$

In these propetries we assume the values $p, q \in \{1, 2, \ldots, k\}$.

We use the algorithm A2 for the proof of the minimal feature of the path which ia created by the following algorithm A3.

4. An algorithm for finding of the shortest path between the labels A and B

The next step in our process of finding the shortest path between vertices with labels A and B is to construct a sequence of vertices labeled $A_0 = A, A_1, A_2, \ldots, A_m = B$. The labels A_i and A_{i+1} have only one different element from n components.

To store intermediate data in the following algorithm A3, we use a stack structure with the LIFO maintenance order.

When we transfer the disks, we use free pegs. In the relevant operations on labels the so-called free number should be used. We should have enough free numbers, temporarily used in the construction of paths.

Algorithm 3.

Source data: The vertices labeled $A = a_1a_2 \ldots a_n$ and $B = b_1b_2 \ldots b_n$, where $[a_i], [b_i] \in \{1, \ldots, k\}, k > n$; a set Z; a set of numbers $W \neq \emptyset$; a stack $S = \emptyset$.

Output data: The shortest path $(A, B) = A_0 = A, A_1, A_2, \ldots, A_m = B$.

1. $A = a_1a_2 \ldots a_n$, $B = b_1b_2 \ldots b_n$, Z, $W \neq \emptyset$, a stack $S = \emptyset$, $A = A_0$, $m = 0$, $i := n$.

2. If $i = 0$, then go to step 5. Otherwise, we go from right to left in A_m. If $a_i \in Z$, then $i := i - 1$ and go to step 2.

Otherwise, we compare $[a_i]$ and $[b_i]$.

The following three situations are possible.

2.1. $[a_i] \neq [b_i] = l$.

If $\forall b_i(b_i \notin Z)\exists j(j < i)([b_i] \neq l)$ and $\forall a_i(a_i \notin Z)\exists j(j < i)([a_j] \neq [a_i])$ and $\forall a_i(a_i \neq l)$, then we change the value of the component a_i and have $[a_i] = l$.

We have a new label A_{m+1}, a new set W and $Z := Z \cup \{ai\}$. We change $i := i - 1$ and go to step 2.

2.2. $[a_i] = q, [b_i] = l, q \neq l$.

To the left of the component $a_i = a_{i1}$, we have the components $a_{i2}, a_{i3}, \ldots, a_{is}$ such that $[a_{i2}] = [a_{i3}] = \ldots = [a_{is}] = q$, and to the left of the component $b_i = b_{i1}$, we have the components $b_{i2}, b_{i3}, \ldots, b_{it}$ such that $[b_{i2}] = [b_{i3}] = \ldots = [b_{it}] = l$.

In this case, we change the content of components $a_i = a_{i1}, a_{i2}, \ldots, a_{i(s-1)}$ twice. The exception is made for the component a_{is}, which content we shall change one-off later. First, we change the content of each component $a_i = a_{i1}, a_{i2}, \ldots, a_{i(s-1)}$ on a free number $w \in W$. Thereafter, we write it down $a_i = a_{i1}, a_{i2}, \ldots, a_{i(s-1)}$ to a stack S and create new labels. Later we change the set $W := W'$ and go to step 4.

2.3. $[a_i] = [b_i] = l$.

Then $[a_i] = l* \in W$. Thereafter, we create a new label A_{m+1}, a new W and modernize a stack S. Then $j = i$ and go to step 3.

3. $j := j - 1$. If $j = 0$, then $i := i - 1$ and go to step 4.

Otherwise, we analyze a_j.

If $[a_j] = [a_i] = l$, we compare $[a_j]$ and $[b_j]$.

If $[a_j] = [b_j] = l$, then $[a_j] = l* \in W$, we create a new label A_{m+1} and new sets W, S.

If $[a_j] \neq [a_i] = l$, we compare $[b_j]$ and $[a_i]$. If $[b_j] \neq [a_i]$, go to step 3.

If $[bj] = [ai]$, then $S := \{a_j\} \cup S$ and go to step 3.

4. If $S = \emptyset$, then $i := i - 1$ and go to step 2.

We analyze the next element from the stack S.

Our actions are similar to those in step 2.

We have new labels and new elements for S, Z, W.

Thereafter, $S := S - \{a*\}$ and go to step 4.

5. If $\mid Z \mid = n$, go to step 6. Otherwise, $i := i - 1$ and go to step 2.

6. Stop. We have the shortest path (A, B) :

$$A = A_0 \to A_1 \to A_2 \to \ldots \to A_m = B.$$

Example. Let us build the shortest path between vertices with labels A and B in the graph $S(10,9)$ with 10^9 vertices.

1. $A = 10\,1\,3\,1\,2\,9\,9\,4\,3$, $B = 8\,1\,2\,1\,2\,4\,4\,4\,3$, $S = \emptyset$, $W = \{5,6,7,8\}$, $Z = \{a_2, a_4\}$.

2. $i = 9$. We analyze a_9. We have $[a_9] = [b_9] = 3$ and $[a_3] = 3$. We should change this component by a free number $[a_9] = 5$. Then $A_1 = 10\,1\,3\,1\,2\,9\,9\,4\,5$, $W = \{6,7,8\}$, $S = (a_9)$, $Z = \{a_2, a_4\}$.

3. We analyze a_3, because $[a_3] = [a_9] = 3$. We cannot change $[a_3] = 2$, because $[a_5] = 2$, $S = (a_3, a_9)$.

4. We analyze the element a_3 from the stack S. We cannot change $[a_3] = 2$, because $[a_5] = 2$. We cannot change $[a_5] = 2$, because $[b_3] = 2$. We have $A_2 = 10\,1\,3\,1\,6\,9\,9\,4\,5$, $W = \{7,8\}$, $S = (a_5, a_9)$, $Z = \{a_2, a_4\}$.

We analyze the element a_3 once again. We can change $[a_3] = 2$ and have $A_3 = 10\,1\,2\,1\,6\,9\,9\,4\,5$, $W = \{3,7,8\}$, $S = (a_5, a_9)$, $Z = \{a_2, a_3, a_4\}$.

We analyze the element a_5 from the stack S and have $A_4 = 10\,1\,2\,1\,2\,9\,9\,4\,5$, $W = \{3,6,7,8\}$, $S = (a_9)$, $Z = \{a_2, a_3, a_4, a_5\}$.

We analyze the element a_9 from the stack S and have $A_5 = 10\,1\,2\,1\,2\,9\,9\,4\,3$, $W = \{5,6,7,8\}$, $S = ()$, $Z = \{a_2, a_3, a_4, a_5, a_9\}$.

2. We analyze a_8. As $[a_8] = [b_8] = 4$, we should change this component on a free number $[a_8] = 5$. Then $A_6 = 10\,1\,2\,1\,2\,9\,9\,5\,3$, $W = \{6,7,8\}$, $S = (a_8)$,
$Z = \{a_2, a_3, a_4, a_5, a_9\}$.

3. We have $[b_6] = 4$ and $S = (a_6, a_7, a_8)$.

4. We cannot change $[a_6] = 4$, because $[a_7] = 9$. We should change $[a_7] = 6$. Then $S = (a_7, a_8)$ and $A_7 = 10\,1\,2\,1\,2\,9\,6\,5\,3$, $W = \{7,8\}$, $S = (a_7, a_8)$, $Z = \{a_2, a_3, a_4, a_5, a_9\}$. Thereafter, we can change $[a_6] = 4$. Then $A_8 = 10\,1\,2\,1\,2\,4\,6\,5\,3$, $A_9 = 10\,1\,2\,1\,2\,4\,4\,5\,3$, $A_{10} = 10\,1\,2\,1\,2\,4\,4\,4\,3$, $A_{11} = 8\,1\,2\,1\,2\,4\,4\,4\,3$, $W = \{5,6,7,9,10\}$, $S = \emptyset$, $Z = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$.

5. $\mid Z \mid = 9$. $d(A,B) = 11$.

From our estimation (1) with $n_0 = 2$ and $n_1 = 3$, the minimal distance between vertices with labels A and B for our example should be equal to $d(A,B) = 2(9-2) - 3 = 11$. With the help of the algorithm A3 we have built the path with the length 11. So the constructed path is the shortest one.

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ARITHMETIC OF QUADRATIC MINIMAL REDUNDANT MODULAR NUMBER SYSTEMS

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Abstract. In this paper, research in the field of modular computing structures defined on sets of Gaussians are presented. The basis of the qualitatively new technique for the organization of high-speed parallel computations in a complex plane is presented by quadratic minimum redundant modular number systems (QMRMNS).

1. Introduction

For the organization of high speed parallel information processing, the quadratic modular number systems (QMNS) are the most adjusted and convenient among number systems with complex ranges. In such systems all the modules $m_1, m_2, \dots, m_k$ can be represented as products of pairs of the complex-conjugate multipliers [1]. Using quadratic minimal redundant modular coding, the essentially higher optimality of computer procedures can be reached in comparison with the use of standard complex modular coding. This is caused by a property of arithmetical closing of the quadratic minimal redundant modular codes, i.e. by the possibility of performance of all arithmetical operations (including not modular) without transition to real components of complex ranges. Our study is aimed at implementation of this distinctive feature of QMNS.

2. Quadratic modular number system

Let us consider QMNS defined by means of the set of pairwise prime natural modules $m_1, m_2, \dots, m_k$ such that all of them are representing as products of two conjugated integer complex numbers (ICN): $m_l = p_l \bar{p}_l$, where

$p_l = p_l^{'} + ip_l^{''}$, $\bar{p}_l = p_l^{'} - i\,p_l^{''}$; $p_l^{'}$, $p_l^{''} \in \mathbf{Z}$; $p_l^{'} > 0$, $p_l^{''} > 0$ and $(p_l^{'},\ p_l^{''}) = 1$; $l = 1, 2, \ldots, k$.

According to the theorem 3 from [1] for modules of type $m = p\,\bar{p}$, the Cartesian product $\langle\cdot|_p \times \langle\cdot|_{\bar{p}}$ is isomorphic to the set

$$|\cdot|_{\|p\|} \times |\cdot|_{\|\bar{p}\|} = |\cdot|_m \times |\cdot|_m,$$

which in one's part is isomorphic to the complete set of residues $\langle\cdot|_m$. Therefore, in the QMNS with modules $m_1, m_2, \ldots, m_k$ every ICN $X = X^{'} + i\,X^{''}$ from a range $\langle\cdot|_{M_k} = |\cdot|_{M_k} \times |\cdot|_{M_k}$ can be uniquely presented by the complex modular code (MC)

$$((\mathrm{X}_1^{'};\ \mathrm{X}_1^{''}), (\mathrm{X}_2^{'};\ \mathrm{X}_2^{''}), \ldots, (\mathrm{X}_k^{'};\ \mathrm{X}_k^{''})), \tag{1}$$

where

$$(\mathrm{X}_l^{'};\ \mathrm{X}_l^{''}) = (R_{p_l}(X); R_{\bar{p}_l}(X)), \tag{2}$$

$$R_{p_l}(X) = R_{p_l}(X^{'}, X^{''}) = |X^{'} + J_l X^{''}|_{\|p_l\|},$$

$$R_{\bar{p}_l}(X) = R_{\bar{p}_l}(X^{'}, X^{''}) = |X^{'} - J_l X^{''}|_{\|\bar{p}_l\|},$$

$$J_l = \left|\frac{p_l^{''}}{p_l^{'}}\right|_{\|p_l\|}, \quad l = 1, 2, \ldots, k. \tag{3}$$

The main advantage which is reached with the use of complex-conjugate modules consists in simplicity of performance of multiplication operations modulo $p_l^{'}$ and $p_l^{''}$. The given operations require only two real multiplications modulo m_l in contrast to standard complex multiplication modulo m_l which demands four real multiplications and two real additions [1]. At the same time the addition operations with respect to the pair of complex modules $(p_l^{'}, p_l^{''})$ and real module m_l are identical with respect to complexity.

3. Quadratic minimal redundant modular number system

Efficiency of QMNS arithmetic essentially rises by use of minimal redundant modular coding which, as it is known, leads to simplification of not modular procedures [2]. According to [2, 3], in order that the MC (1) should be the minimal redundant it is enough that the real and imaginary parts of the coded ICN, $X = X^{'} + i\,X^{''}$, should be elements of the range $\mathbf{D} = \mathbf{Z}_{2M}^{-} = \{-M, -M+1, \ldots, M-1\}$, where $M = \prod_{l=0}^{k-1} m_i$, m_0 is the auxiliary natural module satisfying the condition $m_k >= 2m_0 + \rho$.

Decoding mapping in quadratic minimal redundant modular number system (QMRMNS) is realized by the relations

$$X' = \sum_{l=1}^{k-1} M_{l,k-1} |M_{l,k-1}^{-1}\, \chi_l'|_{m_l} \;+\; I(X')M_{k-1}, \tag{4}$$

$$X'' = \sum_{l=1}^{k-1} M_{l,k-1} |M_{l,k-1}^{-1}\, \chi_l''|_{m_l} \;+\; I(X'')M_{k-1}, \tag{5}$$

where $\chi_l' = |X'|_{m_l}$, $\chi_l'' = |X''|_{m_l}$, $l = 1, 2, \ldots, k$; $I(X')$ and $I(X'')$ are the interval indexes of the ICN X' and X'', respectively [2–4].

Due to the simplicity of realization of the interval-modular forms (IMF) (4) and (5), application of QMRMNS for parallel information processing in the complex plane provides exclusively high efficiency not only on modular segments of computing processes but also on the segments containing not modular operations.

As follows from [1], in the considered QMRMNS the operations of modular addition, subtraction and multiplication for any ICN $A = A' + i\,A''$ and $B = B' + i\,B''$ $(A', A'', B', B'' \in \mathbf{D})$ are realized by the same general rule:

$$\begin{gathered}((\mathrm{A}_1'; \mathrm{A}_1''), (\mathrm{A}_2'; \mathrm{A}_2''), \ldots, (\mathrm{A}_k'; \mathrm{A}_k'')) \circ ((\mathrm{B}_1'; \mathrm{B}_1''), (\mathrm{B}_2'; \mathrm{B}_2''), \ldots, (\mathrm{B}_k'; \mathrm{B}_k'')) = \\ = ((|\mathrm{A}_1' \circ \mathrm{B}_1'|_{m_1};\ |\mathrm{A}_1'' \circ \mathrm{B}_1''|_{m_1}),\ (|\mathrm{A}_2' \circ \mathrm{B}_2'|_{m_2};\ |\mathrm{A}_2'' \circ \mathrm{B}_2''|_{m_2}), \ldots \\ \ldots,\ (|\mathrm{A}_k' \circ \mathrm{B}_k'|_{m_k};\ |\mathrm{A}_k'' \circ \mathrm{B}_k''|_{m_k})),\end{gathered} \tag{6}$$

where A_l', A_l'' and B_l', B_l'' are the digits of quadratic MC of the numbers A and B modulo m_l, accordingly (see (1)–(3)), $l = 1, 2, \ldots, k$; $\circ \in \{+, -, \cdot\}$.

As for the problem of performing not-modular operations in the QMRMNS, the IMF (3) and (4) for the real and imaginary parts of elements $X = X' + i\,X''$ $(X',\, X'' \in \mathbf{D})$ are of importance for its solution.

Theorem. In the QMRMNS with pairwise prime odd modules $m_1, m_2, \ldots,$ m_{k-1}, m_k ; $m_k >= 2m_0 + \rho$ $(m_0 >= \rho)$, the IMF of the real and imaginary components of arbitrary ICN $X = X' + i\,X'' = ((\mathrm{X}_1';\ \mathrm{X}_1''), (\mathrm{X}_2';\ \mathrm{X}_2''), \ldots,$ $(\mathrm{X}_k';\ \mathrm{X}_k''))$ (see (1)–(3)) such that $(X'; X'') \in \mathbf{D} \times \mathbf{D}$ are defined as follows:

$$X' = \sum_{l=1}^{k-1} M_{l,k-1} \left| \frac{\mathrm{X}_l' + \mathrm{X}_l''}{2M_{l,k-1}} \right|_{m_l} \;+\; I(X')M_{k-1}, \tag{7}$$

$$X'' = \sum_{l=1}^{k-1} M_{l,k-1} \left| \frac{\mathrm{X}_l' - \mathrm{X}_l''}{2\,J_l\,M_{l,k-1}} \right|_{m_l} \;+\; I(X'')M_{k-1}, \tag{8}$$

where interval indexes $I(X^{'})$ and $I(X^{''})$ of ICN $X^{'}$ and $X^{''}$, respectively, are calculated in accordance with expressions (15)–(17) from [2], when

$$\chi_l = \chi_l^{'} = |X^{'}|_{m_l} = \left|\frac{\mathrm{X}_l^{'} + \mathrm{X}_l^{''}}{2}\right|_{m_l} \quad (l = 1, 2, \ldots, k), \tag{9}$$

$$\chi_l = \chi_l^{''} = |X^{''}|_{m_l} = \left|\frac{\mathrm{X}_l^{'} - \mathrm{X}_l^{''}}{2\, J_l}\right|_{m_l} \quad (J_l = \left|\frac{p_l^{''}}{p_l^{'}}\right|_{m_l}; \quad l = 1, 2, \ldots, k). \tag{10}$$

Proof. For the proof of formulated theorem it is sufficiently to evaluate the digits $\chi_l^{'}$ and $\chi_l^{''}$ by analogous digits $\mathrm{X}_l^{'}$ and $\mathrm{X}_l^{''}$ of the given number X for all $l = 1, 2, \cdots, k$. Then the received expressions should be substituted into formulas (4), (5) as well as into the calculated relations for the interval indexes $I(X^{'})$ and $I(X^{''})$.

As follows from (2) and (3), the pairs of residues $(\chi_l^{'};\, \chi_l^{''})$ and $(\mathrm{X}_l^{'}; \mathrm{X}_l^{''})$ are connected with each other by the set of equations

$$\begin{cases} |\chi_l^{'} + J_l\, \chi_l^{''}|_{m_l} = \mathrm{X}_l^{'}, \\ |\chi_l^{'} - J_l\, \chi_l^{''}|_{m_l} = \mathrm{X}_l^{''}. \end{cases} \tag{11}$$

Addition and subtraction of the equalities modulo m_l in (11) gives

$$\begin{cases} |2\, \chi_l^{'}|_{m_l} = |\mathrm{X}_l^{'} + \mathrm{X}_l^{''}|_{m_l}, \\ |2\, J_l\, \chi_l^{''}|_{m_l} = |\mathrm{X}_l^{'} - \mathrm{X}_l^{''}|_{m_l}. \end{cases} \tag{12}$$

As the real and imaginary parts of multiplicative components $p_l = p_l^{'} + i p_l^{''}$ and $\bar{p}_l = p_l^{'} - i\, p_l^{''}$ of modules $m_l = p_l\, \bar{p}_l$ satisfy the condition $(p_l^{'},\ p_l^{''}) = 1$, then on the basis of equality $(p_l^{'})^2 + (p_l^{''})^2 = m_l$, by the rule of contraries, it is easy to prove that $(m_l,\, p_l^{'}) = 1$ and $(m_l,\, p_l^{''}) = 1$. This provides an existence of the element $J_l = \left|\frac{p_l^{''}}{p_l^{'}}\right|_{m_l}$ and its multiplicative inverse $\left|\frac{1}{J_l}\right|_{m_l} = \left|\frac{p_l^{'}}{p_l^{''}}\right|_{m_l}$ in the ring $\mathbf{Z}_{m_l} = \{0, 1, \cdots, m_l - 1\}$. Furthermore, in view of the fact that all the modules $m_1, m_2, \ldots, m_k$ are odd, then in the ring $\mathbf{Z}_{m_l}$ there are also the residues $\left|\frac{1}{2}\right|_{m_l}$ and $\left|\frac{1}{2J_l}\right|_{m_l}$ for all $l = 1, 2, \ldots, k$. Hence, from (12) it follows that

$$(\chi_l^{'}\, ;\, \chi_l^{''}) = \left(\left|\frac{\mathrm{X}_l^{'} + \mathrm{X}_l^{''}}{2}\right|_{m_l} ; \left|\frac{\mathrm{X}_l^{'} - \mathrm{X}_l^{''}}{2\, J_l}\right|_{m_l}\right).$$

Therefore substitution of (9) and (10) into (4) and (5), accordingly, gives the desired IMF (7) and (8) for numbers $X^{'}$ and $X^{''}$, correspondingly.

Due to the theorem, the methods of performance of not modular operations in the real MRMNS [3] are propagated in the trivial way to QMRMNS.

Let us consider, for example, the procedure of transformation with scaling of positional code of the ICN $X = X^{'} + i\,X^{''}$ into quadratic minimal redundant MC (QMRMC). In contrast to a case of complex MRMNS [1, 3], in which the required transformation is reduced to corresponding transformations of the positional codes of the real and imaginary parts of the given number X, for forming the resulting quadratic MC both numbers $X^{'}$ and $X^{''}$ should be used jointly (see (1), (2)).

Let for the ICN $X = X^{'} + i\,X^{''}$ be necessary to generate the QMRMC of the approximate value $\hat{X} = \hat{X}^{'} + i\,\hat{X}^{''} = X/S = (X^{'} + i\,X^{''})/S$, where S is some natural scale, for example $S = 2h$, h is a natural number. According to the method of approximation described in [3, p. 240], the value $\hat{X}$ is defined by the equality

$$\hat{X} = \sum_{j=0}^{n-1} (F_j(X_j^{'},\, h) + i\,F_j(X_j^{''},\, h)),$$

where $F_j(X_j^{'},\, h) = \rfloor F_j(X_j^{'})\,2^{-h}\lfloor,\ F_j(X_j^{''},\, h) = \rfloor F_j(X_j^{''})\,2^{-h}\lfloor$; the symbol $\rfloor x \lfloor$ designates the rounding of a real number x; $X_j^{'}$ and $X_j^{''}$ are the additive components of positional codes of the integer real numbers $X^{'}$ and $X^{''}$, n is the number of such components.

Therefore, according to (1)–(3), for digits of QMRMC the following formulas are true:

$$R_{p_l}(\hat{X}) = |\hat{X}^{'} + J_l \hat{X}^{''}|_{m_l} = \left| \sum_{j=0}^{n-1} R_l^{'}(X_j^{'},\, X_j^{''},\, j) \right|_{m_l}, \tag{13}$$

$$R_{\bar{p}_l}(\hat{X}) = |\hat{X}^{'} - J_l \hat{X}^{''}|_{m_l} = \left| \sum_{j=0}^{n-1} R_l^{''}(X_j^{'},\, X_j^{''},\, j) \right|_{m_l}, \tag{14}$$

where

$$R_l^{'}(X_j^{'},\, X_j^{''},\, j) = \Big|(F_j(X_j^{'},\, h) + J_l\,F_j(X_j^{''},\, h))\Big|_{m_l}, \tag{15}$$

$$R_l^{''}(X_j^{'},\, X_j^{''},\, j) = \Big|(F_j(X_j^{'},\, h) - J_l\,F_j(X_j^{''},\, h))\Big|_{m_l}. \tag{16}$$

It is supposed that bit capacity of numbers ${X_j}'$, ${X_j}''$ and j allow the generation of residues (15) and (16) by means of table method. In this case expressions (13) and (14) can be implemented within $\lfloor \log l \rfloor + 1$ modular clock intervals, where $\lfloor x \rfloor$ designates the integer part of a real number x.

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BISIMULATION RELATION FOR SELECTED TYPES OF PROBABILISTIC AND QUANTUM AUTOMATA

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Abstract. The first step to make transitional systems more efficient is to minimize the number of their states. A bisimulation relation is a mathematical tool that helps in searching for equivalent systems, what is useful in the minimization of algorithms. For two transition systems bisimulation is a binary relation associating systems which behave in the same way in the sense that one system simulates the other and vice-versa. The definition for classical systems is clear and simple, but what happens with nondeterministic, probabilistic and quantum systems? This will be the main topic of this article.

1. Introduction

During the last fifty years many scientists have been searching for new computation models. They have developed probabilistic automata, models of finite automata over infinite words, timed automata, hybrid automata, etc. We can find their ontological review in the article [5]. In 1997 Kondacs and Watrous formulated the model of 1-way quantum finite automata (1QFA) [4]; in the same year, independently, Moore and Crutchfield defined the quantum finite automata [6]. Later, the model of quantum automata was evolved by Ambainis in many works (see e.g. [1]). This article present the definition of the bisimulation relation for different types of automata. The main focus will be on a finite reactive probabilistic automaton and a one-way quantum finite automaton.

2. Definitions of models

A transition system is a four-tuple $TS = (S, E, T, s_0)$, where S is a set of states with the initial state s_0, E is a set of events, $T \subseteq S \times E \times S$ is a transition relation (as usual, the transition (s, a, s_1) is written as $s \xrightarrow{a} s_1$) [5].

The more complex example of a transition system is a **nondeterministic finite automaton** which is a tuple $NFA = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states with the start state q_0, Σ is a finite set of input symbols, δ is a transition partial function $\delta : Q \times \Sigma \mapsto 2^Q$, $F \subseteq Q$ is a set of final states [3].

A Markov chain is the transition system, in which the probability of reaching the given state is considered. **A finite Markov chain** is a pair $MC = (Q, \delta)$, where Q is a set of states, δ is a transition function ($\delta : Q \mapsto \mathcal{D}(Q)$, where $\mathcal{D}(Q)$ is a discrete probability distribution) [10].

If $q \in Q$ and $\delta(q) = P$ with $P(q') = p > 0$, then the Markov chain is said to go from the state q to the state q' with probability p. We can find the different notations of the same phenomenon: $q \rightsquigarrow P$, $q \overset{p}{\rightsquigarrow} q'$, $\delta(q) = P$, $\delta(q)(q') = p$. Let us consider further extension of this model. **A finite reactive probabilistic automaton** is a tuple $PA = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite set of input symbols, $\delta : Q \times \Sigma \mapsto \mathcal{D}(Q)$ is a transition partial function, $q_0 \in Q$ is an initial state, $F \subseteq Q$ is a set of final (accepting) states [10].

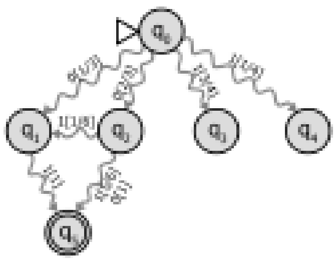

Figure 1: The PA example

After each step, a probabilistic automaton is in a superposition of states: $p_0q_0 + p_1q_1 + ... + p_nq_n$, where $p_0 + p_1 + ... + p_n = 1$.

To define a quantum automaton we need a brief introduction to the theory of quantum computing. In quantum mechanics the possible states of n-level quantum mechanical system are represented by unit vectors (called *"the state vectors"*) residing in a complex Hilbert space H_n (called *"the state space"*).

For the description of this system an ortonormal basis is used: $|x_1\rangle, |x_2\rangle, \ldots, |x_n\rangle$, where the basis vectors $|x_i\rangle$ are called the basis states. Any quantum state can be expressed by a superposition of basis states: $\alpha_1|x_1\rangle + \alpha_2|x_2\rangle + \cdots + \alpha_n|x_n\rangle$, where α_i is a complex number known as a probability amplitude. The probability of observing the state x_i is equal to $|\alpha_i|^2$, with the normalization $|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 = 1$. Time evolution of quantum system is represented by a unitary matrix (it has an inverse equal to its conjugate transpose). This is a stronger condition than that in the probabilistic systems, it causes a phenomenon of interference effects and guarantees that the time evolution of quantum state is reversible [2].

A one-way quantum finite automaton (defined by Kondacs and Watrous) is a tuple $1QFA = (Q, \Sigma, \delta, q_0, Q_a, Q_r)$, where Q is a finite set of states, Σ is a finite set of input symbols, δ is a transition partial function, $q_0 \in Q$ is an initial state, $Q_a \subset Q$ and $Q_r \subset Q$ are sets of accepting and rejecting states.

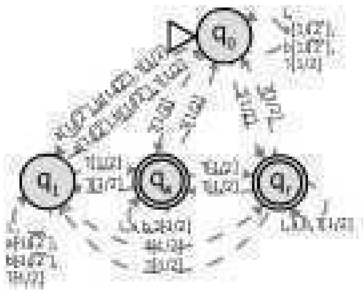

Figure 2: The 1-way QFA example

Q_a and Q_r are called non-halting states; $Q_n = Q \backslash (Q_a \cup Q_r)$. The symbols $\lfloor$ and $\rceil$ mark the beginning and the end of the word on the tape. The working alphabet of automaton is $\Gamma = \Sigma \cup \{\lfloor, \rceil\}$ [4].

The transition function $\delta : Q \times \Gamma \times Q \mapsto \mathbb{C}$ represents the amplitude with which an automaton being currently in a state $|q\rangle$, reading the symbol σ, will change a state to $|q'\rangle$. For $\sigma \in \Gamma$, V_σ is a linear transformation defined by: $V_\sigma(|q\rangle) = \sum_{q' \in Q} \delta(q, \sigma, q')|q'\rangle$ [1], [4].

3. Bisimulation

First, we must ask the question: when are two processes (states) behavioraly equivalent? Secondly, what does it mean for two systems to be equal with respect to their communication structures? The bisiumlation relation will allow us to find the answers.

Two transition systems $TS_1 = (T, \Sigma, \delta_T, t_0)$ and $TS_2 = (S, \Sigma, \delta_S, s_0)$ are bisimilar iff there is a relation $R \subseteq S \times T$ such that $(s_0, t_0) \in R$ and for all pairs $(s, t) \in R$ and for all $\sigma \in \Sigma$ the following holds: whenever $\delta_T(t, \sigma) = t'$, then there exists $s' \in S$ such that $\delta_S(s, \sigma) = s'$ and $(s', t') \in R$, and whenever $\delta_S(s, \sigma) = s'$, then there exists $t' \in T$ such that $\delta_T(t, \sigma) = t'$, and $(s', t') \in R$. The states s and t are called bisimilar which is denoted by $s \approx t$ [7], [10].

There is a simple way to determine whether two systems are bisimilar – by playing a game. This is a game between two persons: the Player and the Opponent. The Player tries to prove that systems are bisimilar, the Opponent intends otherwise. The Opponent opens the game by choosing a transition from the initial state of one of the systems. The Player has to find an equally labelled transition from the initial state of the second system, new states are the starting points for the next turn. If one of the players cannot move – the other wins this turn of the game. The Player loses abundantly if there are no corresponding transition for Opponent's move. The Player wins any infinite turn of the game or any repeated configuration.

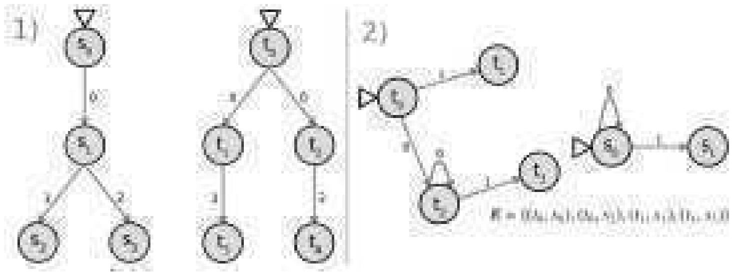

Figure 3: Example of nonbisimilar and bisimilar automata

In the first case, after reading the symbol 0 the Player must be in a state t_1 or t_2, then the road runs out to him, accordingly, or for the symbol 2 or 1. Systems are not bisimilar.

In the second example, we see that for each state and each symbol the Player will always find a corresponding way in the second automaton, so the automata are bisimilar.

4. Bisimulation for probabilistic and quantum systems

To define a bisimulation relation for probabilistic and quantum automata, one can wonder how to compare distributions of probabilities. For this purpose we use the following definitions.

Let $R \subseteq S \times T$ be a relation between sets S and T. Let $P_1 \in \mathcal{D}(S)$ and $P_2 \in \mathcal{D}(T)$ be probability distributions. Define $P_1 \equiv_R P_2$ iff there exists a distribution $Pr \in \mathcal{D}(S \times T)$ such that $Pr(s,T) = P_1(s)$ for any $s \in S$, $Pr(S,t) = P_2(t)$ for any $t \in T$, $Pr(s,t) \neq 0$ iff $(s,t) \in R$ [10].

Let R be an equivalence relation on the set S and let $P_1, P_2 \in \mathcal{D}(S)$ be probability distributions. Then $P_1 \equiv_R P_2 \iff \forall C \in S/R : P_1(C) = P_2(C)$, where C is an abstract class [10].

Let R be an equivalence relation on the set S, A be an arbitrary set, and let $P_1, P_2 \in \mathcal{D}(S)$ be probability distributions. Then $P_1 \equiv_{R,A} P_2 \iff \forall C \in S/R, \forall a \in A : P_1(a, C) = P_2(a, C)$ [10].

An equivalence relation on a set of states Q of a Markov chain (Q, δ) will be a bisimulation relation iff $\forall(q,t) \in R$ the following holds: if $\delta(q) = P_1$, then there exists $\delta(t) = P_2$ such that $P_1 \equiv_R P_2$.

Let $PA_1 = (S, \Sigma, \delta_S)$ and $PA_2 = (T, \Sigma, \delta_T)$ be two probabilistic automata, then there exists a bisimulation relation $R \subseteq S \times T$ if for all pairs $(s,t) \in R$ and for all $\sigma \in \Sigma$ we have: if $\delta_S(s, \sigma) = P_1$, then there exists a probability distribution P_2 such that for some $t \in T$ there exists $\delta_T(t, \sigma) = P_2$ and $P_1 \equiv_{R,\Sigma} P_2$ [10].

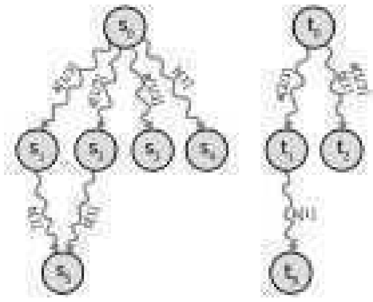

Figure 4: Bisimilar PA

Finally let us go to the bisimulation of the quantum automata, in this case we have to compare the linear operators.

For the given operator V_σ we define $v_\sigma(S) = \sum_{q' \in S} |\delta(q, \sigma, q')|^2$ (the sum of squares of the values of ruthless amplitudes), where $S \subseteq Q$.

Let R be an equivalence relation on the set S, A be an arbitrary set, and V_1, V_2 be unitary operators corresponding to transitions of the quantum system. Then $V_1 \equiv_{R,A} V_2 \iff \forall C \in S/R,\ \forall a \in A : v_{1a}(C) = v_{2a}(C)$.

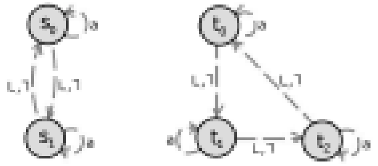

Figure 5: Bisimilar 1-way QFA

Let $1QFA_1 = (S, \Sigma, \delta_S)$ and $1QFA_2 = (T, \Sigma, \delta_T)$ be two one-way quantum finite automata. Then there exists a bisimulation relation $R \subseteq S \times T$ if for all pairs $(s,t) \in R$ and for all $\sigma \in \Sigma$ we have: if $V_{1\sigma}(|s\rangle) = \sum_{s' \in S} \delta_S(s, \sigma, s')|s'\rangle$, then there exists $V_{2\sigma}(|t\rangle) = \sum_{t' \in T} \delta_T(t, \sigma, t')|t'\rangle$ such that $V_1 \equiv_{R,\Sigma} V_2$.

5. Summary

A bisimulation relation can be a great tool to search for systems that simulate each other, and therefore their behavior is analogous to the same symbols, actions, impulses.

The simple way for checking whether or not two classical systems are bisimilar is a game, but for probabilistic and quantum systems we have to consider the sum of probabilities and amplitudes.

Bisimulation can also be a foundation for relations useful, for example, in minimization of systems [8], [9].

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AUTOMATIC SEARCH OF AUTOMORPHISMS OF WITT RINGS

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Abstract. The investigation of strong automorphisms of Witt rings is a difficult task because of variety of their structures. Cordes Theorem, known in literature as Harrison-Cordes criterion (cf. [1, Proposition 2.2], [3, Harrison's Criterion]), makes the task of describing all the strong automorphisms of a given (abstract) Witt ring $W = (G, R)$ easier. By this theorem, it suffices to find all such automorphisms σ of the group G that map the distiguished element -1 of the group G into itself (i.e. $\sigma(-1) = -1$) in which the value sets of 1-fold Pfister forms are preserved in the following sense: $\sigma(D(1, a)) = D(1, \sigma(a))$ for all $a \in G$. We use the above criterion and the well-known structure of the group G as a vector space over two-element field $\mathbb{F}_2$ for searching all automorphisms of this group. Then we check Harrison-Cordes criterion for found automorpisms and obtain all the automorpisms of a Witt ring W.

The task is easy for small rings (with small groups G). For searching of all strong automorpisms of bigger Witt rings we use a computer which automatizes the procedure described above. We present the algorithm for finding strong automorphisms of a Witt rings with finite group G and show how this algorithm can be optimized.

1. Searching of automorphisms of Witt rings

Consider Witt rings in terminology of Marshall [2]. Let $W = (R, G)$ be a Witt ring, where the group G is finite. We are interested in finding all automorphisms of the given finitely generated Witt ring W. By definition, the map σ is an automorphism of a Witt ring W if σ is such an automorphism of the ring R that $\sigma(G) = G$. Cordes in [1] has formulated the usefull criterion for σ to be an automorphism of a Witt ring: any $\sigma \in \mathrm{Aut}(G)$ induces an automorphism of a Witt ring W iff

1) $\sigma(-1) = 1$;

2) $D(1, \sigma(a)) = \sigma(D(1, a))$ for all $a \in G$,

where by $D(1, a)$ we denote the value set of a 1-fold Pfister form $(1, a)$. The above statement, called nowadays the Harrison-Cordes criterion (cf. [3]), allows us to investigate automorphisms of simpler structure of the group G instead of automorphisms of the ring R.

As we know, G is a group of exponent 2, so it can be consisered as a vector space over $\mathbb{F}_2$. Hence, we can consider automorphisms of vector space $G(\mathbb{F}_2)$ (see Algorithm 1). For this purpose we choose a basis $\mathcal{B}$ of that vector space (step 1). If $|G| = 2^n$, then $\mathcal{B}$ of vector space $G(\mathbb{F}_2)$ consists of n elements of G. If we choose another basis $\mathcal{B}'$ (step 2), we can create a map between $\mathcal{B}$ and $\mathcal{B}'$. Finding all such bases we can build all maps from $\mathcal{B}'$ to other bases including their permutations of bases (step 3). Then we extend the obtained maps to a whole group G via known representation of vectors of G as a combinations of elements of the basis $\mathcal{B}$. Finally, we have to check whether the obtained automorphisms of the group G fullfil the Harrison-Cordes criterion. As a result, we get all such automorphisms of the vector space $G(\mathbb{F}_2)$ which can be extended to automorphisms of Witt ring $W = (R, G)$. This is equivalent to the case that we have found all strong automorphisms of W.

Algorithm 1 Search for automorphisms of vector space $G(\mathbb{F}_2)$

INPUT: $\dim A = n$, $|G|$;
OUTPUT: A set of all automorphisms of $G(\mathbb{F}_2)$
1: Finding a basis $\mathcal{B}$ of $G(\mathbb{F}_2)$.
2: Search for every bases of $G(\mathbb{F}_2)$.
3: Make maps from basis $\mathcal{B}$ to every bases of $G(\mathbb{F}_2)$ (including $\mathcal{B}$ and permutations of all bases).
4: Extend created maps to all group G.
5: Check if the obtained maps fulfill the Harrison-Cordes criterion.

The algorithm 1 is easy to handle in the case when the cardinality of the group G generating a Witt ring $W = (R, G)$ is small. Then we can calculate every automorphisms of W by hand (see example 1). When cardinality of G grows up, the task becomes much complicated and takes a lot of time (compare example 2). In order to accelerate calculation, we have written a computer program which realizes the above algorithm. Thanks to the program, we can find automatically all automorphisms for all non-isomorphic Witt rings with the group G being finite. The only limitation is the power of a computer and the time needed for its work.

Example 1. 1. Let $W \cong \mathbb{Z}/2\mathbb{Z}[C_4]$ be the group ring of 4-element cyclic group C_4 with coefficients in the 2-element ring $\mathbb{Z}/2\mathbb{Z}$. Then W is a Witt ring with 4-element group G. The vector space $G(\mathbb{F}_2)$ has two-element basis, and it is easy to calculate all their 6 automorphisms.
2. Take $W \cong \mathbb{Z}/2\mathbb{Z}[C_2]^4$ – a Witt ring, which is the 4th power of a Witt ring $\mathbb{Z}/2\mathbb{Z}$. Then a suitable vector space $G(\mathbb{F}_2)$ has cardinality 16 and 4-element basis. It turns out that $|\mathrm{Aut}(G(\mathbb{F}_2))| = 20160$, and it is rather difficult task to calculate all this automorphisms by hand.

2. Optimalisation of algorithm and experimental results

In this section we shall show how we have optimized algorithm 1 in order to accelerate searching automorphisms with the help of computer. We make some rationalization in the step 1.

We start from the following equivalence relation $\sim$ which determines the equivalence classes of elements of a group G with respect to equicardinality of the value sets of 1-fold Pfister forms. We say that $g_1 \sim g_2$ iff $|D(1, g_1)| = |D(1, g_2)|$. The relation $\sim$ introduces the partition of the set of all elements of group G into the equivalence classes, which we call *types* (of elements) and denote by T (with subscripts when needed). For the sake of simplicity of notation we index them with m consecutive natural numbers, where m is the number of all the equivalence classes. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of the space $G(\mathbb{F}_2)$. Then $\overline{T} = \{T_{j1}, \ldots, T_{jn}\}$ is called the *type of basis B* if the elements b_i are of type T_{ji} for each $1 \leq i \leq n$. In general, a system $(w_1, w_2, \ldots, w_n)$ of elements of G is of type $\overline{T} = (T_{j1}, \ldots, T_{jn})$ if $w_i \in T_{ji}$ for $1 \leq i \leq n$. Clearly, n and m do not have to be identical. We do not assume that the sets T_{ji} are pairwise different for $1 \leq i \leq n$. Repetitions are allowed.

It seems essential to start off with such a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ for which the number of all possible tuples $(w_1, w_2, \ldots, w_n)$ is the smallest possible. Hence, our goal is to find a basis of such a type $\overline{T} = \{T_{j1}, \ldots, T_{jn}\}$ which is *minimal* in the sense that a system of types $\{T_{j1}, \ldots, T_{jn}\}$ has the smallest possible cardinality.

The algebraic program AP (see algorithm 2) searches for a minimal type, in which a basis exist, by computing the determinants. Here the determinants are computed according to the Laplace expansion algorithm. The algebraic program AP takes the dimension n of a vector space $G(\mathbb{F}_2)$ as its input as

Algorithm 2 procedure $\mathbf{AP}(n, \mathcal{T})$

Variable: $min \leftarrow 0$; $j \leftarrow 1$; $j \leftarrow 1$; $\overline{T} \leftarrow \emptyset$;

$<g>$ /* n-tuple of elements in type $\overline{T}$ */

Return values:

$AP():\overline{T}$ /* a minimal type $\overline{T}$ in which a basis exists */

det() : {1,0} /* a value of determinant of matrix $\mathcal{M}$ */

$build\_matrix():\mathcal{M}$ /* a matrix of scalars of n elements of group G */

INPUT: a dimension n and a set $\mathcal{T}$ of all types $\overline{T}$

OUTPUT: a minimal type $\overline{T} = (T_{j1}, T_{j2}, \dots, T_{jn})$ in wich a basis exists

1: **repeat**
2: $\quad \overline{T} \leftarrow \overline{T}_j \in \mathcal{T}$
3: $\quad j \leftarrow j + 1$
4: $\quad min \leftarrow 0$
5: $\quad$ **repeat**
6: $\qquad <g> \leftarrow <g>_i \in \overline{T}$
7: $\qquad \mathcal{M} \leftarrow build\_matrix(<g>)$
8: $\qquad$ **if** $\det(\mathcal{M}) = 1$ **then**
9: $\qquad\quad$ min = 1;
10: $\qquad$ **end if**
11: $\qquad i \leftarrow i + 1$
12: $\quad$ **until** (min = 1)
13: **until** (min = 1)
14: **return** $\overline{T}$

well as a set $\mathcal{T}$ of types $\overline{T}$ in the cardinality increasing order, represented as the lists of 0-1 sequences. The AP program gives an n-tuple of types (step 2), generates n elements of a group G belonging to these types (step 6). Then for each n-tuple it computes the determinant of the $n \times n$ matrix of scalars (step 8 and 7, respectively). If for a given type the determinant of all such matrices is equal 0, the AP program considers the next type. Otherwise, it returns a minimal type in which a basis exists and terminates.

Notice that if there is no linearly independent set of vectors of type $\overline{T}$, the algebraic AP program must perform all the computations of determinant of $|\overline{T}|$ possible systems of n vectors. Since the number of combinations depends exponentially on a dimension of vector space $G(\mathbb{F}_2)$, so the complexity of algebraic program AP depends exponentially on the dimension too.

Our algorithm is implemented in C++ language. Our experiments were carried out on an IBM PC machine with an Intel Pentium IV 3.2 GHz processor, 1024 MB RAM memory and Linux operating system.

Below we present an example involving a Witt ring with rather big group

G, which automorphisms cannot be calculated by hand and the group of all automorpisms is not desribed until now.

Example 2. Consider a Witt ring $W = (R, G)$ with the group $G = G_1 \times G_2 \times G_3$ such that $|G| = 2^9$, where

$G_1 = \{(1,1,1),(1,-1,1),(1,1,x),(1,1,y),(1,1,xy),(1,-1,x),(1,-1,y),(1,-1,xy)\}$,

$G_2 = \{(1,1,1),(1,-1,1),(1,1,x),(1,1,y),(1,1,xy),(1,-1,x),(1,-1,y),(1,-1,xy)\}$,

$G_3 = \{(1,1,1),(-1,-1,-1),(1,1,-1),(1,-1,1),(1,-1,-1),(-1,1,1),(-1,1,-1),(-1,-1,1)\}$.

According to Algorithm 1, we choose some basis of $G(\mathbb{F}_2)$, namely $\mathcal{B} = (g_2, g_3, g_4, g_9, g_{17}, g_{25}, g_{74}, g_{129}, g_{257})$. Let us check what type has that basis. Knowing the value sets of 1-fold Pfister forms we decompose the group G into 5 types. For the sake of simplicity of notation, we denote the type T_j by its index j, for $1 \le j \le m$; for example, the type T_3 will be denoted by 3. Hence, the type T of n-elements of group G will be n-tuple of indices $(j_1, j_2, \dots, j_n)$ of types, where $\forall_{i=1}^{n} 1 \le j_i \le m$ and $n = \dim G(\mathbb{F}_2)$.

Type	1	2	3	4	5
Cardinality	56	336	8	104	8

It follows that the type of our basis $\mathcal{B}$ is $\overline{T} = (1,1,1,4,2,2,5,1,1)$ and we must find all bases in $|\overline{T}| = 178\,862\,731\,407\,360$ possible systems of n vectors.

With the help of algorithm 2, we have found a basis in minimal type. Last 10 from 117 checked types are presented in Table 1. As we can see, the type $(5,4,4,4,4,3,3,3,3)$ has the smallest cardinality between all the types in which basis of $G(\mathbb{F}_2)$ exists.

Type $\overline{T}$	$\lvert\overline{T}\rvert$	AP	
		sec.	result
(5,3,3,3,1,1,1,1,1)	213909696	293.07	NOT EXISTS
(5,5,5,5,4,3,3,1,1)	251130880	269.96	NOT EXISTS
(5,5,5,4,3,3,3,1,1)	251130880	302.70	NOT EXISTS
(5,5,5,5,4,4,4,3,3)	285539072	314.83	NOT EXISTS
(5,5,5,4,4,4,3,3,3)	285539072	165.72	NOT EXISTS
(5,5,5,3,3,1,1,1,1)	287955360	316.66	NOT EXISTS
(5,5,5,5,5,3,2,1,1)	289766400	265.74	NOT EXISTS
(5,5,3,3,3,3,2,1,1)	289766400	343.82	NOT EXISTS
(5,5,5,5,5,4,4,4,4)	321868820	294.82	NOT EXISTS
(5,4,4,4,4,3,3,3,3)	321868820	153.97	EXISTS
	Sum:	3850.03	

Table 1: The results for the group of example 2.1.

The algorithm 1 has been improved yet. In [4] the autors showed that for Witt rings, which have $-1 \neq 1$ in G, we can choose such a basis of minimal type that the distinguished element -1 is one of the vector of that basis. In the next step all bases of minimal type with one fixed vector -1 are searched. Therefore, we can map a choosen basis into all another ones such that the map preserves the vector -1. In that way the first condition of the Harrison-Cordes criterion is fulfilled and we reduce thea number of choosen vectors in each basis to $n-1$.

Another rationalization proposed in [4] consists in coding the problem of searching of minimal type as a propositional formula and using newest SAT-solvers – very effective tool for veryfing satisfiability of that formula. The autors have showed that the time of finding minimal type with the help of SAT-solvers is shorter than in our algorithm 2.

We claim that algorithm 1 can be improved in another way too. We leave it for our future work.

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EVALUATION OF A METHOD FOR ENUMERATING THE MOST PROBABLE PACKET ARRANGEMENTS

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Abstract. The problem of calculating routing probabilities in packet synchronous networks, such as optically-switched packet networks, involves enumerating packet arrangements. Previously we published a method for enumerating the most probable packet arrangements, and in this article we present its novel graph interpretation and evaluate the method for several stop conditions and for the Poisson and geometric probability distributions of arriving packets.

1. Introduction

Optical packet switching (OPS) is a technology that could be deployed in future optical networks with packet switching [1]. In synchronous OPS packets are sent out a node at the beginning of a time slot, and packets last one time slot. Performance evaluation of OPS networks can be used for off-line network planning and on-line evaluation to proactively provision optical resources with OpenFlow [2].

One of the problems in the performance evaluation of synchronous OPS is the enumeration of the possibilities of packet arrivals at a node, which we call *packet arrangements* or just arrangements. Enumerating all arrangements may be impractical and unnecessary, and so in [3] we published an algorithm which efficiently finds any number of the most probable arrangements.

2. Problem statement

A flow is a sequence of packets grouped together according to a given criterion, such as the same source and destination nodes. A flow is described by a probability distribution of the number of packets that arrive at a node in a time slot. There are R flows for which the probability distribution functions are given by vector $F = (f_1, \dots, f_r, \dots, f_R)$. We assume that the distribution functions f_r are independent.

An arrangement takes place at a node in every time slot, and it is described by a discrete random variable $X = (x_1, \dots, x_r, \dots, x_R)$, where x_r is a discrete random variable of the number of packets that belong to flow r. The probability of arrangement X can be calculated as given by Eq. (1):

$$P(X, F) = \prod_{r=1}^{R} f_r(x_r). \tag{1}$$

Arrangement X can be alternatively described by a random variable $Y = (y_1, \dots, y_r, \dots, y_R)$, where the number of packets x_r is the y_rth most probable for flow r. For distribution $f_r(x_r)$, we can find the value of x_r that yields the highest value of $f_r(x_r)$, i.e. the mode, and denote it $\Gamma_r(1)$. The next most probable number is $\Gamma_r(2)$, and $\Gamma_r(y_r)$ for the further y_r-th most probable number of packets.

The number of possible arrangements is the product of the domain sizes of flow distributions. If a distribution for a flow is infinite, the number of arrangements is infinite too. Even if the number of arrangements if finite, enumerating all of them may be unnecessary.

Instead of enumerating arrangements in an arbitrary order, we find a sequence $X_e = (x_{1,e}, \dots, x_{r,e}, \dots, x_{R,e})$ of the most probable arrangements, where X_1 is the most probable one, while the next arrangements X_e have nonincreasing probabilities, i.e. $P(X_e, F) \geq P(X_{e+1}, F)$.

3. Algorithm

The algorithm finds efficiently any number of the most probable arrangements, and stops according to the stop condition provided by the user. In this section we present a novel graph interpretation of the algorithm.

There is given a weighted directed graph. A vertex represents an arrangement, and the label of the vertex represents the probability of the arrangement. The source vertex represents arrangement Y_1. An edge leaving vertex Y_e represents a possible way of obtaining a different arrangement by changing in arrangement Y_e the number of packets of a single flow to the next most

$$\begin{tikzcd}[row sep=tiny]
 & & (1,3) & \cdots \\
 & (1,2) \arrow[ur] \arrow[dr] & & \\
(1,1) \arrow[ur] \arrow[dr] & & (2,2) & \cdots \\
 & (2,1) \arrow[ur] \arrow[dr] & & \\
 & & (3,1) & \cdots
\end{tikzcd}$$

Figure 1: A graph of the possible arrangements for two flows.

probable value. The weight of an edge is negative and represents the decrease in probability of arrangement Y_e caused by the change.

Figure 1 depicts a graph for two flows, but the vertex labels and edge weights are omitted. The source vertex $Y_1 = (1, 1)$ is the most probable, and from it two new arrangements can be obtained: $(1, 2)$ and $(2, 1)$, but which of them is Y_2 depends on their probabilities.

In such a graph, the algorithm finds the longest paths from the source vertex to other vertexes. For this purpose we adapted the Dijkstra algorithm, even though the input data violates the conditions of the Dijkstra algorithm, that the edges are nonnegative and that vertex labels are nondecreasing.

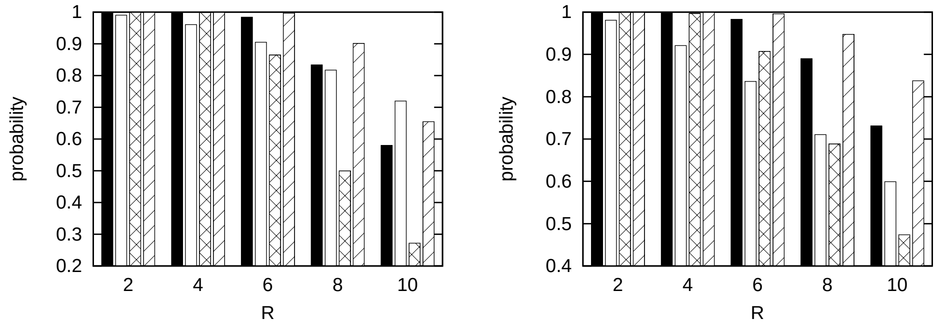
The adaptation involved a number of changes. First, the vertexes in the priority queue are sorted not with increasing labels, but with decreasing labels, as we search for vertexes with decreasing probabilities. Second, the relaxation of an edge is performed not when there is an edge that yields a lower label of a vertex, but when an edge yields a higher label of a vertex. Finally, the source vertex is labeled not with zero, but with the probability of the most probable arrangement.

4. Evaluation

The evaluation discussed in this section was implemented as part of the test suite of the OPUS software. The program file is test/test_arr_queue.cc, and is available for download at [4].

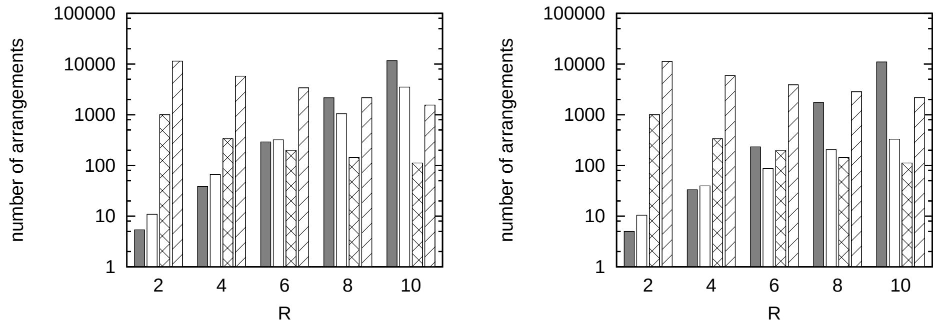
Figures 2, 3, 4 and 5 show results of finding arrangements for five stop conditions. In each of the figures there are two subfigures, one for the flows of the Poisson distribution, the other for the flows of the geometric distribution. The reported values are a function of the number of flows R.

A stop condition, when met, causes the algorithm to stop finding further arrangements. A number of stop conditions can be devised, but we define five stop conditions. They can be used separately or in a compound statement. The algorithm stops when: 1) a number of found arrangements reaches a given value, 2) the aggregate probability of found arrangements exceeds a given value, 3) the ratio of the probability of the found arrangement to



(a) The Poisson case (b) The geometric case

Figure 2: Aggregate probability of considered arrangements.

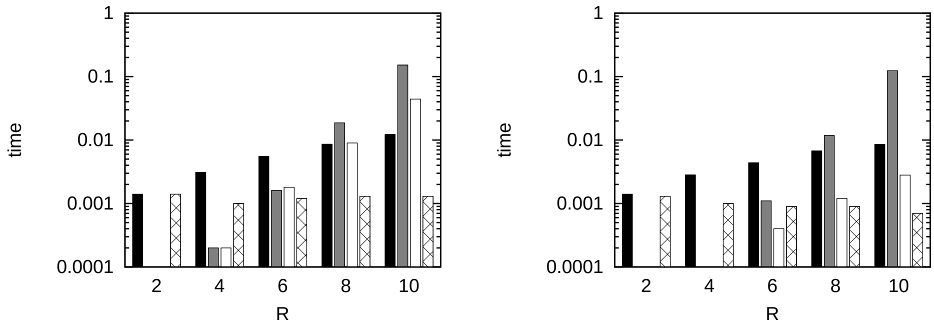


(a) The Poisson case (b) The geometric case

Figure 3: The number of considered arrangements.

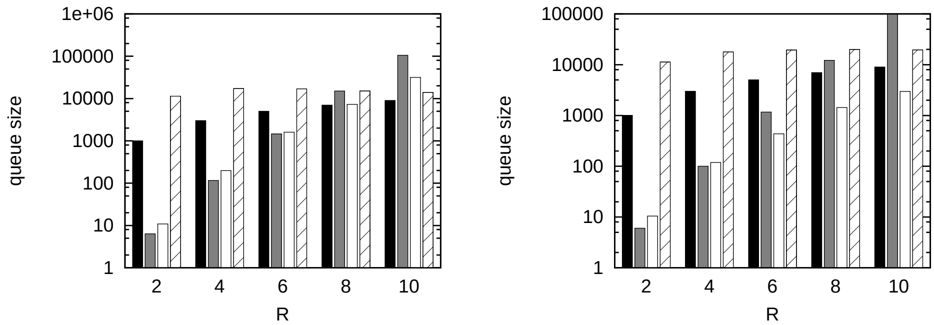
the probability of the most probable arrangement drops below a given value, 4) the size of the priority queue of arrangements exceeds the given value, 5) the execution time exceeds the given value.

Each bar in the figures reports a mean value of a sample of 100 test cases. A test case has the parameters of its flows chosen at random: for the Poisson flows the intensities $\lambda \in (0, 1)$, for the geometric flows the probabilities of success $p \in (0.5, 1)$. The standard errors of the means are small, and they are not reported in the figures. Results for each of the five conditions are shown in each figure with the same bars: the black bars () are for the limit of 1000 on the number of arrangements, the gray bars () are for the limit of 0.5 on the aggregate probability, the white bars () are for the limit of 10^{-2} on the probability ratio, the checked bars () are for the limit of 1000 on the arrangement queue size, and the slashed bars () are for the limit of 10 ms on the execution time.



(a) The Poisson case (b) The geometric case

Figure 4: The time needed to find the arrangements.



(a) The Poisson case (b) The geometric case

Figure 5: The size of the queue.

Figure 2a shows the aggregate probability of the considered arrangements. For up to 6 flows, the number of considered arrangements were enough to result in the aggregate probabilities above 0.9. For 10 flows the probability was slightly above 0.7. Clearly, the stop condition for the number of considered arrangements yields aggregate probability that decreases faster than a linear function as the number of flows decreases. The other conditions gave results of similar characteristic, but the time condition gave the least decreasing aggregate probabilities as the number of flows increased. Figure 2b shows similar results for the geometric flows. Figures 3, 4, 5 show the results on a logarithmic scale. Most of the results are an exponential function of the number of flows, as the bars in the plots raise and fall like a linear function. However, there is one exception: the execution time with the limit on the queue length (Fig. 5, slashed bars) depends slightly on the number of flows, which suggests that the operation of inserting of an element into the queue dominates.

5. Conclusion

The evaluation showed that the proposed algorithm finds efficiently the most probable arrangements, because we adopted the efficient Dijkstra algorithm. However, as the number of flows increases, the number of arrangements required to obtain a given aggregate probability increases exponentially. For a small number of flows the algorithm performs satisfactorily, but for larger number of flows the algorithm is rendered unusable, because the number of arrangements to consider (i.e. the size of the problem) grows exponentially.

In the future work, one could research how the algorithm performed had the distributions been correlated.

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VERIFYING RTECTL PROPERTIES OF A TRAIN CONTROLLER SYSTEMS

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Abstract. In the paper we deal with a classic concurrency problem – a *faulty train controller system* (FTC). In particular, we formalise it by means of finite automata, and consider several properties of the problem, which can be expressed as formulae of a soft real-time branching time temporal logic, called RTECTL. Further, we verify the RTECTL properties of FTC by means of SAT-based bounded model checking (BMC) method, and present the performance evaluation of the BMC method with respect to the considered problem. The performance evaluation is given by means of the running time and the memory used.

1. Introduction

Concurrency is a property of systems that allows to perform multiple computations in parallel and it is ubiquitous in computer science today, for example, it is the core feature of today operating systems. Concurrency is widespread but error prone - typical error includes race conditions and mutual exclusion violations; errors that are unknown in sequential computations. Traditional reliability measures such as simulation and testing fail in the presence of concurrency, due to the difficulties of reproducing erroneous behaviour.

Model checking [3] is an automated technique designed to establish in a formal and precise way that specific properties are satisfied by a given system. Its main idea consists in representing a (finite) state system as a Kripke structure M, expressing a specification by a logical formula φ, and checking automatically whether the formula φ holds in the model M. Unfortunately, the practical applicability of model checking is strongly restricted by the state-space explosion problem, which is mainly caused by representing concurrency of operations by their interleaving. Therefore, there are many different reduction techniques aimed at minimising models. The major methods include application of partial order reductions [10], symmetry reductions [6], abstraction techniques [4], OBDD-based symbolic storage methods [1], and SAT-based bounded [2, 9] and unbounded [8] model checking.

The RTCTL language [5] is a propositional branching-time temporal logic with bounded operators, which was introduced to allow specification and reasoning about time-critical correctness properties. It makes possible to directly express bounded properties like, for example, "property φ will occur in less than 10 unit time", or "property φ will always be asserted between 2 and 8 unit time". Note that properties like above can be expressed using nested applications of the next state operators, however the resulting CTL formula can be very complex and cumbersome to work with. RTCTL, by allowing bounds on all temporal operators to be specified, provides a much more compact and convenient way of expressing time-bounded properties.

In the paper we investigate a finite state systems modelled via a network of finite automata. In particular, we deal with a faulty train controller system (adapted from [7]) – a classic concurrency problem. We model it as a network of finite automata, and verify using a SAT-based bounded model checking (BMC) method for RTCTL properties.

The rest of the article is structured as follows. In the next section we provide the main formalisms used throughout the paper, i.e., finite automata, the RTCTL language together with its universal and existential subsets, and SAT-based BMC for the existential part of RTCTL (RTECTL). In section 3 we show how our SAT-based BMC for RTECTL works by means of the faulty train controller system. In section 4 we conclude our paper.

2. Preliminaries

2.1. Finite automata and parallel composition

Given is a set PV of propositional variables, each of which represents fundamental properties of the system in question. A *finite automaton*, we consider in the paper, is a mathematical structure $\mathcal{A} = (\Sigma, S, s^0, T, V)$ that consists of a finite set of actions (Σ), a finite set of states (S), an initial state (s^0), a transition relation ($T \subseteq S \times S$) defining rules for going from one state to another depending upon the input action, and a valuation function ($V : S \rightarrow 2^{PV}$)

which assigns to every state a set of propositional variables that are assumed to be true at this state.

Typically concurrent systems are designed as collections of interacting computational processes that may be executed in parallel. Therefore, we assume that a concurrent system is modelled as a network of automata that run in parallel and communicate with each other via executing shared actions. There are several ways of defining a parallel composition of a few automata. We adapt the standard definition, namely, in the parallel composition the transitions not corresponding to a shared action are interleaved, whereas the transitions labelled with a shared action are synchronised.

The following definition formalises the above discussion. Let $\mathcal{A}_i = (\Sigma_i, S_i, s_i^0, T_i, V_i)$ be an automaton, for $i = 1, \dots, m$. We take $\Sigma = \bigcup_{i=1}^m \Sigma_i$, and for $\sigma \in \Sigma$ we define a set $\Sigma(\sigma) = \{1 \leq i \leq m \mid \sigma \in \Sigma_i\}$ that gives the indices of the components that synchronise at σ. *A parallel composition of m automata* $\mathcal{A}_i$ is the automaton $\mathcal{A} = (\Sigma, S, s^0, T, V)$, where $\Sigma = \bigcup_{i=1}^m \Sigma_i$, $S = \prod_{i=1}^m S_i$, $s^0 = (s_1^0, \dots, s_m^0)$, $V((s_1, \dots, s_m)) = \bigcup_{i=1}^m V_i(s_i)$, and a transition $((s_1, \dots, s_m), \sigma, (s_1', \dots, s_m')) \in T$ iff $(\forall j \in \Sigma(\sigma))\ (s_j, \sigma, s_j') \in T_j$ and $(\forall i \in \{1, \dots, m\} \setminus \Sigma(\sigma))\ s_i' = s_i$.

2.2. The RTCTL language

Let $p \in PV$, and I be an interval in $\mathbb{N} = \{0, 1, \dots\}$ of the form: $[a, b)$ and $[a, \infty)$, for $a, b \in \mathbb{N}$[1]. Hereafter by *left*(I) we denote the left end of the interval I, i.e. *left*$(I) = a$, and by *right*(I) the right end of the interval I, i.e. *right*$([a, b)) = b - 1$ and *right*$([a, \infty)) = \infty$. The language RTCTL is defined by the following grammar:

$$\varphi := \mathbf{true} \mid \mathbf{false} \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathrm{EX}\varphi \mid \mathrm{AX}\varphi \mid \mathrm{E}(\varphi \mathrm{U}_I \varphi) \mid \mathrm{A}(\varphi \mathrm{U}_I \varphi) \mid \mathrm{EG}_I \varphi \mid \mathrm{AG}_I \varphi$$

U_I and G_I are the operators for bounded "until" and "globally", respectively. E and A are the existential and universal path quantifiers, respectively. The remaining bounded temporal operators are defined in the standard way: $\mathrm{O}(\alpha \mathrm{R}_I \beta) \stackrel{def}{=} \mathrm{O}(\beta \mathrm{U}_I (\alpha \wedge \beta)) \vee \mathrm{OG}_I \beta$, $\mathrm{OF}_I \alpha \stackrel{def}{=} \mathrm{O}(\mathbf{true} \mathrm{U}_I \alpha)$, where $\mathrm{O} \in \{\mathrm{E}, \mathrm{A}\}$.

RTACTL is the fragment of RTCTL such that the formulae are restricted to the positive Boolean combinations of $\mathrm{AX}\varphi$, $\mathrm{AG}\varphi$ and $\mathrm{A}(\varphi \mathrm{U} \psi)$. Negation can be applied to propositions only.

RTECTL is the fragment of RTCTL such that the formulae are restricted to the positive Boolean combinations of $\mathrm{EX}\varphi$, $\mathrm{EG}\varphi$ and $\mathrm{E}(\varphi \mathrm{U} \psi)$. Negation can be applied to propositions only.

[1]Note that the remaining forms of intervals can be defined by means of $[a, b)$ and $[a, \infty)$.

A *model* for RTCTL is the automaton $\mathcal{A} = (\Sigma, S, s^0, T, V)$ as defined in the previous section. Note that this finite automaton can be viewed as a standard Kripke structure or labelled transition system.

Let $\mathcal{A} = (\Sigma, S, s^0, T, V)$ be a model. A *path* of $\mathcal{A}$ is an infinite sequence $\pi = (s_0, s_1, \dots)$ of states such that $(s_j, s_{j+1}) \in T$ for each $j \in \mathbb{N}$. For a path $\pi = (s_0, s_1, \dots)$, we take $\pi(j) = s_j$. By $\Pi(s)$ we denote the set of all the paths starting at $s \in S$, $\mathrm{O} \in \{\mathrm{E}, \mathrm{A}\}$, and $\# = \exists\pi \in \Pi(s)$ if $\mathrm{O} = \mathrm{E}$, otherwise $\# = \forall\pi \in \Pi(s)$. Given the above, the formal semantics of RTCTL is defined recursively as follows:

- $\mathcal{A}, s \models$ **true**,
- $\mathcal{A}, s \not\models$ **false** ,
- $\mathcal{A}, s \models p$ iff $p \in V(s)$,
- $\mathcal{A}, s \models \neg p$ iff $p \notin V(s)$,
- $\mathcal{A}, s \models \alpha \wedge \beta$ iff $\mathcal{A}, s \models \alpha$ and $\mathcal{A}, s \models \beta$,
- $\mathcal{A}, s \models \alpha \vee \beta$ iff $\mathcal{A}, s \models \alpha$ or $\mathcal{A}, s \models \beta$,
- $\mathcal{A}, s \models \mathrm{OX}\alpha$ iff$\#(\mathcal{A}, \pi(1) \models \alpha)$,
- $\mathcal{A}, s \models \mathrm{O}(\alpha \mathrm{U}_I \beta)$ iff $\#(\exists m \in I)[\mathcal{A}, \pi(m) \models \beta$ and $(\forall j < m)\mathcal{A}, \pi(j) \models \alpha]$,
- $\mathcal{A}, s \models \mathrm{OG}_I\alpha$ iff $\#(\forall m \in I)[\mathcal{A}, \pi(m) \models \alpha]$.

We end the section by defining the notions of validity and the model checking problem. Namely, a RTCTL formula φ is *valid* in $\mathcal{A}$ (denoted $\mathcal{A} \models \varphi$) iff $\mathcal{A}, s^0 \models \varphi$, i.e., φ is true at the initial state of the model $\mathcal{A}$. The *model checking problem* asks whether $\mathcal{A} \models \varphi$.

2.3. SAT-based BMC for RTECTL

In this section we give an overview of a SAT-based BMC method for the existential fragment of RTCTL (RTECTL) [11]. As usual, we start by defining k-paths, and (k,l)-*loops*, and then in turn we define a *bounded semantics* for RTECTL, which is later used for translation to SAT.

Given are a model $\mathcal{A} = (\Sigma, S, s^0, T, V)$ and a bound $k \geq 0$. A k-path is the prefix of length k of a path in Π. By P_k we denote a set of all the k-paths. By $P_k(s)$ we denote a set of all the k-paths π_k with $\pi_k(0) = s$. A (k,l)-*loop* is a k-path $\pi_k = (\pi_k(0), \dots, \pi_k(l), \dots, \pi_k(k))$ such that $\pi_k(l) = \pi_k(k)$, for some $0 \leq l < k$. A function $loop : \Pi_k \to 2^{\mathbb{N}}$ identifies these k-paths that are loops and it is defined as: $loop(\pi_k) = \{l \mid 0 \leq l < k$ and $\pi_k(l) = \pi_k(k)\}$.

Definition 1 *Given are a bound $k \in \mathbb{N}$, a model $\mathcal{A}$, and RTECTL formulae α, β. $\mathcal{A}, s \models_k \alpha$ denotes that α is $k-$true at the state s of $\mathcal{A}$. The relation $\models_k$ is defined inductively as follows:*

- $\mathcal{A}, s \models_k$ **true**,
- $\mathcal{A}, s \not\models_k$ **false**,
- $\mathcal{A}, s \models_k p$ iff $p \in V(s)$,
- $\mathcal{A}, s \models_k \neg p$ iff $p \notin V(s)$,
- $\mathcal{A}, s \models_k \alpha \vee \beta$ iff $\mathcal{A}, s \models_k \alpha$ *or* $\mathcal{A}, s \models_k \beta$,

- $\mathcal{A}, s \models_k \alpha \wedge \beta$ iff $\mathcal{A}, s \models_k \alpha$ *and* $\mathcal{A}, s \models_k \beta$,
- $\mathcal{A}, s \models_k \mathrm{EX}\alpha$ iff $k > 0$ and $(\exists \pi \in \Pi_k(s))\mathcal{A}, \pi(1) \models_k \alpha$,
- $\mathcal{A}, s \models_k \mathrm{E}(\alpha \mathrm{U}_I \beta)$ iff $(\exists \pi \in \Pi_k(s))(\exists 0 \leq m \leq k)(m \in I$ and $\mathcal{A}, \pi(m) \models_k \beta$ and $(\forall 0 \leq j < m)\mathcal{A}, \pi(j) \models_k \alpha)$,
- $\mathcal{A}, s \models_k \mathrm{EG}_I\alpha$ iff $(\exists \pi \in \Pi_k(s))((k \geq \mathit{right}(I)$ and $(\forall j \in I)\ \mathcal{A}, \pi(j) \models_k \alpha)$ or $(k < \mathit{right}(I)$ and $(\exists l \in \mathit{loop}(\pi))(\forall \mathit{min}(\mathit{left}(I), l) \leq j < k)\mathcal{A}, \pi(j) \models_k \alpha))$.

A RTECTL formula φ is *valid in model $\mathcal{A}$ with bound k* (denoted $\mathcal{A} \models_k \varphi$) iff $\mathcal{A}, s^0 \models_k \varphi$, i.e., φ is $k-$true at the initial state of the model $\mathcal{A}$. The *bounded model checking problem* asks whether $\mathcal{A} \models_k \varphi$.

The following theorem, which can be proven by induction on the length of a RTECTL formula, states that there exists a bound such that bounded and unbounded semantics are equivalent. This implies that the model checking problem ($\mathcal{A} \models \varphi$) can be reduced to the bounded model checking problem ($\mathcal{A} \models_k \varphi$).

Theorem 1 *Let $\mathcal{A}$ be a model and φ a RTECTL formula. Then, the following equivalence holds: $\mathcal{A} \models \varphi$ iff there exists $k \geq 0$ such that $\mathcal{A} \models_k \varphi$.*

We can also show even the stronger property, namely, we can prove that φ is $k-$true in $\mathcal{A}$ if and only if φ is $k-$true in $\mathcal{A}$ with a number of $k-$paths reduced to $f_k(\varphi)$, where the function $f_k : RTECTL \rightarrow \mathbb{N}$ is defined as follows:

- $f_k(\mathbf{true}) = f_k(\mathbf{false}) = f_k(p) = f_k(\neg p) = 0$, where $p \in PV$,
- $f_k(\alpha \wedge \beta) = f_k(\alpha) + f_k(\beta)$,
- $f_k(\alpha \vee \beta) = max\{f_k(\alpha), f_k(\beta)\}$,
- $f_k(\mathrm{X}\alpha) = f_k(\alpha) + 1$,
- $f_k(\mathrm{E}(\alpha \mathrm{U}_I \beta)) = k \cdot f_k(\alpha) + f_k(\beta) + 1$,
- $f_k(\mathrm{EG}_I\alpha) = (k+1) \cdot f_k(\alpha) + 1$.

Given are a model $\mathcal{A} = (\Sigma, S, s^0, T, V)$, a bound $k \geq 0$, and a RTECTL formula φ. The problem of checking whether $\mathcal{A} \models_k \varphi$ holds can be translated to the satisfiability problem of the following propositional formula:

$$[\mathcal{A}, \varphi]_k := [\mathcal{A}^{\varphi, s^0}]_k \wedge [\varphi]_{\mathcal{A},k} \tag{1}$$

where, the formula $[\mathcal{A}^{\varphi, s^0}]_k$ constrains the $f_k(\varphi)$ symbolic k-paths to be valid k-paths of $\mathcal{A}$, while the formula $[\varphi]_{\mathcal{A},k}$ encodes a number of constraints that must be satisfied on these sets of k-paths for φ to be satisfied. Once this translation is defined, checking satisfiability of a RTECTL formula can be done by means of a SAT-solver.

In order to define the formula $[\mathcal{A}, \varphi]_k$ we proceed as follows. We assume that each state s of $\mathcal{A}$ is encoded by a bit-vector whose length, say r, depends on the number of local states of "component" automata. Thus, each state s of $\mathcal{A}$ we can represent by a vector $w = (u_1, \ldots, u_r)$ of propositional variables (usually called *state variables*), to which we refer to as *symbolic states*. A finite sequence $(w_0, \ldots, w_k)$ of symbolic states is called a *symbolic k-path*. Since, in general, we may need to consider more than one symbolic k-path, we introduce a notion of the j-th symbolic k-path, which is denoted by $(w_{0,j}, \ldots, w_{k,j})$, where $w_{i,j}$ are symbolic states for $0 \leq j < f_k(\varphi)$ and $0 \leq i \leq k$. Note that the exact number of necessary symbolic k-paths depends on the checked formula φ, and it can be calculated by means of the function f_k.

The propositional formula $[\mathcal{A}^{\varphi,s^0}]_k$ is defined over symbolic states $w_{i,j}$, for $0 \leq i \leq k$ and $0 \leq j < f_k(\varphi)$, in the following way:

$$[\mathcal{A}^{\varphi,s^0}]_k := I_{s^0}(w_{0,0}) \wedge \bigwedge_{j=0}^{f_k(\varphi)-1} \bigwedge_{i=0}^{k-1} \mathcal{R}(w_{i,j}, w_{i+1,j}) \tag{2}$$

where $I_{s^0}(w)$ is a formula that encodes the initial state s^0 of $\mathcal{A}$, and $\mathcal{R}(w, w')$ is a formula that encodes the transition relation of $\mathcal{A}$.

The next step is the translation of a RTECTL formula φ into a propositional formula $[\varphi]_{\mathcal{A},k} := [\varphi]_k^{[0,0]}$, where $k \geq 0$ is a bound, $[\varphi]_k^{[m,n]}$ denotes the translation of φ at the symbolic state $w_{m,n}$, and it is defined inductively as follows:

- $[\mathbf{true}]_k^{[m,n]} := \mathbf{true}$,
- $[\mathbf{false}]_k^{[m,n]} := \mathbf{false}$,
- $[p]_k^{[m,n]} := p(w_{m,n})$,
- $[\neg p]_k^{[m,n]} := \neg p(w_{m,n})$,
- $[\alpha \wedge \beta]_k^{[m,n]} := [\alpha]_k^{[m,n]} \wedge [\beta]_k^{[m,n]}$,
- $[\alpha \vee \beta]_k^{[m,n]} := [\alpha]_k^{[m,n]} \vee [\beta]_k^{[m,n]}$,
- $[\mathrm{EX}\alpha]_k^{[m,n]} := \begin{array}{ll} (1) \ \bigvee_{ll=0}^{f_k(\varphi)-1} H(w_{m,n}, w_{0,ll}) \wedge [\alpha]_k^{[1,ll]}, & \text{if } k > 0 \\ (2) \ \mathbf{false}, & \textit{otherwise} \end{array}$
- $[\mathrm{E}(\alpha \mathrm{U}_I \beta)]_k^{[m,n]} := \bigvee_{ll=0}^{f_k(\varphi)-1} (H(w_{m,n}, w_{0,ll}) \wedge \bigvee_{i=0}^{k} ([\beta]_k^{[i,ll]} \wedge In(i, I) \wedge \bigwedge_{j=0}^{i-1} [\alpha]_k^{[j,ll]}))$,
- $[\mathrm{EG}_I \alpha]_k^{[m,n]} := \bigvee_{ll=0}^{f_k(\varphi)-1} H(w_{m,n}, w_{0,ll}) \wedge$

  (1) $\bigwedge_{j=left(I)}^{right(I)} [\alpha]_k^{[j,ll]}$, if $right(I) \leq k$

  (2) $\bigvee_{l=0}^{k-1} (H(w_{k,ll}, w_{l,ll}) \wedge \bigwedge_{j=min(left(I),l)}^{k-1} [\alpha]_k^{[j,ll]})$, otherwise.

The following theorem, which can be proven by induction on the length of a RTECTL formula, expresses the correctness and the completeness of the translation presented above.

Theorem 2 *Let $\mathcal{A}$ be a model, and φ a RTECTL formula. Then for every $k \in \mathbb{N}$, $\mathcal{A} \models_k \varphi$ if, and only if, the propositional formula $[\mathcal{A}, \varphi]_k$ is satisfiable.*

3. A faulty train controller system

To evaluate the BMC technique for RTECTL, we analyse a scalable concurrent system, which is a faulty train controller system (FTC) (adapted from [7]). The system consists of a controller, and n trains (for $n \geq 2$), and it is assumed that each train uses its own circular track for travelling in one direction. At one point, all trains have to pass through a tunnel, but because there is only one track in the tunnel, trains arriving from each direction cannot use it simultaneously. There are colour light signals on both sides of the tunnel, which can be either red or green. All trains notify the controller when they request entry to the tunnel or when they leave the tunnel. The controller controls the colour of the colour light signals, however it can be faulty, and thereby it does not serve its purpose. Namely, the controller does not ensure the mutual exclusion property: two trains never occupy the tunnel at the same time.

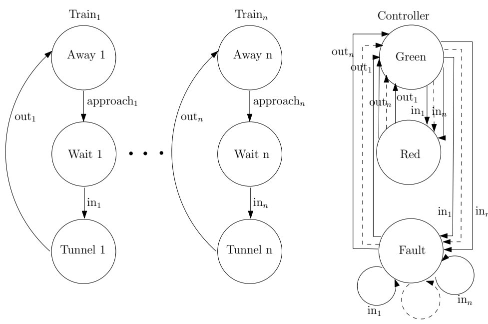

Figure 1: A network of automata for train controller system

An automata model of the FTC system is shown on Figure 1. The specifications for it are given in the universal form, i.e., they are expressed in the RTACTL language:

$$\varphi_1 = \mathrm{AG}_{[0,\infty]}\big(InTunnel_1 \rightarrow \mathrm{AF}_{[1,\infty]}(InTunnel_1)\big),$$
$$\varphi_2 = \mathrm{AG}_{[0,\infty]}\big(\textstyle\bigwedge_{i=1}^{n-1}\bigwedge_{j=i+1}^{n}\neg(InTunnel_i \wedge InTunnel_j)\big),$$
$$\varphi_3 = \mathrm{AG}_{[0,\infty]}\big(InTunnel_1 \rightarrow \mathrm{AF}_{[1,n+1]}\big(\textstyle\bigvee_{i=1}^{n} InTunnel_i\big)\big).$$

The formula φ_1 states that it is always the case that whenever Train 1 is in the tunnel, it will be in the tunnel once again within a bounded period of time, i.e., within n time units for $n \geq 1$. The formula φ_2 represents the fact that trains have exclusive access to the tunnel. The formula φ_3 expresses that it is always the case that if Train 1 is in the tunnel, then either he or other train will be in the tunnel during the next $n+1$ time units.

All the above formulae are not true in the model for FTC, and for every specification given, there exists a counterexample. This was shown by means of the BMC method for RTECTL and testing the formulae $\psi_i = \neg\varphi_i$ (for $i = 1, 2, 3$), which are the negations of the assumed universal specifications and are interpreted existentially.

For the tests we have used a computer equipped with AMD phenom(tm) 9550 Quad-Core 2200 MHz processor and 4 GB of RAM, running Ubuntu Linux with kernel version 2.6.35-28-generic-pae, and we have set the timeout to 3600 seconds, and memory limit to 3072 MB. We have used the state of the art SAT-solver MiniSat 2. The experimental results are shown in Table 1. In particular, we present there the results for the formulae φ_1, φ_2, and φ_3, and the maximum number of trains we were able to model check by means of our BMC method for RTECTL.

To get the experimental results in Table 1, we started with a propositional encoding of a network of automata that models FTC. To this end we have encoded the states of the network, in particular the initial state, and the transition relation. An example of such encoding for two trains and a controller, we present below.

Let $SV = \{p_1, p_2, \dots\}$ be an infinite set of state variables. A Boolean encoding of all the local states of the two automata representing trains is the following:

$Train\ 1$				$Train\ 2$			
$state$	bit_2	bit_1	$formula$	$state$	bit_4	bit_3	$formula$
$away_1$	0	0	$\neg p_1 \wedge \neg p_2$	$away_2$	0	0	$\neg p_3 \wedge \neg p_4$
$wait_1$	1	0	$\neg p_1 \wedge p_2$	$wait_2$	1	0	$\neg p_3 \wedge p_4$
$tunnel_1$	0	1	$p_1 \wedge \neg p_2$	$tunnel_2$	0	1	$p_3 \wedge \neg p_4$

$Controller$			
$location$	bit_6	bit_5	$formula$
$green$	0	0	$\neg p_5 \wedge \neg p_6$
red	0	1	$p_5 \wedge \neg p_6$
$faulty$	1	0	$\neg p_5 \wedge p_6$

Given the above, it is easy to see that each state of the network of automata modelling the FTC system can be represented by a valuation of a symbolic state $w = (p_1, \ldots, p_6)$. Then, a propositional formula $I_{s^0}(w)$, which encodes the initial global state of the considered system, is the conjunction of three formulae that encode all the local initial states, i.e.

$$I_{s^0}(w) = (\neg p_1 \wedge \neg p_2) \wedge (\neg p_3 \wedge \neg p_4) \wedge (\neg p_5 \wedge \neg p_6)$$

Furthermore, let $w = (p_1, \ldots, p_6), w' = (p'_1, \ldots, p'_6)$ be two different symbolic states. A propositional formula $\mathcal{R}(w, w')$, which encodes all the transitions of the considered system is defined as the disjunction of formula that encode single transitions:

$\mathcal{R}(w, w')$	$approach_1 \vee in_1 \vee out_1 \vee approach_2 \vee in_2 \vee out_2$
$approach_1$	$\neg p_1 \wedge \neg p_2 \wedge \neg p'_1 \wedge p'_2 \wedge (p_3 \leftrightarrow p'_3) \wedge (p_4 \leftrightarrow p'_4) \wedge (p_5 \leftrightarrow p'_5)$ $\wedge (p_6 \leftrightarrow p'_6)$
in_1	$\neg p_1 \wedge p_2 \wedge p'_1 \wedge \neg p'_2 \wedge (p_3 \leftrightarrow p'_3) \wedge (p_4 \leftrightarrow p'_4) \wedge (\neg p_5 \wedge \neg p_6 \wedge$ $p'_5 \wedge \neg p'_6 \vee \neg p_5 \wedge \neg p_6 \wedge \neg p'_5 \wedge p'_6 \vee \neg p_5 \wedge p_6 \wedge \neg p'_5 \wedge p'_6)$
out_1	$p_1 \wedge \neg p_2 \wedge \neg p'_1 \wedge \neg p'_2 \wedge (p_3 \leftrightarrow p'_3) \wedge (p_4 \leftrightarrow p'_4) \wedge$ $(p_5 \wedge \neg p_6 \wedge \neg p'_5 \wedge \neg p'_6 \vee \neg p_5 \wedge p_6 \wedge \neg p'_5 \wedge \neg p'_6)$
$approach_2$	$\neg p_3 \wedge \neg p_4 \wedge \neg p'_3 \wedge p'_4 \wedge (p_1 \leftrightarrow p'_1) \wedge (p_2 \leftrightarrow p'_2) \wedge (p_5 \leftrightarrow p'_5)$ $\wedge (p_6 \leftrightarrow p'_6)$
in_2	$\neg p_3 \wedge p_4 \wedge p'_3 \wedge \neg p'_4 \wedge (p_1 \leftrightarrow p'_1) \wedge (p_2 \leftrightarrow p'_2) \wedge (\neg p_5 \wedge \neg p_6 \wedge$ $p'_5 \wedge \neg p'_6 \vee \neg p_5 \wedge \neg p_6 \wedge \neg p'_5 \wedge p'_6 \vee \neg p_5 \wedge p_6 \wedge \neg p'_5 \wedge p'_6)$
out_2	$p_3 \wedge \neg p_4 \wedge \neg p'_3 \wedge \neg p'_4 \wedge (p_1 \leftrightarrow p'_1) \wedge (p_2 \leftrightarrow p'_2) \wedge$ $(p_5 \wedge \neg p_6 \wedge \neg p'_5 \wedge \neg p'_6 \vee \neg p_5 \wedge p_6 \wedge \neg p'_5 \wedge \neg p'_6)$

			number of			BMC		MiniSat 2	
φ	k	$f_k(\varphi)$	trains	variables	clauses	sec	MB	sec	MB
φ_1	4	2	1000	4251246	12747733	217.9	553.5	2081.0	902.0
φ_2	16	1	8	4349	12418	0.1	2.0	1882.3	32.0
φ_3	4	2	240	292798	876949	7.7	39.6	1851.0	676.0

Table 1: Experimental results

4. Conclusions

In this paper we gave a SAT-based symbolic approach to bounded model checking of concurrent systems modelled by network of finite automata. We focused on the properties expressed in RTECTL. The method has been implemented, and tested on the standard benchmark – a faulty train controller system. The benchmark has been carefully selected in such a way as to reveal the advantages and disadvantages of both approaches.

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M/M/n/m QUEUEING SYSTEMS WITH NON-IDENTICAL SERVERS

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Abstract. $M/M/n/m$ queueing systems with identical servers are well known in queueing theory and its applications. The analysis of these systems is very simple thanks to the fact that the number of customers $\eta(t)$ in the system at arbitrary time instant t forms a Markov chain. The main purpose of this paper is to analyse the $M/M/n/m$ system under assumption that its servers are different, i.e. they have different parameters of service time.

1. The analysis of classical $M/M/n/m$ queueing system

Let us consider the classical $M/M/n/m$ queueing system ($m < \infty$). Let a be an arrival rate of customers, μ be the parameter of service time distribution. This system is illustrated in Fig. 1.

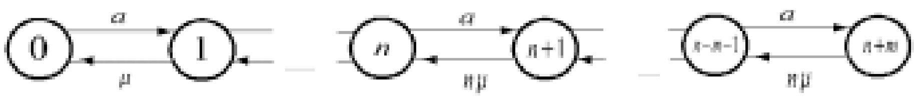

Figure 1: A graph of the classical $M/M/n/m$ queueing system ($m < \infty$)

Denote $\rho = a/(n\mu)$. In the case $\rho < \infty$ the steady state exists, and we have the following equations for steady state probabilities $p_k = P\{\eta = k\}$:

$$\begin{cases} ap_{k-1} = k\mu p_k, & k = \overline{1, n}, \\ ap_{k-1} = n\mu p_k, & k = \overline{n+1, n+m}, \end{cases} \tag{1}$$

where η is the stationary number of customers present in the system, $k = 0, 1, \dots$.

It can be easily shown [1] that the solution of this system has the form:

$$p_k = \begin{cases} \dfrac{(n\rho)^k}{k!} p_0, & k = \overline{1, n}, \\ \dfrac{n^n \rho^k}{n!} p_0, & k = \overline{n+1, n+m}. \end{cases} \tag{2}$$

The value $p_0 = P\{\eta = 0\}$ is obtained from the normalization condition $\sum_{k=0}^{n+m} p_k = 1$. If $m = \infty$, the graph of the system is similar. In this case $\eta(t)$ is a birth-death process and we can also obtain p_k probabilities (if $\rho < 1$) from analogous equations. The final formulas are also very similar.

Assume now that μ_i is a service time distribution parameter of the ith server in the system, $i = \overline{1, n}$ and these parameters may be different. The problem appeared for the first time in [2],[3] and [4]. In this case it is possible that arriving customer has to choose one of free different servers. Hence we must define the discipline of such a choice. We shall analyze the following disciplines:

1. RANDOM CHOICE.

   If there are k free servers, an arriving customer chooses every free server with the same probability $\frac{1}{k}$. Such systems will be denoted as $M/M/n/m$-$NHRC$ (non-homogeneous servers with classical random choice).

2. FASTEST SERVER CHOICE.

   An arriving customer chooses the fastest server. Such systems will be denoted as $M/M/n/m - NHFC$ (non-homogeneous servers with the fastest server choice).

We shall describe the behaviour of the system by the following generalized Markovian process:

$$(\eta(t), i_1(t), i_2(t), \dots, i_l(t)), \tag{3}$$

where $l = \min(\eta(t), n)$ and $i_1(t), i_2(t), \dots, i_l(t)$ is the sequence of the numbers of busy servers written increasingly. If $\eta(t) = 0$, the process reduces to $\eta(t)$.

The process (3) is characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \tag{4}$$

$$P_{kf_1f_2\dots f_l}(t) = P\{\eta(t) = k,\ i_1(t) = f_1, i_2(t) = f_2, \dots, i_l(t) = f_l\},$$
$$k = \overline{1, n+m}. \tag{5}$$

It is clear that if $k \geq n$, then the process $\eta(t)$ can be understood as a birth-death process and we can rewrite functions from (5) as $P_k(t) = P\{\eta(t) = k\}$. If $k < n$, we have

$$P_k(t) = P\{\eta(t) = k\} = \sum_{\{F_k^n\}} P_{kf_1 f_2 \dots f_k}(t), \tag{6}$$

where $\{F_k^n\}$ is the set of all combinations of the set $\{f_1, f_2, \dots, f_k\}$.

Now we shall investigate these systems in some special cases and try to find formulas for p_k in the general case.

2. Analysis of $M/M/2/m$–NHRC and $M/M/3/m$–NHRC queueing systems

Let us consider the generalization of $M/M/n/m$ queueing system with two non-identical servers and random choice of a free server. Denote as μ_1, μ_2 the service time parameters for the servers. We will also use the notation: $\rho = \frac{a}{\mu_1+\mu_2}$. A graph of such a system is presented in Fig. 2.

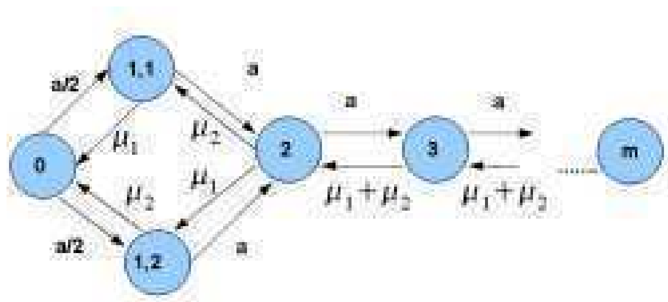

Figure 2: A graph of the $M/M/2/m - NHRC$ queueing system ($m < \infty$)

To describe a behavior of the system we introduce the following Markovian process:

$$(\eta(t), i_1(t), \dots, i_l(t)), \tag{7}$$

where $l = \min(\eta(t), 2)$.

The process (7) is characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \tag{8}$$

$$P_{11}(t) = P\{\eta(t) = 1,\ i_1(t) = 1\}; \tag{9}$$

$$P_{12}(t) = P\{\eta(t) = 1,\ i_1(t) = 2\}; \tag{10}$$

$$P_k(t) = P\{\eta(t) = k\},\ k = \overline{2, m+2}. \tag{11}$$

The Kolmogorov equations for these functions have the form:

$$P_0'(t) = -aP_0(t) + \mu_1 P_{11}(t) + \mu_2 P_{12}(t); \tag{12}$$

$$P_{11}'(t) = -(a + \mu_1)P_{11}(t) + \frac{a}{2}P_0(t) + \mu_2 P_2(t); \tag{13}$$

$$P_{12}'(t) = -(a + \mu_2)P_{12}(t) + \frac{a}{2}P_0(t) + \mu_1 P_2(t); \tag{14}$$

$$P_2'(t) = -(a + \mu_1 + \mu_2)P_2(t) + a(P_{11}(t) + P_{12}(t)) + (\mu_1 + \mu_2)P_3(t); \tag{15}$$

$$P_k'(t) = -(a+\mu_1+\mu_2)P_k(t)+aP_{k-1}(t)+(\mu_1+\mu_2)P_{k+1}(t),\ k = \overline{3, m+1}; \tag{16}$$

$$P_{m+2}'(t) = -(\mu_1 + \mu_2)P_{m+2}(t) + aP_{m+1}(t); \tag{17}$$

$$P_0(t) + P_{11}(t) + P_{12}(t) + \sum_{k=2}^{m+2} P_k(t) = 1. \tag{18}$$

In the steady state (if $\rho < \infty$) we obtain the following equations for probabilities p_0, p_{11}, p_{12} and p_k that are limits of functions (8)–(11) for $t \to \infty$:

$$ap_0 = \mu_1 p_{11} + \mu_2 p_{12}; \tag{19}$$

$$(a + \mu_1)p_{11} = \frac{a}{2}p_0 + \mu_2 p_2; \tag{20}$$

$$(a + \mu_2)p_{12} = \frac{a}{2}p_0 + \mu_1 p_2; \tag{21}$$

$$(a + \mu_1 + \mu_2)p_2 = a(p_{11} + p_{12}) + (\mu_1 + \mu_2)p_3; \tag{22}$$

$$(a + \mu_1 + \mu_2)p_k = ap_{k-1} + (\mu_1 + \mu_2)p_{k+1},\ k = \overline{3, m+1}; \tag{23}$$

$$(\mu_1 + \mu_2)p_{m+2} = ap_{m+1}; \tag{24}$$

$$p_0 + p_{11} + p_{12} + \sum_{k=2}^{m+2} p_k = 1. \tag{25}$$

The solution has the form:

$$p_{11} = \frac{a}{2\mu_1}p_0; \tag{26}$$

$$p_{12} = \frac{a}{2\mu_2}p_0; \tag{27}$$

$$p_1 = p_{11} + p_{12} = \frac{ap_0}{2}(\frac{1}{\mu_1} + \frac{1}{\mu_2}); \tag{28}$$

$$p_k = \frac{ap_0}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\rho^{k-1},\ k = \overline{2, m+2}. \tag{29}$$

Formulas (28)–(29) can be rewritten as

$$p_k = \frac{ap_0}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\rho^{k-1},\ k=\overline{1,m+2}, \tag{30}$$

where the value of p_0 can be obtained from the normalization condition (25):

$$p_0 = \left[1+\frac{a}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\sum_{k=1}^{m+2}\rho^{k-1}\right]^{-1}. \tag{31}$$

If we investigate $M/M/2/\infty - NHRC$ system in the steady state ($\rho < 1$), formulas (30) for probabilities p_k will be the same, but the relation (31) will have a simpler form (in this case we use formula for the sum of geometric series).

If we analyse $M/M/3/m - NHRC$ system ($\rho = \frac{a}{\mu_1+\mu_2+\mu_3}$) and write analogous equations, we will obtain the following results:

$$p_1 = \frac{ap_0}{3}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}\right); \tag{32}$$

$$p_k = \frac{a^2p_0}{6}\left(\frac{1}{\mu_1\mu_2}+\frac{1}{\mu_1\mu_3}+\frac{1}{\mu_2\mu_3}\right)\rho^{k-2},\ k=\overline{2,m+3}; \tag{33}$$

$$p_0 = \left[1+\frac{a}{3}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}\right)+\frac{a^2}{6}\left(\frac{1}{\mu_1\mu_2}+\frac{1}{\mu_1\mu_3}+\frac{1}{\mu_2\mu_3}\right)\sum_{k=2}^{m+3}\rho^{k-2}\right]^{-1}. \tag{34}$$

Analysis of $M/M/n/m - NHRC$ for $n > 3$ is more complicated because the number of equations is increasing exponentially and it is difficult to find the exact solution. But obtained results let predict the general solution for the arbitrary number of servers in the following form:

$$p_k = \begin{cases} \dfrac{a^k(n-k)!p_0}{n!}\displaystyle\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_n^k}\mu_{x_i}}, & k=\overline{1,n-1}, \\ \dfrac{a^k(n-k)!p_0}{n!}\displaystyle\sum_{\{F_n^n\}}\frac{1}{\prod_{x_i\in F_n^n}\mu_{x_i}}\rho^{k-n}, & k=\overline{n,n+m}, \end{cases} \tag{35}$$

where F_k^n denotes a k-element subset of an n element set.

3. Analysis of $M/M/2/m-NHFC$ queueing system

In this case, if we assume that $\mu_1 > \mu_2$, then we will not have the transition $0 \to (1,2)$ in the graph presented in Fig. 2, and the transition $0 \to (1,1)$ parameter will be equal to a.

So in this case we have the following equations in the steady state:

$$ap_0 = \mu_1 p_{11} + \mu_2 p_{12}; \tag{36}$$

$$(a+\mu_1)p_{11} = ap_0 + \mu_2 p_2; \tag{37}$$

$$(a+\mu_2)p_{12} = \mu_1 p_2. \tag{38}$$

The rest of equations for the functions p_k will be the same as in (22)–(25).

The solution of this system has the form:

$$p_k = \frac{a(a+\mu_2)(\mu_1+\mu_2)p_0}{\mu_1\mu_2(2a+\mu_1+\mu_2)}\rho^{k-1},\ k = \overline{1, m+2}; \tag{39}$$

$$p_0 = \left[1 + \frac{a(a+\mu_2)(\mu_1+\mu_2)}{\mu_1\mu_2(2a+\mu_1+\mu_2)}\sum_{k=1}^{m+2}\rho^{k-1}\right]^{-1}. \tag{40}$$

The general analysis for $n \geq 3$ is complicated and predicting general formulas for p_k is not easy.

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PART III

MATHEMATICS TEACHERS TRAINING

Jan Długosz University in Częstochowa
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CHANGES OF ATTITUDES TOWARDS MATHEMATICS

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Abstract. Mathematical competence is the ability to develop and use mathematical thinking for solving various problems in everyday situations. The process of acquiring mathematical competence is strongly determined by attitudes to mathematics. In this contribution, we examine the attitudes of students towards mathematics during their transition to post-secondary institutions.

1. Introduction

Mathematics is a part of human culture. Its pieces of knowledge, methods and procedures are transferred to all parts of human activity. One of those parts is, obviously, education. Mathematical education is not an independent part of education. Mathematics undoubtedly develops cognition of every student, and with its methods and devices it is predetermined to become an instrument for the development of ability for constructing knowledge.

Attitudes are the variables that play an important role in teaching mathematics. They represent an assumption about an educational and learning process, about students and their knowledge. Thus attitudes function as cognitive and emotional filters through which we are able to explain and judge our knowledge and experience. For this reason we consider attitudes and their development as a key variable in the continuing education process. It should be taken into consideration that students of the mathematics teaching programme are coming with attitudes towards mathematics and its teaching. However, many studies have confirmed the fact that teachers eventually use the teaching methods which have been used in their own educational process. Attitudes of students are continuously changing and developing. This change of attitudes and the way of thinking are reflected in the way they learn.

2. Survey of students' attitudes towards mathematics

In the process of grant project realization we carried out the survey of students' attitudes towards mathematics during an initial phase of their studying. For the survey, the students of three among the seven faculties of the University of Žilina were selected. The faculties differ not only in their field of study and practical orientation but also in entrance exam requirements.

- ◇ The first faculty orientation is technical, and students are accepted without maths entrance test.
- ◇ The second faculty focuses on economic sectors, and students have to write an entrance test with relatively low math requirements.
- ◇ Specialized areas of study of the third faculty are economics of transport and communications, and students have to write an entrance test with higher math requirements in comparison with the second faculty.

Among the two hundred survey respondents, seventy were from the first faculty, seventy from the second faculty and sixty respondents from the third faculty. Multidimensional conception of attitudes, which recognizes 3 components – emotional reactions, confidence connected with an object and behaviour towards an object, had been used as a theoretical basis of a questionnaire [2]. The test consisted of 15 questions and students could choose one from five or six answers. The offered answers were scaled so that it could be possible to sort out positive and negative attitudes towards teaching mathematics, self-evaluation of individual mathematical abilities, student's ideas of a good way of teaching secondary school mathematics and changes in attitudes towards mathematics after entering the university.

In this paper we present some of the results of our survey. One of them is the answer to the question concerning students applying for entry to particular faculties and their secondary school results. It is reasonable to assume that the faculties which require entrance exams in the form of tests (also tests from mathematics) accept more students with higher preconditions for study success, i.e. with better results. The results of our survey, which confirm this fact, are presented in Table 1 and Figure 1.

Evaluation of results from secondary school mathematics	**Faculty I**	**Faculty II**	**Faculty III**
a) excellent	10.00%	10.00%	15.00%
b) excellent or very good	15.71%	42.86%	33.33%
c) very good or good	37.14%	20.00%	36.67%
d) good	28.57%	20.00%	13.33%
e) fair/sufficient	8.57%	7.14%	1.67%

Table 1: Secondary school study results

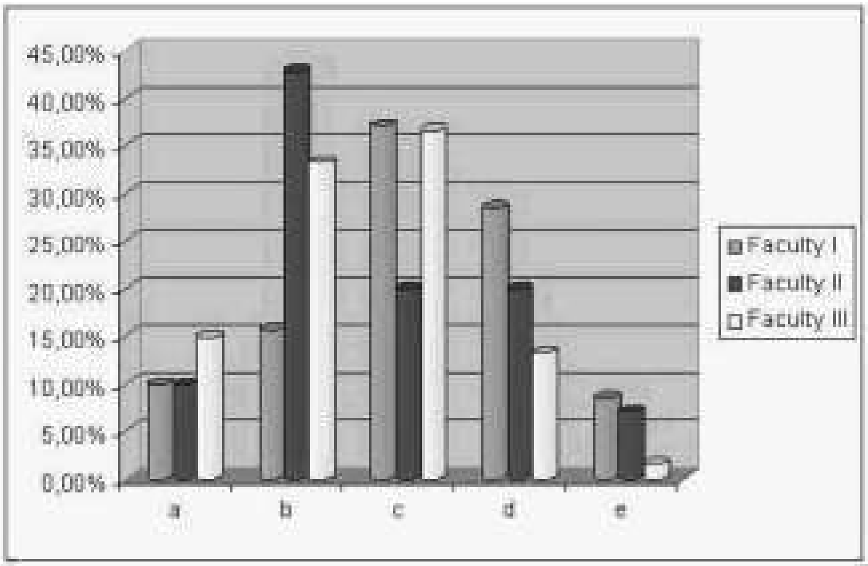

Figure 1:

Positive and negative self-evaluation of attitudes of students is analyzed by another test item, results of which are presented in Table 2 and Figure 2. It can be seen from the results that regardless of the entrance exams, mathematics is a difficult subject for applicants and they are not able to learn it with ease.

Mathematics is among	**Faculty I**	**Faculty II**	**Faculty III**
a) the most difficult subjects	25.71%	27.14%	30.00%
b) difficult subjects	42.86%	48.57%	43.33%
c) semi-difficult subjects	28.57%	20.00%	23.33%
d) easy subjects	0.00%	4.29%	3.33%
e) the easiest subjects	2.86%	0.00%	0.00%

Table 2: Difficultness of mathematics learning

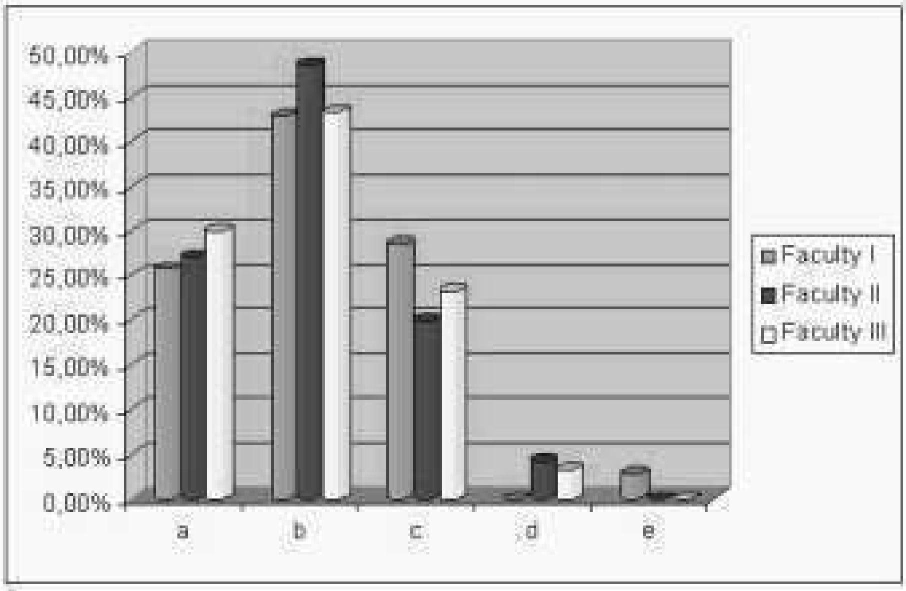

Figure 2:

The problems in learning mathematics are closely related to its popularity. For this reason we included question in the survey designed to analyze the popularity of mathematics among students. The results are summarized in Table 3 and Figure 3.

What is your relationship to mathematics?	**Faculty I**	**Faculty II**	**Faculty III**
a) I really do not like mathematics	12.86%	2.86%	3.33%
b) I do not like mathematics because I often do not understand it	14.29%	14.29%	16.67%
c) It depends on a current topic	40.00%	45.71%	35.00%
d) I enjoy mathematics if I understand the current topic	28.57%	34.29%	45.00%
e) Mathematics lessons belong to my most favourite ones	4.29%	2.86%	0.00%

Table 3: Popularity of mathematics

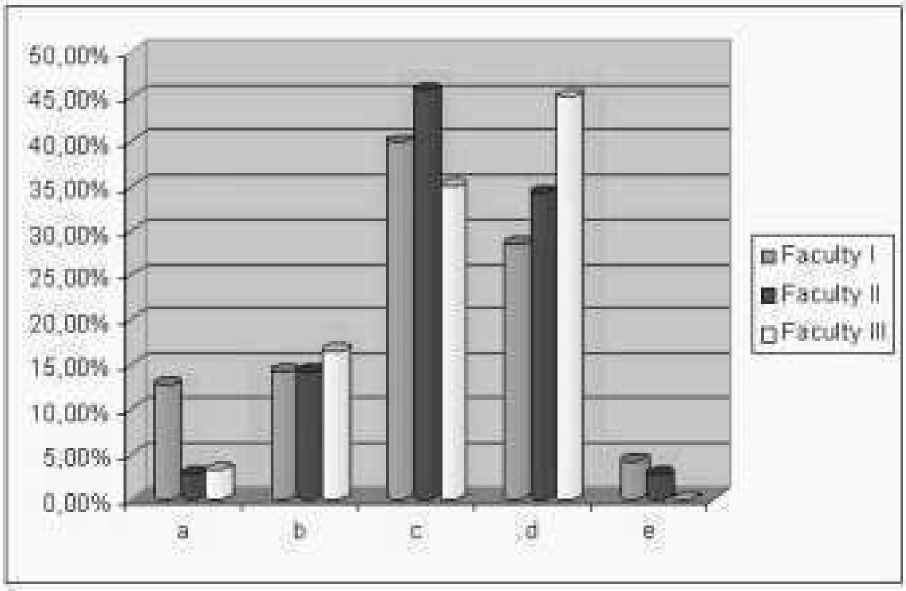

Figure 3:

It has been shown by our survey that the negative attitudes are predominantly observed among the students who did not have to deal intensively with mathematics during their senior year at a secondary school. Very surprising is the fact that also positive attitudes towards mathematics prevail among students of this group.

Views of importance of mathematics in future career	**Faculty I**	**Faculty II**	**Faculty III**
a) Mathematics is very important for my future career	12.86%	2.86%	3.33%
b) I will certainly need mathematics from time to time	14.29%	14.29%	16.67%
c) I will use mathematics only marginally	40.00%	45.71%	35.00%
d) I do not think I will need mathematics in my future career	28.57%	34.29%	45.00%
e) I hope I will not need mathematics in my future career	4.29%	2.86%	0.00%

Table 4: The necessity of mathematics

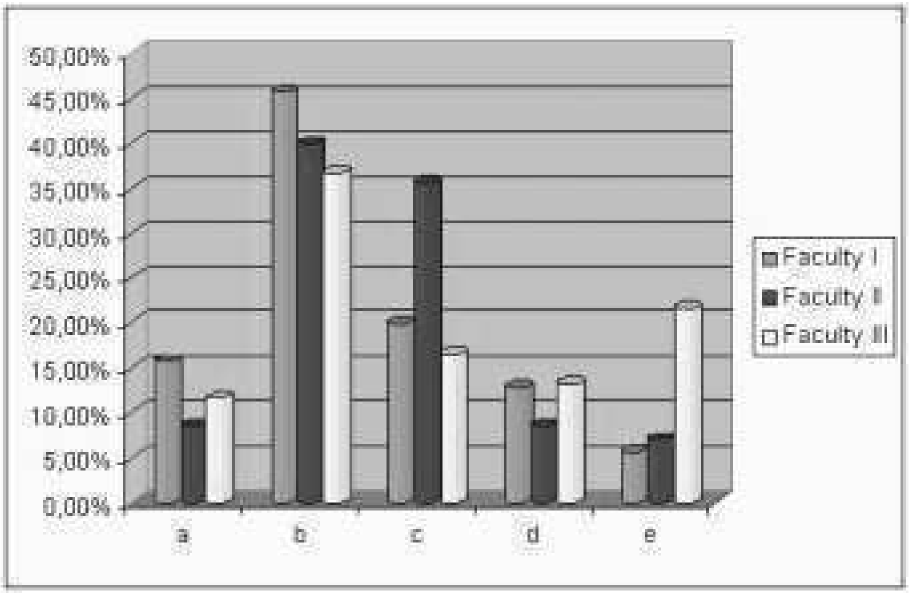

Figure 4:

The question of importance of mathematics in student's perspective job is presented by Table 4 and Figure 4. According to the presented results, positive attitudes of Faculty I and Faculty II students and negatives attitudes of students of Faculty III prevail in the survey. We did not expect this conclusion.

In Introduction we have mentioned that the attitudes of students are in the continuing development and change during their study. Thus we also analyzed which period of their studies they consider to be the point when their attitudes changed. This is presented in Table 5 and subsequently by Figure 5.

When did the change happen?	**Faculty I**	**Faculty II**	**Faculty III**
a) During the first four years of basic school	1.43%	1.43%	1.67%
b) During the second four/five years of basic school	12.86%	4.29%	3.33%
c) At secondary school	50.00%	45.71%	41.67%
d) It did not happen	24.29%	25.71%	41.67%
e) I am not able to decide when it happened	11.43%	22.86%	11.67%

Table 5: The period of changes of attitudes towards mathematics

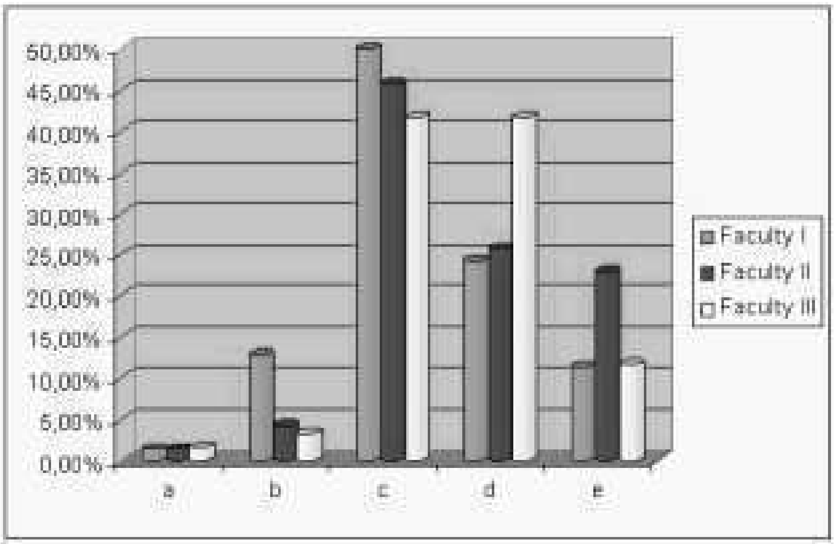

Figure 5:

As we tested the opinions of students at the very beginning of their university studies, only the periods of primary and secondary schools were included. The survey shows that the changes of attitudes towards mathematics (or no changes) are characteristic for students of all three types of faculties.

3. Conclusion

In this paper not all the results and conclusions of all tested items are presented. In spite of this fact, the presented selection indicates that study results of university students are strongly determined by their prior schooling, during which their attitudes towards mathematics usually worsen. It has been shown that too demanding tasks, incorrectly-chosen rate of teaching, the choice of inadequate language and negative attitudes of mathematics teachers have a bad impact on students. At the same time, the periods of primary and secondary school are key periods for the formation of students' attitudes towards mathematics.

We note that the results presented in this paper are, to some degree, influenced by the comparatively small number of respondents of the survey. But in spite of this fact, they represent an incentive for further study of students' attitudes towards mathematics and its teaching during the secondary-tertiary transition, and for the research of methods positively influencing these attitudes. This includes, for instance, activating methods developing motivation and creativity of students during the process of mathematical education [1] and positive attitudes of mathematics teachers at universities.

Acknowledgements

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Jan Długosz University in Częstochowa
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DIDACTIC PC GAME AS A TOOL OF CURRICULUM ENRICHMENT OF MATHEMATICALLY GIFTED STUDENTS EDUCATION

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Abstract. Curriculum enrichment principle is one of the approaches in education of mathematically gifted students. Usage of didactic computer games could be one of the possibilities of accomplishment.

1. Introduction

Curriculum enrichment principle is one of the ways of creating proper conditions at school, that is to say by modifying educational content, helping the gifted pupil's to develop their capacities in an optimal way. It brings enrichment of knowledge, interests and abilities beyond the regular curriculum.

The enrichment should be divided in three levels (broadening, deepening, and enriching the curriculum) and should be focused mainly on developing higher order thinking processes, self-reliance in problem solving and creativity. The enriched curriculum should follow the abilities, knowledge and needs of a gifted pupil [1, 3].

Current curricular documents designated for elementary schools support using computers for education of mathematics. They also constitute problem solving and solving of nonstandard application problems as one of the four main topics of mathematical education at elementary schools. A nonstandard application problem is called a problem that has to be solved by using not frequent methods or different methods that pupils know in their classes. One of the ways of reaching this intent is the application of didactic computer games to mathematics teaching.

There are a lot of programs or didactic games based on a mathematical problem, so we picked one of them.

2. Didactic computer game "4COLORS"

This is one of the older (a DOS program) but very usefull game based on the four colour theorem. It states that four colors are sufficient to color any map so that regions sharing a common border receive different colors.

It is very easy to demonstrate the mathematical induction proof to elementary school pupils by using the "4COLORS". We have to keep in mind the complexity of this kind of proof. During solving the next problem we can see that the pupils gifted with mathematics can comprehend. Mathematical induction is a deductive process which is specific for mathematicians but not for pupils (the mathematician in the first steps of his heuristic strategy solves using induction; in the case he finds "something" he proves by inference).

Let us solve a problem:

Is it possible to color all of the regions in the plane devided by the given number of lines using only two colors, so that regions colored by the same color share only points – vertices? If it is possible, what is the number of lines that accomplish requirements?

There is a geometric character of this problem, problem solving by using "4COLORS" is very rational. We have to draw a new map. Because of some software problems we have to draw a rectangle at first. That is the plane. Then draw a line in this rectangle. Now the plan is divided in two areas. It is easy to color this map by using only two colors. Because of the software problems we have to use the third color to color the area outside the rectangle. After that the program displays that we managed the task. Then we can draw a new map (a new rectangle) with two lines inside. The plane is divided into four areas. Again, there is no problem to color the map using only two colors. We are able to continue with drawing three lines, four lines, etc. and each time we are able to solve the problem. At this moment the pupil could dispute. What if this four lines would be placed in a different way? How many qualitatively different situations are possible when we place four lines inside the plane? Do we have to solve each situation particularly? Considering this questions, this stadium is very suggestive for each pupil because they notify: I am not evidently able to verify my hypothesis (there is always a solution for any number of lines despite their positions in a plane) in the way of solving the problem, for example, for $1, 2, \ldots, 10$ lines. That is pupil's dilemma that allowes us to perform mathematical induction. We have to notify that the most of statement characters valid for all natural numbers introduced to pupils are easily provable. In the most cases the pupil "proves" for $n = 1, 2, 3$ and then he declares: it is proved for all natural numbers.

Mathematical induction (sometimes called full induction) is connected with mathematical axiomatics, so it is complicated for beginners. Hence we can present it to the pupils as a fact:

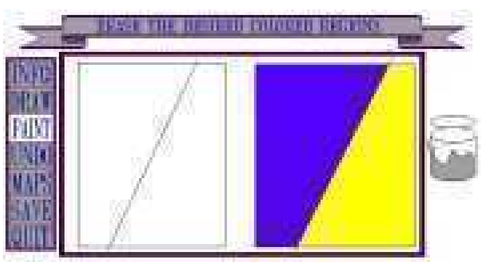
If any mathematical statement is true for some natural number k and if validity for $\mathbf{n+1}$ follows from the validity of the statement for any natural number n, then the statement is valid for all natural numbers greater than or equal to number k.

Now we reword the first task:

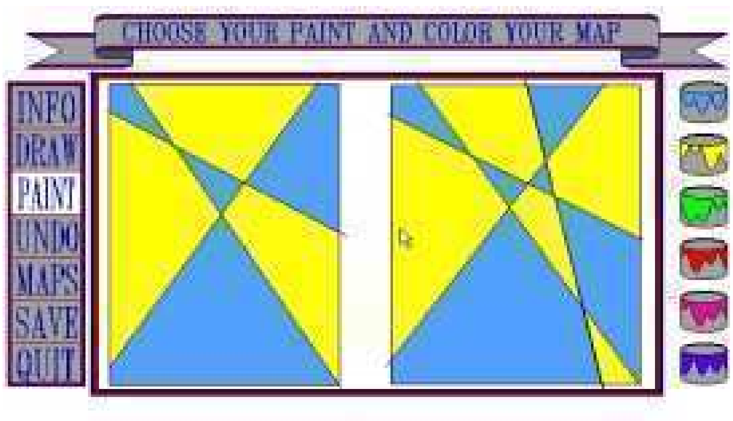
Demonstrate that it is possible to color all areas of a plane so that any pair of areas colored by the same colur contacts only in their vertices, without regards to the number of the lines dividing the plane.

The solving process is demonstrated by using "4COLORS".

1. The plane is divided by one line and these two areas are colored by different colors. The statement is valid for $k = 1$.



2. Let us draw two rectangle planes in one picture. Draw some lines into the left plane and then draw the lines the same way into the right plane and add one more line. Now we have two maps. Color by two colors the left map. In this case we manage to do it – we do not have to be affraid of failure caused by different positions of the lines. Why it is so we demonstrate later. So in the left map the statement is valid for n. Let us color the right map. The added line devides the rectangular plane into two areas. One of them is colored by the same color as in the left map. The other area has to be colored by the other color used in the left map. The result has to be the requested coloring.



In the case we draw the "added line" from the right map into the left colored map, there would be some areas where the "added line" goes through (there are the requirements broken) and some areas where does not (there are the requirements accomplished). But there is simple solution how to accomplish the requirements in the areas with "added line". We have to use the other color in one of the two areas divided by the "added line" and colored by the same color.

In the left map there are $n+1$ lines, on the ground of verity of the left map (verity for n) we proved verity of the right map (for $n+1$). Finally, the "added line" was placed randomly and so we have solved the problem of different positions of the lines – it does not matter at all!

On the ground of the full induction, the statement is valid for any random natural n.

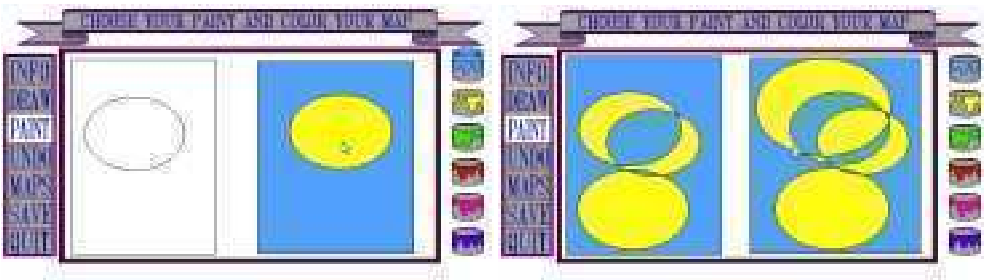
The usage of "4COLORS" for this problem solving is very apposite because pupils have an oppurtunity to go through the second induction step many times, as long as they understand its logical structure. After that they can understand deductive procedure of mathematical induction proof.

They also understand that validity for k (in this case $k = 1$) is basic; the validity of statement for $n + 1$ following from the validity of statement for n does not suffice in order to verify the validity for $n \geq k$ or for all n. At this moment we can give the pupils an idea of the mathematical induction by using domino: let us have a row of domino stones, the induction step means that if the first stone falls, the next stone is hit and falls too (the step means that the first (or kth) stone really falls).

In the task the crucial facts are that a line devides a plane into two disjoint areas and two lines can share only one point. So if we mention this facts, we can generate the next task.

Demonstrate that all the areas of a plane divided by a random number of circles/ellipses are colorable by two colors, so that two areas colored by the same number contacts just in vertices.

Solution of this problem is analogous to the solution of the first task and is clearly shown in Figures.



Acknowledgements

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Jan Długosz University in Częstochowa
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INNOVATION IN THE MATHEMATICS COMPONENT OF THE FUTURE PRIMARY SCHOOL TEACHERS' PREGRADUATE CURRICULUM

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Abstract. The aim of the paper is to present some of the results of an analysis of educational needs which was carried out among students of the Primary school teacher training study programme at Palacký University Olomouc and which was focused on the mathematical and didactical competencies of those students.

1. Introduction

Since the beginning of this year, the Department of Mathematics at Palacký University, Faculty of Education has been dealing with an ESF (European Social Fund) project called IMAKOS. The project aim is to enhance the quality of the mathematics component of the future primary school and special school teachers professional training. It is to be achieved by means of the innovation of the educational content of all the subjects, technical, mathematical as well as didactic ones.

To meet the project goals, it was above all necessary to carry out an analysis of the educational needs of Primary school teacher training and Primary school teacher training and special pedagogy study modes students, with respect to their mathematical and didactic competencies. The data were obtained by means of a non-standardized questionnaire and a didactic test in mathematics assigned to the students.

Along with the analysis of the mathematical subjects educational contents within the abovementioned study programmes taught at our faculty, as well as at other faculties, e.g. at the Faculty of Education of the University of Prešov or at the Faculty of Education of Matej Bel University in Banska Bystrica, we also took a deep look at the situation in the "Mathematics and its applications within the framework of primary education" training area.

At the level of primary education, the training area of "Mathematics and its applications" is mostly based on active learning activities, i.e. the work with mathematical objects and the use of mathematics in real-life situations. This provides students with the knowledge and skills needed in everyday life and thus makes them mathematically literate. Due to the vital role which it plays, mathematics is present at every stage of the primary education and represents a prerequisite for further studies of any kind. Within the training, a deep comprehension of the elementary thinking processes and mathematical notions and the interconnections between them are emphasized. Step by step, the pupils adopt particular notions, algorithms, terminology, symbolics and their applications (see FEP BE [2]).

Students graduated from grammar schools apply and are accepted, above all, for the studies in the abovementioned study programmes, but there are also many fresh students coming from pedagogy institutes as well as from other vocation schools, i.e. nursing, agriculture, technical schools, and apprentice training centers. At the beginning of the studies they are all surprised by the necessity of learning mathematics as well as the didactics of mathematics and one frequently runs across the statement such as "I've never been much keen on or good at mathematics." At start they do not quite grasp the significance of the tasks being a part of their introduction to mathematics at the faculty. That is why we aim at the development of the key competencies which students should dispose of, in compliance with the creation and development of the key competencies set in FEP BE [2].

2. Analysis of students' educational needs through an investigation questionnaire

During the initial attempts to identify the educational needs of the students, we used a questionnaire focused on the usefulness of the professional training within the future teachers pregraduate preparation (compiled by Kalhous and Horak [1]), which we have slightly modified to serve the purposes of our research. The questionnaire was handed out to 54 first-year students and to

44 students in the second year of the Primary school teacher training study programme. It comprised 20 items (statements). On the basis of their personal experiences, assumed knowledge, thoughts and feelings connected with their dealing with mathematics as a school subject, on the one hand, and their being taught mathematics within the framework of the didactic preparation, on the other hand, the students were asked to assess the usefulness of the given skills (knowledge) in their future occupation, i.e. that of a primary school teacher of mathematics. The assessment was carried out by choosing one of the numbers on a 5-degree scale, whereas the number 1 meant topmost quality, the number 2 – quality, 3 – relative quality, 4 – no quality, 5 – absolute lack of quality, N – unable to assess).

Chart No 1 for the 1^{st} and 2^{nd} year students

t1	Mastery of technical elements of mathematics
t2	Appropriate use of the mathematical terminology and symbolics
t3	Ability to solve a learning task in mathematics
t4	Ability to manage pupils' activities connected with solving a learning task in mathematics
t5	Ability to formulate (create) learning tasks in mathematics in compliance with the teaching goals
t6	Ability to compile a quality didactic test in mathematics
t7	Ability to set the educational goals
t8	Ability to motivate a pupil in an appropriate way
t9	Ability to work with the material didactic instruments (teaching aids, computers)
t10	Ability to assess a textbook used in teaching mathematics
t11	Acquaintance with and ability to use adequate teaching methods
t12	Ability to asses pupil's performance
t13	Ability to identify the internal as well as the external conditions making for effective learning mathematics on the side of the pupil
t14	Understanding the necessity of a permanent self-education in mathematics and its realisation in practice
t15	Awareness of the most wide-spread learning disabilities (dyscalculia) and ability to deal with them
t16	Ability to recognize a mathematically-gifted pupil and to develop his/her gift
t17	Ability to make on the spot decisions regarding typical as well as unusual pedagogical situations
t18	Ability to project (plan) one's own pedagogical activities
t19	Awareness of the alternative didactic procedures and ability to apply them
t20	Ability to communicate with the pupils

To illustrate the selected sample of students, we attached hereunder a chart demonstrating the response rate across the sample. In the first line, labelled valid, the number of the students having answered the particular question is stated. In the missing line, on the other hand, the number of the students who, for one reason or another, did not respond to the particular item of the questionnaire or did not feel competent to assess the extent to which a certain ability might be important for their future job, is stated.

Chart No 2: Response rate across the sample

Σ	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10	t11	t12	t13	t14	t15	t16	t17	t18	t19	t20
Valid	97	98	97	93	94	95	95	96	98	91	95	96	92	98	92	96	95	95	96	98
Missing	1	0	1	5	4	3	3	2	0	7	3	2	6	0	6	2	3	3	2	0

The chart makes it obvious that with some items students had no difficulties considering the level of necessity of the given abilities for their future job. This is true for the following abilities: t2, t9, t14, t20; possibly also: t1, t3. All of them may be regarded as the premises for the basic abilities and competencies every teacher should possess. However, coming to other premises, a large number of the students found it impossible to respond. They were as follows: t10, t13, t15. In our opinion, the reason why some students chose the *N* answer in response to these premises probably was connected with their limited professional experience, i.e. limited knowledge in the given area of interest, which is demonstrated by the chart below showing the response rate in particular years of study.

Chart No 3: Response rate across the whole sample

1st year	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10	t11	t12	t13	t14	t15	t16	t17	t18	t19	t20
Valid	54	54	53	50	51	52	52	53	54	47	51	52	48	54	48	52	51	51	52	54
Missing	0	0	1	4	3	2	2	1	0	7	3	2	6	0	6	2	3	3	2	0

Chart No 4: Response rate across the whole sample

2nd year	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10	t11	t12	t13	t14	t15	t16	t17	t18	t19	t20
Valid	43	44	44	43	43	43	43	43	44	44	44	44	44	44	44	44	44	44	44	44
Missing	1	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0

The results show that with increasing age, which brings about more knowledge from the studies, the ability to shape one's opinion grows.

Let us have a more detailed look at the structure of the responses.

The table shows that with all items (except items 1 and 14) students of higher years tend to prefer new skills and knowledge which they acquired in the course of their studies and which they had not run across before.

Chart No 5: Response rate in per cents

		of topmost quality	of quality	of relative quality	of no quality	absolute lack of quality	unable to assess
t1	1st year	40. 74	37.04	20.37	1.85	0	0
	2nd year	22.73	43.18	29.55	2.27	0	2.27
t2	1st year	25.93	50	18.52	5.56	0	0
	2nd year	29.55	40.91	25	4.55	0	0
t3	1st year	44.44	37.04	12.96	3.7	0	1.85
	2nd year	47.73	40.91	11.36	0	0	0
t4	1st year	40.74	33.33	16.67	1.85	0	7.41
	2nd year	68.18	20.45	6.82	2.27	0	2.27
t5	1st year	22.22	35.19	27.78	9.26	0	5.56
	2nd year	40.91	38.64	13.64	4.55	0	2.27
t6	1st year	27.78	44.44	22.22	1.85	0	3.7
	2nd year	63.64	25	6.82	2.27	0	2.27
t7	1st year	35.19	31.48	25.93	3.7	0	3.7
	2nd year	43.18	34.09	13.64	6.82	0	2.27
t8	1st year	64.81	22.22	9.26	1.85	0	1.85
	2nd year	79.55	13.64	2.27	2.27	0	2.27
t9	1st year	38.89	40.74	18.52	1.85	0	0
	2nd year	63.64	27.27	9.09	0	0	0
t10	1st year	9.26	50	22.22	5.56	0	12.96
	2nd year	13.64	43.18	36.36	6.82	0	0
t11	1st year	35.19	37.04	18.52	3.7	0	5.56
	2nd year	43.18	52.27	4.55	0	0	0
t12	1st year	61.11	27.78	7,41	0	0	3,7
	2nd year	81.82	18.18	0	0	0	0
t13	1st year	22.22	44.44	20.37	1.85	0	11.11
	2nd year	36.36	47.73	11.36	4.55	0	0
t14	1st year	11.11	53.7	31.48	3.7	0	0
	2nd year	6.82	54.55	34.09	4.55	0	0
t15	1st year	37.04	25.93	18.52	5.56	1.85	11.11
	2nd year	68.18	13.64	15.91	2.27	0	0
t16	1st year	35.19	33.33	24.07	3.7	0	3.7
	2nd year	50	36.36	13.64	0	0	0
t17	1st year	38.89	37.04	16.67	1.85	0	5.56
	2nd year	52.27	43.18	4.55	0	0	0
t18	1st year	40.74	40.74	12.96	0	0	5.56
	2nd year	56.82	34.09	9.09	0	0	0
t19	1st year	12.96	57.41	16.67	9.26	0	3.7
	2nd year	29.55	45.45	25	0	0	0
t20	1st year	77.78	16.67	5.56	0	0	0
	2nd year	95.45	4.55	0	0	0	0

A very interesting phenomenon is also the increase in the number of respondents who do not believe that it is absolutely necessary to possess technical knowledge. This illustrates quite precisely the current situation in education. Students do not require any deepening of the existing technical knowledge which, according to them, will be of no use in their job and which they consider as sufficient for them. At the same time, they emphasize deficiencies in other areas. They also regard further progress in mathematics coming with every year of studies as less important.

3. Conclusion

Another basis for the analysis of the educational needs was a didactic test, again assigned to the 1^{st} and 2^{nd} year students of Primary school teacher training study programme. Our aim was to get the students acquainted with some tasks which primary school pupils solve in the 5^{th} grade, right from the start of their studies. Their own performance should help the students realize the necessity of self-training, of an independent logical thinking, and of developing confidence in one's own abilities, etc.

Through the analysis of investigation questionnaires and of tests results we acquire valuable information which, along with the analysis of the content of curricula documents, conceptual and methodical materials (i.e. current syllabi of the particular study modes) relating to the teaching mathematical subjects, enable us to identify the educational needs of the survey group.

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GEOMETRICAL IMAGINATION AND KNOWLEDGE OF TRIANGLES AT ELEMENTARY SCHOOLS

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Abstract. A pilot study concerning rudimentary knowledge of triangles and spatial imagination is submitted to 7th-grade pupils of an elementary school and students of the same age at a grammar school. The theory is important but practical problems are more interesting for students. Problems in spatial imagination allow a teacher to specify the level of students' mathematical abilities. The aim of this pre-research is to compare knowledge of pupils of an elementary school and a grammar school.

Nowadays more practice than theory is getting to the center of interest in education at schools. Pupils need to find some bridges between theory and practice.

In mathematics, geometrical problems are a good connection between theory and practice. Pupils use their theoretical knowledge to construct concrete tasks. At first, many questions in geometry seem to be theoretical and pupils need to apply a spatial imagination to resolve these problems.

27 pupils of a grammar school in Lanškroun and 32 pupils of an elementary school in Letohrad were tested on 20 April 2011. Pupils were of the same age (2nd grade of the grammar school, 7th grade of the elementary school). Pupils answered ten questions concerning basic geometrical knowledge about triangles. The results of the research are summarized in the tables below with the following symbols used:

+ ... correct answer
− ... incorrect answer
0 ... missing answer

Question 1: The point of intersection of medians separates medians:

a) exactly in the middle
b) shorter parts of medians are halves of longer parts
c) noway, because the medians do not intersect themselves
d) without rules

Results of question 1	7th grade of the elementary school	2nd grade of the grammar school
+	16%	33%
−	81%	67%
0	3%	0%

Question 2: The midpoint of the inscribed circle of a triangle lies in the point of intersection of:

a) bisectors of angles
b) bisectors of sides
c) altitudes
d) medians

Results of question 2	7th grade of the elementary school	2nd grade of the grammar school
+	53%	30%
−	47%	70%
0	0%	0%

Question 3: The altitude of a triangle is:

a) the straight line going through the vertex of a triangle and orthogonal to the opposite side
b) the line with outside points – the vertex of a triangle and the heel of orthogonal going through the vertex of a triangle to an opposite side
c) the length of a segment as in item b)
d) the line with outside points – the vertex of a triangle and the midpoint of the opposite side

Results of question 3	7th grade of the elementary school	2nd grade of the grammar school
a (+)	65%	44%
b (+)	13%	19%
c (+)	3%	0%
d (−)	16%	33%
0	3%	4%

The third question is a special question. Three answers are correct. Only answer d is incorrect.

Question 4: The midpoint of the circumcircle of a triangle lies at the point of intersection of:

a) bisectors of angles
b) bisectors of sides
c) altitudes
d) medians

Results of question 4	7th grade of the elementary school	2nd grade of the grammar school
+	34%	37%
−	63%	63%
0	3%	0%

Question 5: The point of intersection of altitudes separates altitudes:

a) exactly in the middle
b) noway, because the altitudes do not intersect themselves
c) shorter parts of altitudes are halves of longer parts
d) without rules

Results of question 5	7th grade of the elementary school	2nd grade of the grammar school
+	22%	26%
−	66%	74%
0	12%	0%

Question 6: There are two interior angles of 40° in a triangle. This triangle is:

a) equilateral
b) obtuse-angled and isosceles
c) obtuse-angled and equilateral
d) acute-angled and isosceles

Results of question 6	7th grade of the elementary school	2nd grade of the grammar school
+	16%	56%
–	84%	41%
0	0%	3%

Question 7: The sum of the measure of acute angles in a right-angled triangle is:

a) 80°
b) 90°
c) 100°
d) 180°

Results of question 7	7th grade of the elementary school	2nd grade of the grammar school
+	44%	67%
–	56%	33%
0	0%	0%

Question 8: It is not true that in an obtuse-angled isosceles triangle:

a) one of the medians is perpendicular to the opposite side
b) it is axially-symmetrical
c) a bisector of one of angles is perpendicular to the opposite side
d) the midpoint of the circumcircle of a triangle is the same as the midpoint of inscribed circle, the centre of gravity and the point of intersection of altitudes

Results of question 8	7th grade of the elementary school	2nd grade of the grammar school
+	25%	26%
–	56%	70%
0	19%	4%

Question 9: The central transversal line of a triangle is:

a) the line with outside points – the vertex of a triangle and the midpoint of its opposite side
b) the line which is the connecting line of the midpoints of two sides of a triangle
c) the line which is the connecting line of the vertex of a triangle and the midpoint of its opposite side
d) the line going through the midpoint of a triangle

Results of question 9	7th grade of the elementary school	2nd grade of the grammar school
+	25%	48%
−	47%	45%
0	28%	7%

Question 10: The measure of interior angles in the obtuse-angled isosceles triangle is:

a) different, depending on the size of sides
b) two of these angles are the same and the third one is the complement to 180°
c) we are not able to determine it
d) 60°

Results of question 10	7th grade of the elementary school	2nd grade of the grammar school
+	44%	89%
−	47%	11%
0	9%	0%

The pupils of the grammar school were more successful than the pupils of the elmentary school in the majority of questions. The better group was formed by the pupils of the grammar school, but there was nobody with full score – 10 points, while at the elementary school there was one pupil with 10 points.

The group of pupils of the elementary school was better only in two questions (number 2 and 3). The worst answers in both groups were to question 5. The least good answers at the elementary school were to questions number 1 and 6 and at the grammar school to questions number 5 and 8. Mistakes in question 8 occurred probably because this question was in negation. To question number 3 there were three good answers, but nobody marked all of them. The majority of pupils chose the answer a (an altitude of a triangle is a straight line going through the vertex of triangle and orthogonal to the opposite side).

Different results at two schools are influenced by more factors. Some problems could be caused by inattention of the part of pupils, nontraditional form of validation of knowledge, and lack of maturity for the correct, independent and accurate understanding of basic concepts of plane geometry. Each of the schools uses different school texts and there are different teachers, too. The better results had pupils of a grammar school, what was expected because a grammar school is the selective school.

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COMPUTER PROOFS IN PLANE GEOMETRY

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Abstract. Over the past 25 years highly successful methods for geometry theorem proving have been developed. We will use elementary and understandable examples to show the nature of the techniques for verification of geometric constructions made with interactive geometry environment and for proving geometric statements. In addition to some informations about the WinGCLC software with specific language, we look at the system GeoThms that integrates Automatic Theorem Provers, Dynamic Geometry Tools and a database. The abovementioned system provides an environment suitable for new ways of studying and teaching geometry at different levels.

1. Introduction

Dynamic geometry software (DGS) is the most widely used software for mathematics in education. DGS allows the user to create complex geometric constructions step by step using free objects such as free points, construct new objects depending on the existing ones (for instance, the line passing through two distinct points) and then move the starting points to explore how the whole construction changes. The corresponding figure is updated in real time. There exist a large number of free and commercial software[1] (e.g. Baghera, Cabri, Cinderella, Dr. Geo, Eukleides, WinGCLC, GeoGebra, Geometer's Sketchpad, Geometrix, Geometry Expert (GEX), Geometry Explorer, Géoplan, GeoNext, GeoProof, KGeo, KIG, Non-Euclid, OpenEuclide, WinGeom). Interactive geometry software can help teachers to illustrate abstract concepts in geometry and students may explore and understand the secret of plane geometry on their own. Therefore, DGS systems are used for two activities:

[1] http://en.wikipedia.org/wiki/Dynamic_geometry_software

(1) to help a student to create geometric constructions; (2) to help a student to explore a figure, invent conjectures, and check facts.

From the beginning, various kinds of DGS have been the paradigm of new technologies applied to mathematics education, area where they have found their most applications. Their convenience in the classroom is almost unanimously praised by education experts. However, questions have been raised on the influence or interaction of the use of DGS on the development of the concept of proof in school curricula [2]. Sometimes, formal proofs have been replaced by the construction of a great number of examples of a configuration, what has come to be known as a visual proof.

Geometry is also an important area for automatic theorem proving (ATP), the field of using automated methods for creating mathematical proofs. The exactness and broad theoretical foundation that is present in geometry and the beauty and elegance of geometry make it a wonderful platform for experimentation and testing for new algebraic and other methods.

Several DGS systems with proof-related features can be roughly classified into two categories [5]:

- systems that permit one to build proofs;
- systems that permit one to check facts using an automated theorem prover.

A breakthrough in automated geometry theorem proving (AGTP) is made by Wen-Tsün Wu. Restricting himself to a class of geometry statements of *equality type*, in 1977 Wu introduced a method which can be used to prove quite difficult geometry theorems efficiently. Here we would like to remind that Wu's method cannot deal with theorems involving inequalities.

AGTP has two major lines of research [4, 9]: the synthetic proof style and the algebraic proof style. *Algebraic proof* style methods are based on reducing geometric properties to algebraic properties expressed in terms of Cartesian coordinates. *Synthetic methods* attempt to automate traditional geometry proof methods. The synthetic methods provide traditional (not coordinate-based), human-readable proofs. In both cases (algebraic or synthetic) we claim that the AGTPs can be used in the learning process.

2. WinGCLC software

WinGCLC package is a tool which enables producing geometrical figures (i.e. digital illustrations) on the basis of their formal descriptions. This approach is guided by the idea of formal geometrical constructions. A geometrical construction is a sequence of specific, primitive construction steps (*elementary constructions*). Figure descriptions in WinGCLC are usually made by a list of definitions of several (usually very few) fixed points (defined in terms of Cartesian plane, e.g. by pairs of coordinates) and a list of construction steps based on that points.

WinGCLC uses a specific language for describing figures. The GCLC language consists of the following groups of commands: *definitions*, *basic constructions*, *transformations*, *drawing commands*, *marking and printing commands*, *low level commands*, *Cartesian commands*, *commands for describing animations*, *commands for the geometry theorem prover*. These descriptions are compiled by the processor and can be exported to different output formats. There is an interface which enables simple and interactive use of a range of functionalities, including making animations.

The theorem prover (GCLCprover) built into WinGCLC is based on Chou's algorithm for proving geometry theorems (*area method*, see [1]). This method belongs to the group of synthetic methods. The main idea of the method is to express hypotheses of a theorem using a set of constructive statements, each of them introducing a new point, and to express a conclusion by an equality of expressions in geometric quantities such as *ratio of directed parallel segments* $\overline{AB}/\overline{CD}$ (where $\overline{AB}$ denotes the *signed length*[2] of a segment AB), *signed area* S_{ABC} (the area of a triangle ABC with a sign depending on the order of the vertices A, B and C[3]) and *Pythagoras difference* $P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2$ as a generalization of the Pythagoras equality (for details see [8]).

The proof is then based on eliminating (*in reverse order*) the points introduced before, using for that purpose a set of appropriate lemmas. After eliminating all introduced points, the current goal becomes a trivial equality that can be simply tested for validity. At all stages, different expression simplifications are applied to the current goal.

Let us take next elimination lemma and one example:

Lemma 1. Let S_{ABY} be the signed area of a triangle ABY for distinct points A, B and Y. For collinear points Y, U and V it holds

$$S_{ABY} = \frac{\overline{UY}}{\overline{UV}} S_{ABV} + \frac{\overline{YV}}{\overline{UV}} S_{ABU}.$$

Example 1 (of elimination technique). Let Y be a point on a line passing through a given point W and parallel to a line UV, such that $\overline{WY} = r\overline{UV}$,

[2]If we prescribe a direction from A to B as positive, then $\overline{AB} = |AB|$ and $\overline{BA} = -|AB|$.

[3]S_{ABC} is positive if we move along the perimeter of a triangle from the vertex A to B and C anti-clockwise.

where r can be a rational number, a rational expression in geometric quantities, or a variable. Then it holds:

$$S_{ABY} = S_{ABW} + r(S_{ABV} - S_{ABU}).$$

The constructions accepted by GCLCprover are: construction of a line given by two points; an intersection of two lines; the midpoint of a segment; a segment bisector; a line passing through a given point, perpendicular to a given line; a foot from a point to a given line; a line passing through a given point, parallel to a given line; an image of a point in a given translation; an image of a point in a given scaling transformation; a random point on a given line.

Let us consider the triangle area theorem as an example:

Example 2 (Triangle area theorem). Each median divides the triangle into two smaller triangles which have the same area.

Proof (using the method). Let ABC be a triangle, and M be a midpoint of AB. We first translate the goal into its equivalent using the signed area:

$$S_{AMC} = S_{MBC}.$$

The proof is actually to eliminate a point M. Using Example 1, the above equality of signed areas can be reduced to the expressions as follows:

$$S_{AMC} = S_{CAM} = S_{CAA} + \frac{1}{2}(S_{CAB} - S_{CAA}),$$

$$S_{MBC} = S_{BCM} = S_{BCA} + \frac{1}{2}(S_{BCB} - S_{BCA}).$$

The new goal is:

$$\frac{1}{2}S_{CAB} = \frac{1}{2}S_{BCA}.$$

The proof is completed as $S_{CAB} = S_{BCA}$.

We can use WinGCLC to validate the previous statement by describing the construction and proving the property for given three fixed distinct points A, B, C with M being the midpoint of AB. The WinGCLC code for this construction and the corresponding illustration (LaTeX output), are shown in Figure 1. It can be checked (using GCLCprover) that a median CM divides a triangle ABC into two smaller triangles ($\triangle AMC$ and $\triangle MBC$) which have the same area, i.e. $S_{AMC} = S_{MBC}$. This statement can be given in the code of GCLC language by the following line:

```
point  A 10 10
point  B 70 10
point  C 55 35
midpoint  M A B

drawsegment  A B
drawsegment  B C
drawsegment  A C
drawsegment  C M

cmark_b  A
cmark_b  B
cmark_t  C
cmark_b  M
```

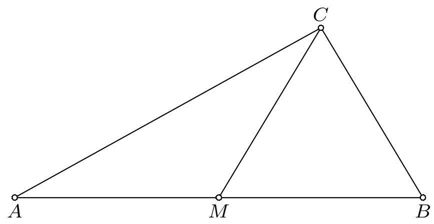

Figure 1: Example 1

```
prove { equal { signed_area3 A M C } { signed_area3 M B C } }
```

The prover produces a short report of information on number of steps performed, on CPU time spent and whether or not the conjecture has been proved. For our example we have:

```
The theorem prover based on the area method used.
 Number of elimination proof steps:  2
 Number of geometric proof steps:    7
 Number of algebraic proof steps:    9
 Total number of proof steps:        18
Time spent by the prover: 0.004 seconds
The conjecture successfully proved.
The prover output is written in the file triangle_area.tex.
```

The prover also generates a proof in LaTeX form (in the file `proof.tex`). We can control the level of details given in the generated proof. The proof consists of *proof steps*. For each step, there is an explanation and its semantic counterpart. This semantic information is calculated for concrete points used in the construction. For our example (in Figure 1), we will get the following:

(1)	$S_{AMC} = S_{MBC}$	, by the statement
(2)	$S_{CAM} = S_{BCM}$	, by geometric simplifications
(3)	$\left(S_{CAA} + \left(\frac{1}{2}(S_{CAB} + (-1 \cdot S_{CAA}))\right)\right) = S_{BCM}$	, by Lemma 29 (M eliminated)
(4)	$\left(0 + \left(\frac{1}{2}(S_{CAB} + (-1 \cdot 0))\right)\right) = S_{BCM}$	, by geometric simplifications
(5)	$\left(\frac{1}{2}S_{CAB}\right) = S_{BCM}$	, by algebraic simplifications
(6)	$\left(\frac{1}{2}S_{CAB}\right) = \left(S_{BCA} + \left(\frac{1}{2}(S_{BCB} + (-1 \cdot S_{BCA}))\right)\right)$	, by Lemma 29 (M eliminated)
(7)	$\left(\frac{1}{2}S_{CAB}\right) = \left(S_{CAB} + \left(\frac{1}{2}(0 + (-1 \cdot S_{CAB}))\right)\right)$	, by geometric simplifications
(8)	$0 = 0$	, by algebraic simplifications

Q.E.D.

3. GeoThms

GeoThms[4], is a framework that links DGS (GCLC and Euklides), AGTP (GCLCprover), and a repository of geometry problems (GeoDB), providing a common web interface for all these tools (see Figure 2).

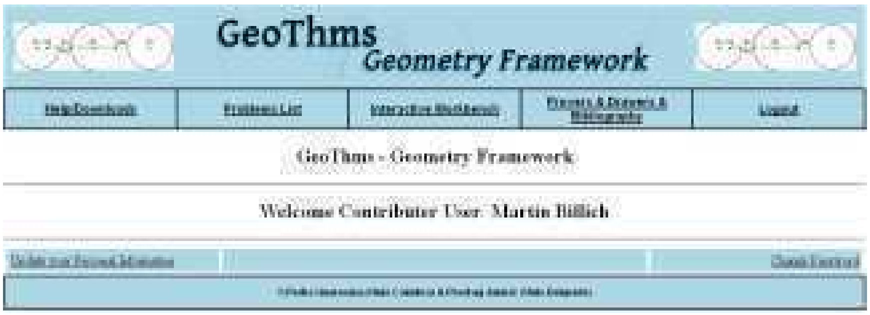

Figure 2: GeoThms – Regular Users Page

Integration of GeoThms with dynamic geometry software and automatic theorem provers and its repository of theorems, figures and proofs give the user the possibility to browse easily through the list of geometric problems, their statements, illustrations and proofs, and also to use interactively the drawing and proving programs (see Figure 3).

[4]GeoThms is a set of PHP scripts of top of a MySQL database and is accessible from http://hilbert.mat.uc.pt/GeoThms.

Figure 3: GeoThms – Theorem Report

As a web service GeoThms emphasizes [6]: (1) a simple interface based on using geometrical specification languages of the underlying geometrical tools; (2) a low communication burden. A basic communication, concerning describing geometrical constructions and conjectures, is based on formal languages of the underlying geometrical tools. Within GeoThms, data are presented in *textual form* as GCLC code, or as XML rendered as HTML, and *graphical form* as JPEG image, or as SVG image. When adding new geometrical tools, it will be sufficient to develop converters from its format to XML and vice versa. This enables converting from any format to any other, and consequently makes usable the whole of the repository to any geometrical tool.

4. Conclusion

In this paper we present some advantages of interactive geometry system WinGCLC, automated theorem prover GCLCprover, and geometry framework GeoThms. The built-in module is based on the area method for Euclidean geometry. The main advantage of this method is that each step of the generated proof has clear geometric meanings and the proofs are generally elegant. The computer program based on the area method has produced proofs of more than 500 geometry theorems, some of which are even shorter than those given by geometry experts. A drawback is that the students must be taught the "area axioms" instead of the standard Euclidean axioms.

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RANDOM WALK – FUZZY ASPECTS

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Abstract. Some beautiful and powerful mathematical ideas are hard to present to students because of the involved abstract language (notation, definitions, theorems, proofs, formulas) and lack of time. Animation and "mathematical experiments" provide a remedy. In the field of stochastics, the Galton board experiment presents several fundamental stochastic notions: a random event, independent random events, the binomial distribution, limit distribution, normal distribution, interpretation of probability, and leads to their better understanding. Random walk is a natural generalization of the Galton board. We use random walks as a motivation and presentation of basic principles of fuzzy random events and fuzzy probability. Fuzzy mathematics and fuzzy logic generalize classical (Boolean) mathematics and logic, reflect everyday experience and decision making and have broader applications. Experimenting with random walks also sheds light on the transition from classical to fuzzy probability.

1. Introduction

Probability and statistics are considered to be important and useful components of the general mathematical education. Unfortunately, due to the abstract language of mathematics (notation, definitions, theorems, proofs, formulas) and lack of time it is usually hard to present students with some beautiful and powerful stochastic ideas. A possible remedy is to work with animation and "stochastic experiments".

Internet sources provide numerous and detailed information about experiments with the Galton board, see for example:

http://en.wikipedia.org/wiki/Bean_machine,

http://animation.yihui.name/prob:bean_machi,

http://www.jcu.edu/math/isep/Quincunx/Quincunx.html,

http://mathworld.wolfram.com/GaltonBoard.html.

Experimenting with the Galton board enables us to present several fundamental stochastic notions and laws, for example, a random event, independent random events, the binomial, normal, and limit distributions, interpretation of probability and so on, in a natural way, and leads to their better understanding. Our goal is to study the Galton board and some of its generalizations from the viewpoint of random walks and fuzzy probability. We believe that our approach provides a vehicle to convey to students basic ideas of both classical and nonclassical fields of stochastics and sheds some light on the transition from classical to fuzzy probability (cf. [1], [4]). The latter one reflects everyday experience and decision making and has broader applications.

In this paper we concentrate on finite random walks, but the infinite ones constitute another important topic to be included into "stochastic experiments", see for example

http://en.wikipedia.org/wiki/Random_walk.

Here we would like to point out a surprising fact that even a very small change of the probability $p(l) = p(r) = 1/2$ (going left or right) in the symmetric one-dimensional random walk to $p(l) = 1/2 + 0.01$, $p(r) = 1/2 - 0.01$ leads to a very nonsymmetric behaviour, see

http://artax.karlin.mff.cuni.cz/ macim1am/pub/antoch/pdf.

The next steps in animation should be relationships between the Galton board and the Moivre-Laplace limit theorems leading to the normal distribution, various laws of large numbers, and limit theorems. "But that's another story", as Rudyard Kipling would say.

2. The Galton board

According to http://mathworld.wolfram.com/GaltonBoard.html *the Galton board, also known as a quincunx or bean machine, is a device for statistical experiments named after English scientist Sir Francis Galton. It consists of an upright board with evenly spaced nails (or pegs) driven into its upper half, where the nails are arranged in staggered order, and a lower half divided into a number of evenly-spaced rectangular bins. The front of the device is covered with a glass cover to allow viewing of both nails and slots. In the middle of the upper edge, there is a funnel into which balls can be poured, where the diameter of the balls must be much smaller than the distance between the nails.*

The funnel is located precisely above the central nail of the second row so that each ball, if perfectly centered, would fall vertically and directly onto the uppermost point of this nail's surface. Each ball follows a path starting at the center nail, then bounces either right or left and so on, and ultimately lands in one of the bins.

Schematically, it can be visualized via a graph starting with one vertex v_{00} on the level zero, continuing with two vertices v_{10}, v_{11} on the level one and so on, ending with $N+1$ vertices (bins) $v_{N0}, v_{N1}, \ldots, v_{NN}$ on the level N, see Figure 1.

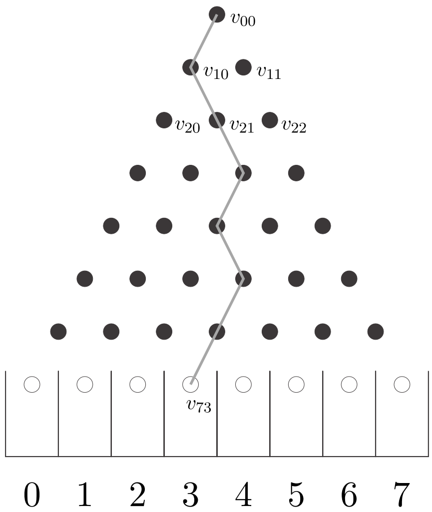

Figure 1: Visualization of of a ball path via a graph

The Galton board is connected to the binomial distribution in the following way. Each time a ball hits one of the nails, it can bounce left (or right) with some probability $p(l)$ (right with the probability $p(r) = 1 - p(l)$). For symmetrically placed nails, balls will bounce left or right with equal probability, so $p(l) = p(r) = 1/2$. The probability that a ball (after hitting $N-1$ nails) ends in the nth bin, $n = 1, 2, \ldots, N$, is

$$P(n) = \binom{N}{n} p(l)^n p(r)^{N-n}.$$

Even a novice in probability should be able to appreciate how experimenting with the Galton board is related to random walks. Indeed, the path of a ball can be viewed as a random walk on the graph of Galton board. The paths constitute a discrete probability space and we offer an alternative way how to calculate *the probability of a path.* To this end, we recall the notion of a conditional probability.

Let us repeat some random experiment N times independently (the outcomes do not depend on the previous experiments). We consider two events A and B, n_B is the number of occurrence of B (we assume $n_B > 0$), and $n_{A\cap B}$ is the number of their joint occurrences (we count only occurrences of A when B has occurred). Then

$$\frac{n_{A\cap B}}{n_B} = \frac{n_{A\cap B}/N}{n_B/N}$$

means that the *conditional probability* of A given B should be defined as

$$P(A|B) = \frac{P(A\cap B)}{P(B)}.$$

Of course, we assume that $P(B) > 0$ and the definition is based on the *interpretation of probability via relative frequency.*
QUESTION: What is the probability of the path $(v_{00}, v_{10}, \ldots, v_{N0})$?

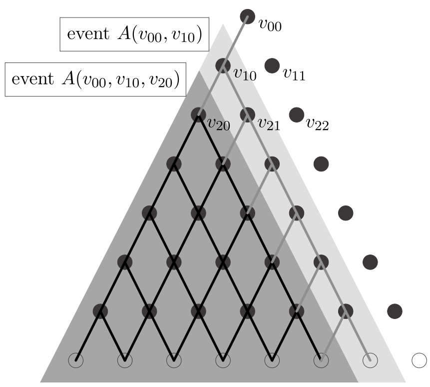

Figure 2: Events as sets of paths

Denote by $A(v_{00}, v_{10})$ the set of all paths going trough v_{10}, $A(v_{00}, v_{11})$ the set of all paths going trough v_{11}, and $A(v_{00}, v_{10}, v_{20})$ the set of all paths going trough v_{10} and then through v_{20}.

It is easy to see (Figure 2) that $P\big(A(v_{00}, v_{10})\big) = P\big(A(v_{00}, v_{11})\big) = 1/2$ and $A(v_{00}, v_{10}, v_{20}) \subset A(v_{00}, v_{10})$. Further,

$$P\big(A(v_{00}, v_{10}, v_{20})|A(v_{00}, v_{10})\big) = P\big(A(v_{00}, v_{10}, v_{21})|A(v_{00}, v_{10})\big) = 1/2.$$

Thus

$$P\big(A(v_{00}, v_{10}, v_{20})|A(v_{00}, v_{10})\big) = \frac{P\big(A(v_{00}, v_{10}, v_{20}) \cap A(v_{00}, v_{10})\big)}{P\big(A(v_{00}, v_{10})\big)}$$

implies that $P\big(A(v_{00}, v_{10}, v_{20})\big) = (1/2)^2$.

Repeating the reasoning, we arrive to the conclusion that the probability of the path $(v_{00}, v_{10}, \ldots, v_{N0})$ equals $(1/2)^N$. Analogously, we can calculate the probability of any other path: it is equal to $(1/2)^N$. So, we ended up with a classical (discrete) probability space the elementary events of which are exactly the paths of balls in the Galton experiment.

3. Walking on the Galton board

The random walk on *the classical Galton board* is rather simple. Each vertex v_{Nk}, $k = 0, 1, 2, \ldots, N$, is absorbing, other vertices are not. From any other vertex a ball can proceed to two adjacent vertices on the next level with equal probability 1/2. We can study "two step walks" or "k step walks" and ask about the corresponding conditional probabilities. In such cases *combinatorics suffices.* On a more complicated board, a ball at each vertex can proceed to more than two points on the next level and the conditional probabilities can vary from one level to the next level, and then *combinatorial methods do not suffice.* We believe (see the next section) that fuzzy probability offers a natural approach to such random experiments.

The original Galton board can be used for less traditional experiments. We mention two of them. First, let us imagine that inside the board there is another funnel pointing to some fixed vertex v_{ij} which forces all the balls to through it. We can study the relationships between the (discrete) probability spaces describing the modified experiment and the original one. Second, let us imagine that behind the board a magnet is placed to influence the fall of balls. This time it is impossible to calculate the probabilities of individual paths but, using statistical tests, we can carry out a large number of experiments and test whether the magnet has an impact on the experiment. Similar "statistical activities" can be carried out in the case of the first modified experiment.

A less traditional approach to walking on the Galton board is to study the transition of balls from a given level to the next one. Each level can be viewed as a discrete probability space, the transition can be studied as a transformation of one probability space into another, and the consecutive transitions can be chained as the compositions of transformations.

4. Transformations

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ be a finite set, let p be a probability function on Ω, i.e. $0 \leq p(\omega_i) \leq 1$ and $\sum_{i=1}^{m} p(\omega_i) = 1$. Then (Ω, p) is said to be a *discrete probability space.* Note that to each probability function p on Ω there corresponds a probability measure P defined on subsets of Ω and, for discrete probability spaces, there is a natural one-to-one correspondence between probability functions and probability measures. *In what follows, all the probability spaces will be discrete.*

Definition 1. *Let* (Ω, p) *and* (Ξ, q) *be probability spaces. Let* T *be a map of* $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ *into* $\Xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ *such that*

$$q(\xi_j) = \sum_{\omega_i \in T^{\leftarrow}(\xi_j)} p(\omega_i) \quad \textit{for all} \quad j \in \{1, 2, \ldots, n\} \quad \textit{such that} \quad q(\xi_j) > 0.$$

Then T *is said to be a* transformation *of* (Ω, p) *to* (Ξ, q). *If* Ξ *is a set of real numbers, then* T *is said to be a* random variable.

Each transformation T can be visualized as a system of pipelines going from Ω to Ξ, through which p flows and results in q, see Figure 3 (cf. [1], [2]).

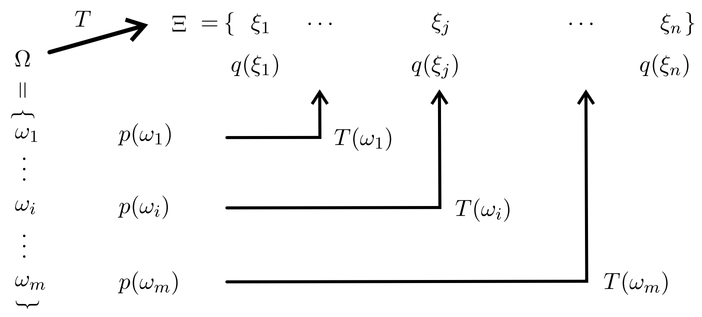

Figure 3: Transformation of a probability space

Let (Ω, p) and (Ξ, q) be probability spaces. It is natural to ask the following question: Does there always exist a transformation of (Ω, p) to (Ξ, q)? The answer is NO.

Indeed, for probability spaces (Ω, p) and (Ξ, q), if Ξ has more points than Ω, $p(\omega_i) = 1/m$ for all $i = 1, 2, \ldots, m$, and $q(\xi_j) = 1/n$ for all $j = 1, 2, \ldots, n$, then there is *no transformation* of (Ω, p) to (Ξ, q).

As shown in [1] and [2], if we replace the classical pipeline (sending the whole amount of each $p(\omega_i)$ to exactly one ξ_j) by a more complex pipeline (sending each $p(\omega_i)$ proportionally to several/all points of Ξ), then the answer is YES. The solution, called a “fuzzy transformation”, is based on Figure 4.

Observe that our complex pipeline is determined by a special matrix $\mathbf{A} = (a_{ij})_{m \times n}$, and the corresponding “fuzzy transformation” of p on Ω into q on Ξ can be described as follows: q (as a vector) is the (matrix) product of p (as a vector) and $\mathbf{A}$. The ith row of $\mathbf{A}$

$$q_i = (a_{i1}, a_{i2}, \ldots, a_{in})$$

is a probability function on Ξ, and a_{ij} can be interpreted as the probability of “transition” from ω_i to $\xi_j \in \Xi$, see [3].

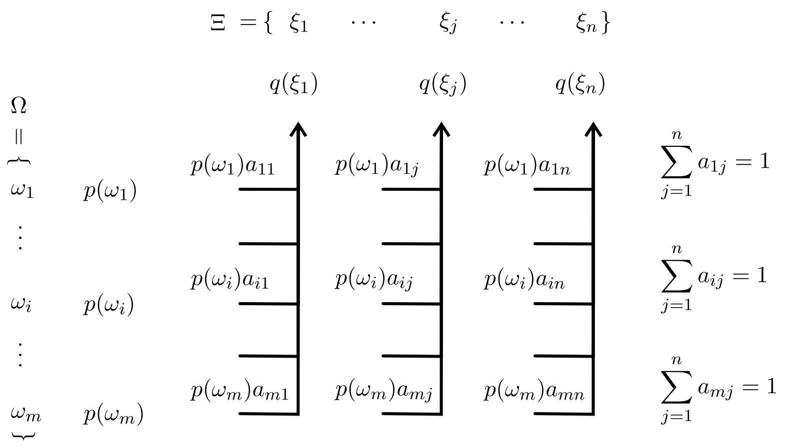

Figure 4: "Fuzzy transformation" of a probability space

Definition 2. *Let* $\mathbf{A} = (a_{ij})_{m\times n}$ *be an* m*-by-*n *matrix such that all* a_{ij} *are non-negative and* $\sum_{j=1}^{n} a_{ij} = 1$ *for all* i*,* $1 \le i \le m$*. Then* $\mathbf{A}$ *is said to be a* generalized stochastic matrix*. Further, if* $a_{i,j} \in \{0, 1\}$ *for all indexes, then* $\mathbf{A}$ *is said to be a* crisp generalized stochastic matrix*. If* $m = 1$*, then* $\mathbf{A}$ *is condensed to* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and is called a* stochastic vector*. If* $m = n$*, then* $\mathbf{A}$ *is called a* stochastic matrix*.*

Note that the product of two generalized stochastic matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{n\times l}$ is a generalized stochastic m-by-l matrix (in particular, the product of a stochastic vector and a generalized stochastic matrix is a stochastic vector).

Definition 3. *Let* (Ω, p)*,* $\Omega = \{\omega_1, \ldots, \omega_m\}$*, and* (Ξ, q)*,* $\Xi = \{\xi_1, \ldots, \xi_n\}$*, be probability spaces. Let* $\mathbf{A} = (a_{ij})_{m\times n}$ *be a generalized stochastic matrix. Let* $T_{\mathbf{A}}$ *be a map of* Ω *into the set of all probability functions on* Ξ *defined by*

$$T_{\mathbf{A}}(\omega_i) = (a_{i1}, a_{i2}, \ldots, a_{in}), \quad i = 1, 2, \ldots, m.$$

If $q = p\mathbf{A}$*, then* $T_{\mathbf{A}}$ *is said to be a* fuzzy transformation *of* (Ω, p) *to* (Ξ, q)*.*

Let (Ω, p) and (Ξ, q) be probability spaces. Define $\mathbf{A} = (a_{ij})_{m\times n}$ as follows:

$$q = (a_{i1}, a_{i2}, \ldots, a_{in}), \quad i = 1, 2, \ldots, m.$$

Let $T_{\mathbf{A}}$ be the corresponding map of Ω sending each ω_i into q.

Lemma 1. $T_{\mathbf{A}}$ *is a fuzzy transformation of* (Ω, p) *to* (Ξ, q)*.*

Note that there are other (non trivial) fuzzy transformations of (Ω, p) to (Ξ, q), and fuzzy transformations are related to probability functions on the product $\Omega \times \Xi$ such that p and q are marginal probabilities, see [3].

Let (Ω, p), $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, (Ξ, q), $\Xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$, and (Λ, r), $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ be discrete probability spaces. Let $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{n\times l}$ be generalized stochastic matrices such that $T_{\mathbf{A}}$ is a fuzzy transformation of (Ω, p) to (Ξ, q) and $T_{\mathbf{B}}$ is a fuzzy transformation of (Ξ, q) to (Λ, r). Let $\mathbf{C} = (c_{ij})_{m\times l} = \mathbf{A} \times \mathbf{B}$ and let $T_{\mathbf{C}}$ be the corresponding map of Ω into probability functions on Λ.

Lemma 2. *$T_{\mathbf{C}}$ is a fuzzy transformation of (Ω, p) to (Λ, r).*

5. Generalized random walk

A *generalized random walk* can be viewed as a finite series of successive fuzzy transformations "governed" via the product of constituent matrices. Indeed, for $l = 1, 2, \ldots, N$, let (Ω_l, p_l), $\Omega_l = \{\omega_{l1}, \omega_{l2}, \ldots, \omega_{lm_l}\}$, be discrete probability spaces and, for $l = 1, 2, \ldots, N-1$, let $\mathbf{A}_l$ be generalized stochastic matrices such that $T_{\mathbf{A}_l}$ is the corresponding fuzzy transformation of (Ω_l, p_l) to (Ω_{l+1}, p_{l+1}).

The starting point (top vertex) can be viewed as a trivial probability space (Ω_0, p_o), where Ω_0 consists of just one point $\{\omega_{00}\}$ and $p_0(\omega_{00}) = 1$. Formally, p_1 can be viewed as the "fuzzy image" of p_0 and (Ω_1, p_1) can be viewed as the fuzzy transformation of (Ω_0, p_o) (via T_{p_1}). Consequently,

$$p_N = p_1 \mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_{N-1}$$

enables us to calculate the probability of "a generalized random walk starting at ω_{00} ends up at ω_{Nk}, $k = 1, 2, \ldots, m_N$".

The fact that we send a point (elementary event) to a probability measure has definitely a quantum nature and characterizes the transition from classical to fuzzy transformations [1].

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VIEW ON SOME THEORIES OF MATHEMATICS EDUCATION

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Abstract. In this article we present some theoretical approaches in mathematics education, especially in Poland and Hungary, with practical examples. We also describe some aspects of using ICT in mathematics education and cooperative learning.

1. Introduction

Didactics of mathematics is very young research field which will find its position in the system of research subjects. According Szendrei [10], didactics of mathematics is a specific research field which develops actual questions of mathematics education, understanding and character of mathematical notions in teaching process. A lot of questions are connected with other research subjects such as psychology, pedagogy, sociology, philosophy and so on. For this reason we can use in didactics of mathematics the results and methods of other research subjects. Szendrei formulates the following goals for research in mathematic education:

- Theoretical background of teaching and learning mathematics.
- Construction and structure of mathematics and their influence on process of gaining knowledge in mathematics education.
- Difficulties of the process of gaining knowledge in mathematics education.

- Education of mathematics as a cognitive process.
- Heuristics and discovering as a tool of knowledge and process of education.
- Relationship between mathematics, culture and community.
- Social processes in teaching and learning mathematics.
- Behavior, attitudes and thinking of pupils in mathematic education.
- Modern technologies, their possibilities and borders in mathematics education.

2. View of Tamás Varga

Tamás Varga in his work [12]:

- would like to find suitable models for teaching set theory, algebra, functions and logic,
- would like to find complex view by building of notions,
- recommends to use open problems in which a pupil tries to find suitable model for problem solving,
- recommends that the teaching process must be based on the internally motivation of pupil through the suitable motivational tools (games, problems of real live, using history of mathematics and so on).

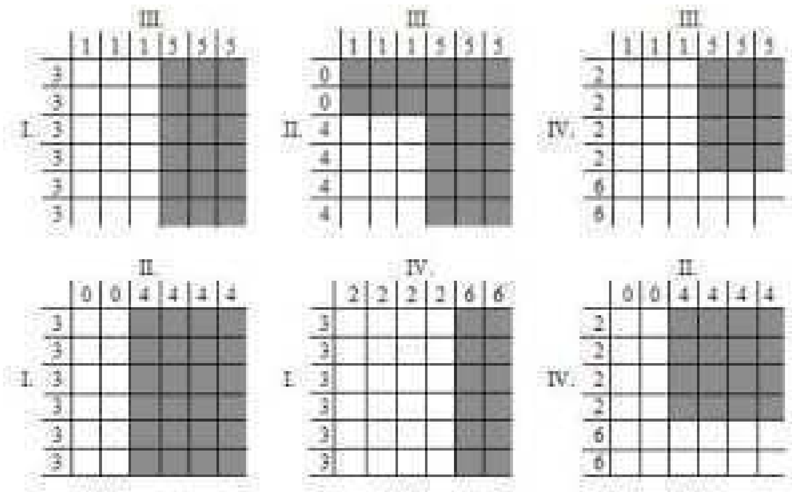

Figure 1

Pálfalvi [7] shows one example used by Tamás Varga from school probability: We have four cubes with following signing of the walls:

I: 3, 3, 3, 3, 3, 3
II: 0, 0, 4, 4, 4, 4
III: 1, 1, 1, 5, 5, 5
IV: 2, 2, 2, 2, 6, 6

We have now the game for two players. The player wins, if he throws the bigger number. Game situations according to different cubes can be shown by the following schemata (the grey squares are for the situations when cubes in columns win, see Figure 1).

From the tables we see that the cube IV is "better" than the cube I and the cube II is "better" than the cube IV, but the cube I is "better" than II. (The relation is not transitive). The relations between cubes are shown by the following schema (the arrow is oriented to the "better" cube):

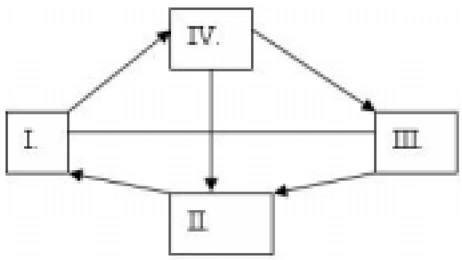

Figure 2

This example illustrates how it is possible to meet paradoxes in stochastics, which was also shown by Płocki [8].

3. Cooperative learning in mathematics education

Cooperative learning has many applications in mathematical education. According to Jablonský [3], this learning has five essential principles:

1. Development of positive interdependence.
2. Assurance of face-to-face promotive interaction.
3. Strengthening of individual accountability and personal responsibility.
4. Improvement of interpersonal and small groups skills.
5. Reflection of group processing.

Nowadays this type of learning is very important for development of social skills of pupils. Some educational projects concerning creation of teaching materials for GeoGebra Wikipedia can be realized in effective way only in the case if these materials are prepared by groups of pupils.

4. ICT in mathematics education

According to Oldknow and Taylor [6], we can identify at least three reasons for promoting the integration of Information and Communication Technologies (ICT) in mathematics teaching at schools:

Desirability: From the students side, the use of ICT may stimulate their motivation and curiosity; encourage them to develop their problem-solving strategies. From the teachers side, the use of ICT may improve their efficiency, release more time to address students individually, stimulate re-thinking their approach to teaching and understanding.

Inevitability: Many fields of publishing have moved from printing to electronic form. This applies to conference proceedings, reference works such as encyclopaedias, small-circulation textbooks, special journals, etc.

Public policy: In Slovak National Curriculum ISCED 1, 2 and 3 there is defined that Mathematics as a subject belongs to the group "Mathematics and Working with Information".

According Gunčaga, Fulier and Eisenmann [2], implementation of ICT in education brings more open questions which are common for mathematics education:

- The use of which technological tools in learning mathematics are important and relevant to different groups (primary schools, high schools and universities)?
- What role can different ICT instruments play in making education more efficient?
- What effects do the new tools and technologies have concerning the cognitive processes?
- What has to be changed in the curriculum of school mathematics?
- How has the use of ICT changed the curriculum of school mathematics so far?

In last years the open source software GeoGebra has been often used. Nowadays it is accessible in 50 language versions. This system joins together the computer algebra system, dynamical geometrical software and spreadsheet. Its big advantage consists in user friendly character and possibility to create dynamical HTML WebPages with interactive pictures (see www.geogebra.org). Teaching materials developed by this software is possible to find on the GeoGebra Wiki. Its Slovak version is step by step created now, and we are interested in cooperation in this developing process with teachers in schools, pupils and students – future teachers at universities. Some examples can be found in [1, 4, 5, 11].

5. Conclusions

In the teaching process, the teacher plays an important role. ICT does not change this, because the computer and software are only tools, not goals. Pólya [9] formulated the following ten practical rules for the mathematics education. The teacher should

1) have interest in professional character of his teaching.

2) know professional character of his teaching.

3) know the center of attraction of teaching and the fact that the best way for the teacher is the way which he finds himself.

4) know the images of the pupils. What do they expect? What is difficult for them?

5) not only give the pupils professional knowledge, but also develop their working abilities (correctness of the algorithm and its steps).

6) teach pupils how to discuss.

7) teach pupils how to prove.

8) develop using by pupils heuristic methods for problem solving, find non-visible general rules.

9) not show the pupils the solution of the problem, but motivate them to find their own solution.

10) not overextend the pupils by the lot of teaching material, but motivate them to learn with understanding.

Acknowledgements

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STUDENT'S AUTOEVALUATION IN THE FIRST YEAR OF STUDY, ESPECIALLY CONCERNING MATHEMATICAL ABILITIES

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Abstract. Significant differences are observed in secondary education of mathematics in the Czech Republic. It makes significant obstacles during tertiary education studies, because many of students do not finish its first study year. Therefore there is a need to continuously evaluate their mathematic abilities and perform appropriate changes in study courses.

1. Introduction

In the frame of the curricular reform in the Czech Republic there are developing and implementing frame curricula (RVP) preceded by more strict standard curricula. In the scope of particular schools they are called school curricula (SVP) which are derived from RVP. The first RVP for secondary schools was accepted in 2007, and from 2009 first groups of schools started to teach according to school curricula[1]. In comparison with old curricula, although they describe compulsory and optional topics, they are more flexible than the old ones. This leads to high imbalance of students' mathematical knowledge from school to school. In the Table 1 we can observe continuity of old curricula and new RVP. Particular types of schools are the following: G – Gymnasium (preparation for university studies), SPS – industrial school (preparation for practice in several technical branches), SOS – integrated school (specialized branches for practice), OA – secondary business school (economically oriented for practice), SOU – secondary education for practice in a trade.

[1] Introduction of RVP and SVP is dividend into 4 phases that should be finished by 2012.

Topics by RVP		SOS, SPS, SOU (2000)	Gymnasium (1999)
SPS, SOS, OA, SUS	Gymnasium		
	Basic notions (definition, proposition, theorem, proof), sets, propositional logic..		Basic notions, numeric fields.
Numbers and expressions transformations.	Numeric fields, power functions, variable expressions, equations, inequations.	Algebraic expressions. Powers, roots.	Algebra.
Function and its behaviour. Equations and inequations solving.		Linear, quadratic function, in/equations. Functions and their properties. Logarithmic and exponencial functions.	Functions.
	Functions and series.	Goniometry, trigonometry.	Goniometry, trigonometry.
Progressions and their usage.		Progressions.	Progressions.
Problem solving using application of functions, progressions and trigonometry. [1]			
Planimetry.	Geometry in plane, space, trigonometry, analytic geometry in plane, space, conic sections.	Planimetry.	Planimetry.
Stereometry.		Stereometry.	Stereometry.
Analytic geometry in plane.		Analytic geometry in plane.	Analytic geometry in plane.
Analytic geometry conic sections[1]		Analytic geometry conic sections[1]	
Combinatorics, probability and statistics.	Combinatorics, probability and data processing.	Combinatorics and statistics.	Combinatorics, probability and statistics.
Complex number's operations and quadratic equations in the field of complex numbers. [1]		Complex numbers.[1]	Complex numbers.[1]
			Analytic geometry in space. [1]
		Differential and integral calculus. [1]	Differential and integral calculus. [1]

[1]Optional recommended topics..

Table 1: Comparison beetween topics of RVP on gymnasium and specialized schools and old curricula from 1999/2000.

In Table 2 there is the number of hours for the whole period of education on the type of school.

	gymnasium	SPS	OA	SOS	SUS
Number of hrs	10	10 - 12	8	8 - 10	8

Table 2: Minimal hours of mathematics according to the school type by RVP.

According to the mentioned above we can predict these results:

1. Differences between gymnasium and other types of schools should be find especially in the topic “propositional logic”.

2. Further differences should be not only in topics but particularly in deep of the knowledge.
3. All school leavers should have worse knowledge in topics "analytic geometry, complex numbers and differential and integral calculus". In the contrary, they should have strong knowledge of "number operations, equations, functions, series, statistics and geometry".

We performed research among students of first year university study through questionnaire.

2. Research and hypotheses

Do correspond differences between types of secondary schools with the distribution of topics in RVP? Which topics are taught well in secondary education and which we have to improve in the first year of university studies? Does knowledge of students depend on the type of secondary education (several types of secondary schools)? Does knowledge of students depend on the type of study field attended?

We performed the abovementioned research by the form of questionnaire given to students of first year at Faculty of Science, University of Ostrava. The sample includes 787 students, we generalized results also for minor subset, where we are only observing mainstream fields for Faculty of Science. Questions were grouped according to mathematical topics into the following ones: sets, numerical fields (SET), propositional logic, proofs (LOGIC), functions (FUNCTION), equations, inequations (EQUATION), fundamentals of the differential and integral calculus (MA), combinatorics (COMB), probability, statistics (STAT), analytical geometry (GEOM).

We have devised questions in these topics and we performed analysis upon the scale. Every student can evaluate knowledge in the scale 1 – 5 (subtopics of above), where 1 is the minimal knowledge and 5 is the maximal knowledge. The method Analysis of variance (ANOVA) has been used. We analyzed results upon the secondary school attended and study fields [2].

At first, we divided answers only into two classes: "I've never hear about it" and "other answers". We got the reply to the first question. According to partition of topics in RVP/old curricula, differences between secondary schools providing preparation for practice (SPS, SOS, OA, SOU) and secondary schools providing preparation for university studies (G) would be especially in the topic LOGIC and next differences would be rather at an intensity of knowledge. But results of the research does not correspond with the assumptions (see Table 3). Likewise strong consciousness in the topic MA does not correspond to the fact that this topic is not obligatory at secondary schools

[2]They are marked as field's codes in the following text.

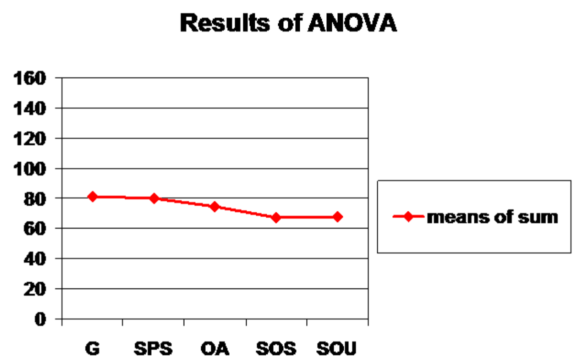
already since year 1999. On the contrary, in the long term fixed subjects at all secondary schools, probability and statistics have weak position in the consciousness of respondents.

%	SET	LOGIC	FUNCT ION	EQUATI ON	MA	COMB	STAT	GEOM
G	81	91	98	100	92	91	79	91
SPŠ	82	75	99	100	98	93	78	95
OA	62	80	92	100	88	90	65	83
SOŠ	54	45	89	97	76	73	58	81
SOU	54	42	90	96	76	80	52	90
sum	71	73	95	99	88	86	71	89

Table 3: Percentage of students' knowledge.

3. Data analysis and results

Results of ANOVA showed statistically significant differences between schools in overall results - F-ratio = 24.96, i.e. Prob. level < 0.000001.

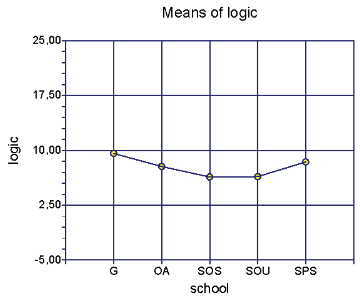


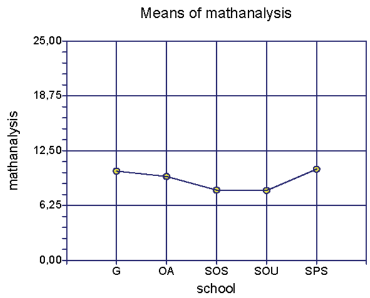
Another interesting result is showed by the Tukey-Kramer Multiple Comparison Test:

Group	Count	Mean	from Groups
SOS	197	67, 11675	OA, SPS, G
SOU	49	68, 10204	SPS, G
OA	60	74, 76667	SOS
SPS	130	80, 03846	SOS, SOU
G	346	80, 93642	SOS, SOU

From the base test we can conclude that hypothesis about difference between schools has been accepted (zero-hypothesis rejected). From additional tests given above, we can see that the level of math knowledge is similar at G, OA and SPS schools. Contrary, SOS and SOU schools have statistically significant differences from the first mentioned school.

If we analyse precisely the topics mentioned, we get these results against factor of school. We used the MANOVA Test. We can also reject zero hypotheses for every particular topic. From following graphs we can observe that differences (not statistically proved) are especially in MA and LOGIC, where practice schools attendants have slightly worse results against other schools.





In the end we verified whether knowledge of students depends upon the study field. We used the Bonferroni (All-Pairwise) Multiple Comparison Test and obtain the following results:

Group	**Count**	**Mean**	**Different From Groups**
AI-KS	1	46	
AE-KS	7	57,14286	AM, IS-KS
GRR-KS	2	67	
AE-PS	28	67,5	AM
KGI	10	69,7	
CHEM	42	71,35714	AM
GRR	74	72,24324	AM
OTK	26	72,69231	
SBE	27	73,18519	
AI-dist	7	73,42857	
FGG	30	73,73333	
PKG	36	75,30556	
BIOF	8	76,25	
AI	177	76,36723	
INF	21	76,42857	
EXB	23	81,04348	
AME	9	81,77778	
IP	25	82,36	
AM	18	90,61111	AE-KS, AE-PS, CHEM, GRR
FJ2	1	92	
KIP	2	104	
IS-KS	3	105	AE-KS

So we cannot reject the zero hypothesis that all students of different programmes have the same results.

4. Conclusions

The analysis of the results shows significant differences between the type of secondary schools. Practically oriented schools have low level of mathematical preparation, so we can establish special courses for these students. Also topics are not balanced (but not statistically). This means that we have to focus to logic and mathematical analysis.

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ANALYSIS OF THE FACTORS AFFECTING STUDENT'S ATTITUDES TO MATHEMATICS

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Abstract. The problem of students' understanding and mastering mathematics is currently one of the most emphasized topics of expert public discussion. New ways of teaching mathematics more effectively and attractively are being searched for. The information society brings new alternatives nowadays; it enables us to change established and rigid forms and methods in teaching. While searching an answer to the question why the student attitudes toward mathematics are so often negative and what could influence this situation in a positive way, we conducted a survey realized under the program ITMS – Flexible and attractive study at the University of Žilina for the needs of the market and knowledge-based society.

The paper presents some of the outcomes which have ensued from the data obtained in the project realization. 200 students from three faculties of the University of Žilina were addressed in the survey (Faculty of Civil Engineering, Faculty of Operation and Economics of Transport and Communications, Faculty of Special Engineering), in which their attitudes toward mathematics were detected as well as the factors determining these attitudes.

1. Introduction

Nowadays there is a lot of disscussion among scientific community about students' problems with understanding and comprehending mathematics.

Sometimes, it is considered that mathematics cannot be learnt, you simply either have apriori talent or not. The aim of the survey, which we have performed during the project ITMS – Flexible and attractive study at the University of Žilina for the needs of the market and knowledge-based society, is to analyze the factors that affect students ability to understand Mathematics. During the project we surveyed 200 students of University of Žilina studying at Faculty of Civil Engineering, Faculty of Social Engineering and Faculty of Operation and Economics of Transport and Communications.

We focused on the analysis of attitudes and opinions of students of mathematics which they expressed in the submitted questionnaire. This questionnaire had 15 questions while responses scaling has 3 levels. Students expressed how they like the style of teaching mathematics and where or not mathematics is an important and interesting subject, how much time do they spend for preparing for lessons and how they actually understand the discussed curriculum, how difficult is that curriculum for them.

We got a lot of interesting material which was subjected to the qualitative statistical analysis.

In this article we present only a few partial results and our main target is to identify the factors which have positive impact on understading mathematics and study results of students in this subject.

2. Analysis of the qualitative characters

We used χ^2- test for contingency table $k \times m$ to verify dependence of each pair of the qualitative characters A and B. The character A was acquiring k categories and the character B was acquiring m categories.
We tested the null hypothesis:

H_0 : the characters A and B are independent, versus

H_1 : the characters A and B are dependent.

The test statistics is

$$\chi^2 = \sum_{i=1}^{k} \sum_{j=1}^{m} \frac{(f_{ij} - o_{ij})^2}{o_{ij}}, \tag{1}$$

where f_{ij} are observed frequencies, $o_{ij} = \dfrac{f_i^A f_j^B}{n}$, $i = 1, 2, ..., k$; $j = 1, 2, ..., m$. The rejection region is $\chi^2 > \chi^2_\alpha((k-1)(m-1))$, where $\chi^2_\alpha((k-1)(m-1))$ is the critical value of χ^2- distribution with $(k-1)(m-1)$ degrees of freedom.

The degree of statistical dependence between the observed qualitative characters A and B is assessed using the contingency coefficient C and the Cramer coefficient V which are defined as

$$C = \sqrt{\frac{\chi^2}{n+\chi^2}}, \qquad V = \sqrt{\frac{\chi^2}{n\,(h-1)}}, \tag{2}$$

where n is a sample size and $h = \min(k,\, m)$.

$A \backslash B$	B_1	B_2	...	B_m	f_i^A
A_1	f_{11}	f_{12}		f_{1m}	f_1^A
A_2	f_{21}	f_{22}	...	f_{2m}	f_2^A
$\vdots$			...		$\vdots$
A_k	f_{k1}	f_{k2}	...	f_{km}	f_k^A
f_j^B	f_1^B	f_2^B	...	f_m^B	n

Table 1. The contingency table of the observed frequencies.

3. Verification of formulated hypotheses

We formulated 3 hypotheses.

Hypothesis 1. **The style of teaching mathematics determines the level of its understanding.**

To determine where or not the style of teaching mathematics and the level its understanding by students are independent we used the χ^2-test of independence. We use $\alpha = 0.05$. We observed the following characters: the character A – the style of teaching matemathics and the character B – the level of understanding mathematics by students. The character A takes the categories: A_1 = very good, A_2 = sometimes convenient, sometimes inconvenient, A_3 = mostly inconvenient. The character B takes the categories: B_1 = the subject matter is understood most of the time, B_2= the subject matter is understood at 50% of lessons, B_3= the subject matter is rarely understood. We tested the null hypothesis:

H_0 : the style of teaching mathematics and the level of its understanding are independent, versus

H_1 : the style of teaching mathematics and the level of its understanding are dependent.

The test statistics is $\chi^2 = 5.5089$. The critical value with $(k-1)(m-1) = 4$ degrees of freedom is $\chi^2_{0.05}(4) = 9.49$. The rejection region is $\chi^2 > 9.49$. Since $\chi^2 = 5.5089 \leq 9.49$, the hypothesis H_0 is **not rejected**. It is evident that the style of teaching mathematics and the level of its understanding are independent (Figure 1). The value of the contingency coefficient is $C = 0.1638$ and the value of the Cramer coefficient is $V = 0.1174$. The values of these coefficients indicate that between the analyzed qualitative characters A and B there exists the small degree of connection.

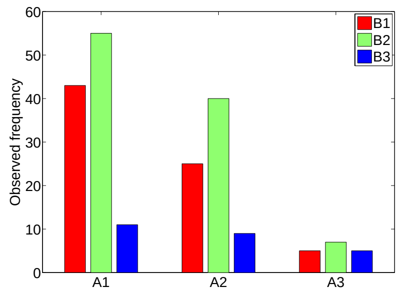

Figure 1: The bar chart of observed frequencies of characters A and B.

Hypothesis 2. **The time devoted to training for lessons of mathematics significantly affects the level of understanding mathematics.**

To determine where or not the time spent for training for the mathematics lessons and the level of understanding mathematics by students are independent we used the χ^2-test of independence. We observed the following characters: the character A – the time spent for training for the mathematics lessons and the character B – the level of understanding mathematics. The character A takes the categories: A_1 = more of time spent in comparison with others subjects, A_2 = the same time as for other lessons, A_3 = less time in comparison with other subjects. The character B takes the categories: B_1 = the subject matter is understood most of the time, B_2 = the subject matter is understood at 50% of lessons, B_3 = the subject matter is rarely understood. We tested the null hypothesis:

H_0 : the time spent for training for the mathematics lessons and the level of understanding mathematics are independent, versus

H_1 : the time spent for training for the mathematics lessons and the level of understanding mathematics are dependent.

The test statistics is $\chi^2 = 9.6552$. Since $\chi^2 = 9.6552 > \chi^2_{0,05}(4) = 9.49$, the hypothesis H_0 is **rejected**. It is evident that the time spent for training for the mathematics lessons and the level of understanding mathematics are dependent (Figure 2). The value of the contingency coefficient is $C = 0.2146$ and the value of the Cramer coefficient is $V = 0.1554$. The values of these coefficients indicate that between the analyzed qualitative characters A and B there exists the small degree of the connection.

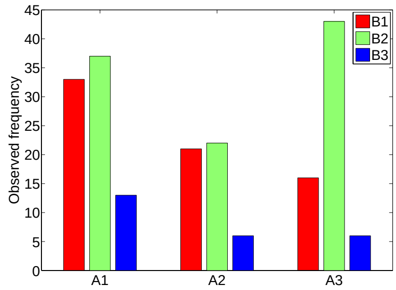

Figure 2: The bar chart of observed frequencies of characters A and B.

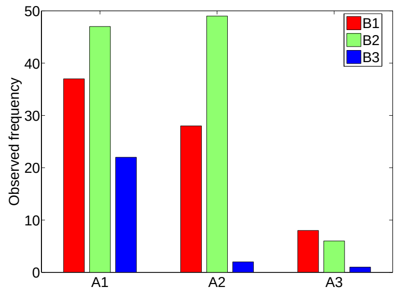

Figure 3: The bar chart of observed frequencies of characters A and B.

Hypothesis 3. **Positive attitude to mathematics significantly affects the level of understanding mathematics.**

To determine where or not the positive attitude to mathematics and the level of understanding mathematics by students are independent we used the χ^2-test of independence. We observed the following characters: the character A – the positive attitude to mathematics and the character B - the level of understanding mathematics. The character A takes the categories: A_1 = I like mathematics only if I understand the curriculum, A_2 = my attitude to mathematics depends on a topic which we learn, A_3 = I do not like mathematics. The character B takes the categories: B_1 = the subject matter is understood most of the time, B_2 = the subject matter is understood at 50% of lessons, B_3 = the subject matter is rarely understood. We tested the null hypothesis:

H_0 : the positive attitude to mathematics and the level of understanding mathematics are independent, versus

H_1 : the positive attitude to mathematics and the level of understanding mathematics are dependent.

The test statistics is $\chi^2 = 16.89$. Since $\chi^2 = 16.89 > \chi^2_{0.05}(4) = 9.49$, the hypothesis H_0 is **rejected**. It is evident that the positive attitude to mathematics and the level of understanding mathematics are dependent (Figure 3). The value of the contingency coefficient is $C = 0.2791$ and the value of the Cramer coefficient is $V = 0.2055$. The values of these coefficients indicate that between the analyzed qualitative characters A and B there exists the small degree of connection.

4. Conclusion

Our research brings two interesting results. Especially, we have verified the generally-known fact: the intrinsic motivation of students to carry out any activity is necessary. For mathematics teachers this means to focus their efforts on creating and applying the chosen teaching methods in such a way that students can understand the importance of mathematics for everyday practice and they will be more motivated intrinsically. The time spent for preparing for mathematics lessons is an important factor for understanding mathematics. It was proved that not only the time spent by students for preparing for lessons, but also their accessibility are important.

The second result proved that in mathematics education the motivation is more important than the teaching method itself. The replies of the respondents also confirmed that the attitude to mathematics is already formed in high school. Students' success in learning mathematics at the university depends mainly on the knowledge which they acquired in high school. Verification of the hypothesis number 1 confirmed this fact. The methods used in teaching mathematics at the university are not so important as the motivation of students and the time of preparing for mathematics.

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THE DIFFICULTIES IN INTEGRATING MATHEMATICS WITH OTHER TEACHING SUBJECTS

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Abstract. In contemporary concepts of school education one suggests a far-reaching integration of teaching contents. The integration is aimed to help school children to gain a comprehensive world picture, stimulate their activeness, develop some creative attitudes of the school children toward mathematics and elaborate various forms of organizing classes. According to the experience with integrating mathematics with other teaching subjects, there are difficulties in realizing the aims mentioned above. In this article some reasons for these difficulties will be discussed at large.

1. The main assumptions of integration

The term "integration" in its broadest sense suggested by *The Dictionary of Foreign Words* means to build a whole from separate parts. The "integration" is a process of joining something to one whole, bringing together, completing. In turn, the integration of teaching according to *The Popular Encyclopedia* is a teaching method aimed to highlight the relations between all the teaching subjects and to present science as a whole.

The concept of integration in teaching is not new. Its development was made due to a new breeding trend in the 19th and 20th centuries which was opposed to *The Herbart School of Pedagogy*.

A new concept, called *integrated teaching*, which canceled the traditional division of teaching into teaching subjects and the lesson system, was elaborated for the needs of elementary education. The assumptions of the concept were premises to a new approach to the system of teaching contents relying on the main-topic centered integration of teaching stuff.

The first attempts of teaching integration on a larger scale were made in Poland after the World War II. In 1990s A. Szyszko-Bohusz introduced the term of holistic pedagogy [6].

According to the holistic teaching one needs to understand oneself in surrounding reality rather than to gain encyclopedic knowledge. This leads to a change of subject-based teaching into searching for correlations between teaching subjects. As a consequence, the overall concept of reality instead of partial knowledge of single phenomena is to be presented to school children.

The introduction of the concept of fully integrated teaching in the curriculums of 1–3 levels and block teaching in 4–6 levels of elementary education has been in progress for more than 10 years. The idea of realizing the integration of teaching by intersubject educational paths is suggested to be applied to all educational levels. Correlated subject-based teaching is an appropriate base for transsubject teaching [1] which relies on getting the borders between the classical branches of science disappeared and focusing on the analyzed problem, phenomenon or process.

The theoretical assumptions of integrated teaching are properly elaborated and fully acceptable. The other thing is their factual, not only declarable realization.

2. Mathematics and the concept of integrated education

Mathematics is a subject which is seriously difficult to be integrated with other teaching subjects.

- Although the introduction of integrated teaching in 1–3 levels has been in progress for 10 years, some publishing houses keep publishing separate course books for the stage of education.

- The Great-Poland's gymnasium and high school students realized some chosen topics as projects within the program e-Szkoła Wielkopolska for one year [7]. Basically, the projects were to integrate mathematical-natural subjects. Only 16 out of 350 completed projects partially concerned mathematics. Moreover, the so called "mathematical" projects, on the contrary to the so-called "natural" projects, enriched students' mathematics knowledge to a minimal extent. They regarded some well known problems presented in a more attractive form, e.g. symmetry in architecture, fashion and art, percentages in everyday life, our neighborhood in numbers.

- The project “Preschoolers” including some very interesting suggestions of educational paths integrating the contents of natural sciences for 6-year-old children was one of the three rewarded projects of e-Szkoła Wielkopolska. What is symptomatic, the project lacks mathematical contents despite of its authors’ creativity. Apparently, the authors claim that teaching mathematical terms cannot be correlated with other contents in an attractive way for children.

- The connection of mathematical and natural contents offers difficulties even to the authors extensively describing their correlation [3]. Although the authors note that there is a number of intersubject relations within mathematical-natural subjects, they directly mention only physics, astronomy, chemistry, biology, and geography.

3. The reasons for the difficulties

Why is mathematics not prone to integration with other subjects? It seems that there are several reasons.

First of all, mathematics is formal on the contrary to all natural subjects, i.e. physics, chemistry, geography, and nature.

One can describe it in the following literary form:

Let’s imagine a crazy tailor who keeps sewing all possible clothes. He knows nothing about people, birds or plants. He isn’t interested in the world and its exploration. He keeps sewing clothes. He doesn’t know for whom, he doesn’t think of it. The tailor takes care of only one thing: he wishes to be consistent. Every time he starts sewing a new piece of cloth, he makes certain assumptions. They aren’t always the same but he proceeds according to the assumptions and wishes not to make them contradictory. There always have to be clothes, not bunches of blindly sewed tatters. He brings ready clothes to a big storage. If we could get there, we would find the clothes for people, centaur, unicorn and for the creatures which haven’t been invented yet. The great number of clothes would be of no use. Everyone admits that the never-ending job of the tailor is sheer madness. Mathematics works as the tailor does. Mathematics builds structures, but no one knows for whom. Perfect models, but a mathematician doesn’t know of what the models are. He isn’t interested in it. He does what he does because such an activity is possible (...) [4].[1]

[1] Author’s individual translation of Stanisław Lem’s quotation.

Therefore, both the subject of the research and the methods used in mathematics and natural sciences are completely different. The essence of mathematical creativity is to operate in the world of abstraction. A physical experiment, observations and conclusions are sufficient only in the elementary stages of mathematics teaching and then they can be used to a limited extent. Eventually, one always needs a formal explanation.

Secondly, mathematics is a language. The statement *The world of nature is written in the language of mathematics* formulated by Galileus nearly four centuries ago is presently considered to be obvious. Nevertheless, one needs to have at least a very basic command in a language to be able to speak it. Besides, the knowledge of the language is prior, not simultaneous, to its use in describing various phenomena, whereas the school children mathematics knowledge is little. What is more, mathematical contents get reduced in every change of the general education curriculum. High school final year students do not know, e.g. that *a velocity is the derivative of distance with respect to time* because they do not know the term “derivative”. Student can get to know numerous impressive examples of fractals, but their knowledge is reduced to the ability to recognize the shapes because the school knowledge of mathematics is insufficient to enable them to understand the rules of constructing fractals.

Another difficulty in the integration defined as a correlation of mathematics with other teaching subjects is a construction of the school children mathematics knowledge. In mathematics one constructs “new terms by means of the previosly introduced terms”. It resembles a construction of an inverted pyramid. One should systematically build floor by floor and it is time-consuming. School children need to know some mathematical terms, e.g. in physics or geography, much earlier than they learn them during mathematics lessons.

The terms which school children use in everyday life, e.g. binominal numbers and percentages, are also discussed in more advanced stages of mathematics teaching at school. Real everyday needs and school mathematics are closely related in early educational stages. Four basic operations on rational numbers, measuring, weighing, time calculations, calcualting the area and the perimeter of a simple geometric figure, percentages and proportions are necessary skills in everyday life. The mathematical operations mentioned above tend to be often done with calculators. The applications of mathematics to everyday life are numerous but trivial. In more advanced stages of mathematics teaching there is a larger discrepancy between school mathematics and everyday needs.

In everyday life one makes no use of e.g. polynominals and quadratic equations. On the other hand, the mathematics knowledge of school children remains too little to show its more complicated applications. The real applications of mathematics generally require much more advanced mathematics tools than the ones of a final year high school student. Therefore, the possibilities to motivate school children to learn mathematics by showing its applications are confined.

Aside from the problem-centered and teaching-content-centered integration, one also considers a key-competence centered integration [5]. Mathematics is traditionally thought to be leading in teaching logical reasoning, whereas the role of mathematics in teaching creative attitudes is unappreciated.

The literature concerning creativity is confined to present literary, plastic, music or technical creativity of school children. The manifestations of such creativity are much easier to be documented and exposed than the manifestations of mathematical creativity. The rule is to regard mathematical creativity as an area of advanced competences. It is mentioned at the very end of the list of achievements of mathematical education, whereas creativity cannot be the end of mathematics learning but the way of handling it. Unfortunately, the requirement of mathematical creativity is not often respected by specialists in mathematics education, authors of course books and teachers [2].

4. Conclusion

The specific nature of mathematics as a formal branch of science is partially reflected in the specific nature of school mathematics among the group of mathematical-natural subjects. This specific nature causes numerous difficulties in integrating mathematics with other teaching subjects. Nevertheless, one should not refrain from supporting the integration because it is the only way to overcome innumeracy.

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COMPUTER USE IN THE DISCOVERY OF PROPERTIES OF FUNCTIONS AND THEIR PROOFS

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Abstract. In this paper we present the results of research which was conducted in years 2006/2007 and 2007/2008 on groups of students of Pedagogical University of Cracow. The results are related with the role of the computer in the process of solving the tasks on the functions with parameter, discovering some properties of functions, formulating statements describing observed regularity and finding the method of proof.

1. Introduction

The second half of the XX century and the beginning of the XXI century are the period of rapid development of computer technology. Its importance in various fields of human life constantly increases so it is no wonder that it also gains entrance to education. Throughout the world, many people carry out research on the effective use of computer technology in teaching and learning mathematics. One of the issues is an attempt to answer the questions: How to use effectively computer technology to shape the image of mathematical concepts? and: How to help learners to develop creative activity? The work in this field was also carried out at the Pedagogical University of Cracow. The results of this research can be found in the papers [2], [3], [4], [6] and [8], among others.

2. Methodological notes

Our results presented in this paper were obtained during attempts to use the computer in the formation of some mathematical concepts in the process of solving certain tasks. In these studies, we used the computer to help students:

- to find the solutions to the problem concerning the properties of functions with the parameter,
- to justify the correctness of the obtained solutions,
- to discover certain properties of functions and formulate them in mathematical language and discover methods of proof.

We were interested in finding answers to the following research questions:

- How can a computer with appropriate software help understanding the content of the mathematical problem?
- Can a computer be useful in discovering the solution to a mathematical problem?
- How much can a computer help to deepen understanding the mathematical concepts?
- How much can a computer be useful in discovering methods of proof of observed facts?

Some results concerning these issues have already been signaled in our paper in 2005. In this paper we present results of our research which was conducted in years 2006–2008 among three groups of postgraduate students (Group A – a group of 14 people in the academic year 2006/2007, Group B – a group of 18 people in the academic year 2006/2007, Group C – a group of 17 people in the academic year 2007/2008). The research involved 49 students. Most of them were teachers who widened their mathematical knowledge to get the right to teach this subject. Their study lasted three semesters. During their studies they had to learn many new mathematical contents. We expect that the use of computer can be a positive motivation for them to try to solve more difficult tasks.

As the research tools we used the sets of tasks with the parameter which relate to the properties of elementary functions and questionnaire which contains twelve questions on the previous experience of using a computer and an opinion on the role of the computer in the class of mathematical analysis and future plans related to the use of the computer. We use the following working methods: observation of students' work, discussion with students, analysis of homework solutions and questionnaires. Research were conducted according to the following plan:

1. Students work on solutions sets of tasks in pairs without the aid of a computer.

2. Common discussion about the obtained solutions with use of a computer during the discussion.

3. Students working alone – solving homework.

4. Students complete the questionnaire.

5. Analysis of the collected material and putting the research hypotheses.

Research started with work on tasks without a computer. The idea was to check whether students will feel the need to use a computer. In our research we used three tasks. The first of them concerned a linear function $f_m(x) = (2m-3)x+3$, where m is a real parameter. The idea was to investigate whether students perceive that a change of the location of plots of these functions depends on a change of the value of the parameter m. We will not analyze this task in current paper, because it is already described in other papers. In the task 2 the goal was to answer the question: what can be said – depending on the parameter m – about the domain and the set of values for the following functions: $g_m(x) = \frac{1}{f_m(x)}$ and $h_m(x) = \sqrt{f_m(x)}$, where f_m is defined as in Task 1. We chose this task to research how the respondents understand the concept of composite functions and if they are able to indicate those functions' domains and sets of values. In addition, we were interested to see whether they can designate the limit cases. Task 3 deals with the description of the position of a plot of the function $f(x) = ax^2 + bx + c$ depending on the real values of the parameters a, b, c. The solution of this task can help you to analyze graphs of functions drawn on a computer. In our research we used programs Derive 5.0 and Wykresy 3 (Graphs 3). We wanted to check our assumption that the location of the graph when you change the coefficient b will be difficult to describe for students. The role of homework and use of computers in solving problems will be discussed in another paper.

Wykresy 3 is a simple program useful for drawing graphs of functions. This program allows us to enter formulas and draw graphs of functions with a parameter. Using it one can observe how the position of the graph depends on the parameters. The program is available at
http://www.up.krakow.pl/mat/komputery/prokomp.html.

The questionnaire was anonymous and consisted of 12 questions divided into three groups. The first group includes general questions about computer use by students during independent problem solving or during classes in school. The second group consists of questions concerning the role of a computer during the exercise and in the process of solving problems. Students have to indicate one from five situations which have occurred:

a) a student independently discovered the solution and the computer did not help in any way,

b) a student independently discovered the solution and a computer confirmed the obtained result,

c) a student independently discovered a way to solve the problem and computer simulations helped him to find an error in reasoning,

d) a student has not discovered the solution and understand how to solve them after the analysis using computer,

e) a student has not discovered the solution and a computer did not help him in any way.

In this group there was also a question which presentation helped them better to discover and formulate the observed hypothesis. The last group of questions concerned the respondents subjective feelings about computer-aided teaching. They were asked about whether:

- more topics should be implemented in a similar way?
- the participant is satisfied that the classes were conducted using a computer?
- the respondent would like to use a computer in the future in the classroom?

3. Analysis of results

Students considering the family of functions $f_m(x) = (2m - 3)x + 3$ should observed that the point $(0, 3)$ belongs to the graphs of all functions from this family. For $m > 1.5$ these functions are increasing, for $m < 1.5$ these functions are decreasing, and for $m = 1.5$ the function f_m is constant. Research has shown that the greatest difficulties for students were caused by constant function. Most respondents generally did not include it in the solutions. Only after using a computer one noticed that it meets the conditions of tasks 1 and 2. This is probably because the respondents do not have yet well formed concept image of linear function (see [7]). They did not see that in the family under consideration there is a constant function which has slightly different properties from other functions of this family. Therefore, in a situation complicated by the use of a parameter, students did not reflect on the limiting case which corresponds to the function $f(x) = 3$ ([1]). Regarding to the task 2, it should be noticed that the domain of the function g_m is the set of real numbers without the root of the function f_m for $m \neq 1.5$ or the set $\mathbb{R}$ for $m = 1.5$. However, the domain of the function h_m is the set of real numbers for which the function f_m takes the non-negative values, and the whole $\mathbb{R}$, when $m = 3$. The answers to general questions show that almost 45% of respondents use a computer while solving tasks independently, and 67% use it at their lessons. This fact can arouse optimism, but let us notice here that the most of the teachers were not learners of mathematics. The most frequently reported methods used were:

- Drawing graphs of functions;
- Search the web definitions, theorems and examples of the solutions;
- Perform calculations and check the results of various activities, in particular, for operations on matrices, calculating determinants, solving systems of equations or the tasks from statistics and economics.

Analysis of responses to the question about the role of a computer during solving the tasks shows that 6% of the students solved the task without the help of a computer and 30% sought confirmation of the correctness of their solution by computer. Approximately 17% of respondents knew the way to solve the task, but using a computer has detected errors in their solution. The largest group (about 47%) were people who could not solve the task and understood the solution after the analysis of drawings on the computer screen viewing. A group of students, representing 11% of respondents, were not able to solve a task and usage of a computer did not help them.

From these data we can conclude that the appropriate use of a computer can contribute to understanding the solution of the problem. It is still an open issue whether a student can independently use a computer to find a solution.

Observation of students' work while solving the task 3 showed that they do not really know how to graph the function changes when the parameter b is changing and the numbers a and c are constant. The first way to use a computer in the class was to show students an image made using the program Derive 5.0 (see Figure 1).

Analysis of this Figure does not lead the students closer to the discovery of a complete solution, although there are some hypotheses. Illustration using Wykresy 3 which allows observation of changes in variable values has led the students to the formulation of appropriate hypotheses. In this case, the plot moves along a parabola with the equation $y = -ax^2 + c$, as shown in Figure 2. Analysis of the vertex coordinates led one of the respondents to justify the correctness of an appropriate solution.

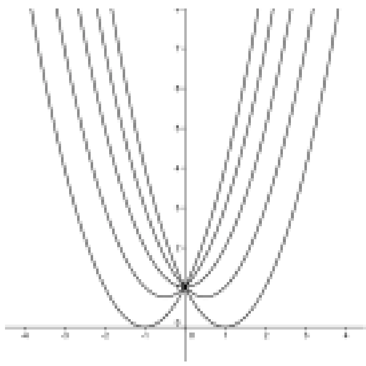

Figure 1.

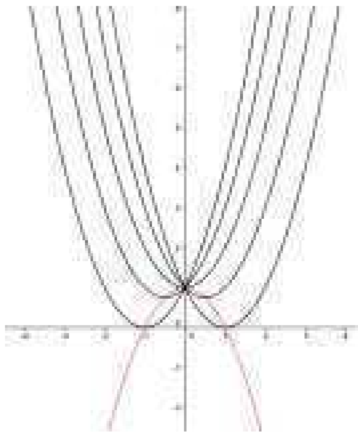

Figure 2.

The fact that this result was confirmed by a mathematical operations surprised 55% of respondents. At the same time, almost 80% of study participants were in favor of the use of dynamic presentations that really helped them to understand the solution. Moreover 98% of the final workshop participants expressed their satisfaction with this form of work and 90% would like to participate in similar activities in other subjects. There occurred some surprising answers to questions about use of a computer in classes conducted by respondents. Only 33% of respondents expressed the willingness to use a computer in their lessons. Comparing this response with answers to general questions, we can see a percentage decrease of the number of people who need to support their teaching with computers.

4. Conclusions and research hypotheses

The conducted observations have revealed a certain behavior of students during solving tasks with a computer. Most respondents sought for a solution by empirical observations, i.e. by examining the correlation between the graph and the specific parameter value. Thereby, obtaining a correct solution is not the intuitive feel of proving the correctness of their observations. Such a behavior is already known and also occurred in other studies of this type (cf. [3]). Answering research questions, it should be noticed that a computer with appropriate software can be helpful in finding a solution, but requires that the user knowledge and skills matches to software suitable for the problem under consideration. The knowledge of mathematical concepts related with the problem which occurs and some experience of students in the creative action are also necessary.

During the research students do not choose the software by themselves, but they use the software suggested by the teacher. Moreover, in the case of difficulty one could ask the teacher for the help. It was also observed in some tasks that there occurred difficulties in interpreting data visualization on the screen. Possible causes of this phenomenon consist in incomprehension terms in the task and inexperience in the use of parameter expressions. In independent work on a task one may also do not receive computer application skills as a tool for exploration of creative solutions.

We note that nearly half of study participants said that they understood the task solution by working with computer. In our view, it is an argument for the use of this tool in the teaching process.

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A STOCHASTIC GRAPH AS A SPECIFIC TOOL OF MATHEMATIZATION AND ARGUMENTATION

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Abstract. The article presents a stochastic graph as a tool enabling us to show the equality of the event probability without calculating the probability as such. A very important factor here is that the discussed events come from different probabilistic spaces being models of specific Markov chains.

Introduction. Let us consider a random board game g_{x-y}. The board consists of two circles: o_v and o_w. At the beginning, there are x coins inside the circle o_v and y coins inside the circle o_w. Let us assume that $x + y = 3$ and $y \neq 3$. There are two players, G_a and G_b, in the game. They take turns and toss 3 coins placed on the game board. The coins that show heads stay in the circle they were originally placed in. The coins that show tails change their circle. If all the coins end up in the circle o_w after the toss, the player who tossed them wins. Let us assume that the player G_a takes the first run (see [2]).

Later, in the article we will answer the question: which of the games g_{3-0}, g_{2-1} and g_{1-2} is the best (gives the best chance to win) for the player G_a and which is the best for the player G_b.

We will mark the experiment conducted in the game as δ_{x-y}. Let A_{x-y} means that the player G_a wins and B_{x-y} means that the player G_b wins.

The random experiment δ_{x-y} is conducted in several phases. Each single phase consists of a coins toss and placing them in the circles. The experiment status after the nth phase is a pair (v_n, w_n), where v_n means the number of coins placed in the circle o_v and w_n means the number of coins placed in the circle o_w after this phase. As $v_n + w_n = 3$, the experiment status after the nth phase is defined by the number v_n. The possibilities here make a set $S = \{0, 1, 2, 3\}$. We can interpret them as the graph loops. The beginning of the game becomes the start loop and the experiment status before the game (the 0 stage) becomes the edge loop (see [1]). Let us mark the probability of the experiment going from the j to the k status as p_{jk}. If $p_{jk} > 0$, we connect the j and k loop points on the graph with a line. Then we write the number p_{jk} next to the line. This way we get a stochastic graph and a game board simultaneously.

I. Let us start with the game g_{3-0}. The graph and the stages of constructing it are shown in Figure 1.

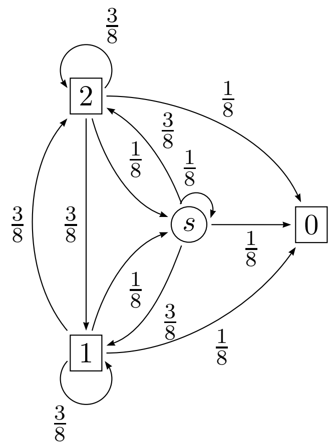

Figure 1.

This graph is a particularly useful tool of argumentation while calculating the probability of a certain player victory in the game.

We call the phases $0, 1$ and 2 the inner ones. We can notice that once the experiment δ_{3-0} gets to one of the inner stages, the next toss will lead it either to the 0 state with the probability $\frac{1}{8}$ or to another inner state with the probability $\frac{7}{8}$. These symmetries prove that the graph from Figure 1 reduces to that from Figure 2.

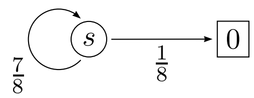

Figure 2.

The course of the game and its result can be registered if we include the time which it takes. Figure 3 shows the graph of the experiment δ_{3-0} after this modification.

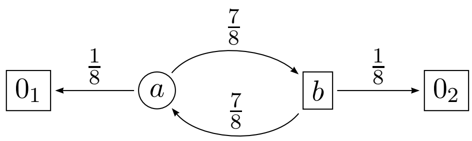

Figure 3.

The player G_a can win only if the experiment after an even toss takes the $\boxed{0}$ stage of the graph from Figure 2 or, which is really the same, the $\boxed{0_1}$ stage of the graph from Figure 3. The player G_b can win if the experiment after an odd toss takes the $\boxed{0}$ stage of the graph from Figure 2 or, which is really the same, the $\boxed{0_2}$ stage of the graph from Figure 3. From the above interpretations we can see that:
1) In the case of the graph from Figure 2 we get

$$P(A_{3-0}) = \frac{1}{8} + \left(\frac{7}{8}\right)^2 \cdot \frac{1}{8} + \left(\frac{7}{8}\right)^4 \cdot \frac{1}{8} + \ldots = \frac{\frac{1}{8}}{1-(\frac{7}{8})^2} = \frac{8}{15} \quad \text{and} \quad P(B_{3-0}) = \frac{7}{15}.$$

2) Let $P(A_{3-0}) = x$ and $P(B_{3-0}) = y = 1 - x$. We know from the graph shown in Figure 3 that

$$x = \frac{1}{8} + x \cdot \left(\frac{7}{8}\right)^2, \quad \text{so} \quad x = \frac{8}{15} \quad \text{and} \quad y = \frac{7}{15}.$$

So the player who starts the game has a better chance to win it.

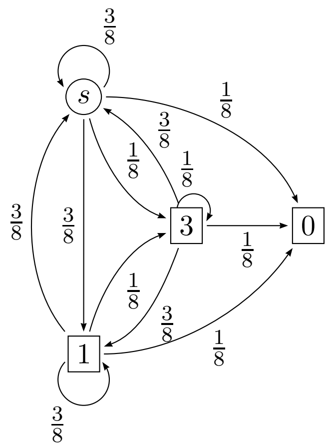

Figure 4.

II. Let us consider the game g_{2-1}. The δ_{2-1} experiment status at the beginning of the game is 2. Figure 4 shows the stochastic graph of the experiment δ_{2-1}.

Just like in the previous game g_{3-0}, once the experiment δ_{2-1} gets to one of the inner stages, the next coins toss leads it either to the 0 stage with the probability $\frac{1}{8}$ (the player who started the game wins it) or to another inner stage with the probability $\frac{7}{8}$. So we can see that the graph of this experiment is isomorphic to the one from Figure 2 (and, considering the time, to the graph from Figure 3), so

$$P(A_{2-1}) = \frac{8}{15} \quad \text{and} \quad P(B_{2-1}) = \frac{7}{15}.$$

III. It is easy to see that when we consider the game g_{1-2} we get a graph (as its board) isomorphic to the graphs of the games g_{3-0} and g_{2-1}. Hence,

$$P(A_{1-2}) = \frac{8}{15} \quad \text{and} \quad P(B_{1-2}) = \frac{7}{15}.$$

Finally we get:

$$P(A_{3-0}) = P(A_{2-1}) = P(A_{1-2}) = \frac{8}{15}$$

and

$$P(B_{3-0}) = P(B_{2-1}) = P(B_{1-2}) = \frac{7}{15}.$$

Summary. The final conclusion of our deliberation is surprising: the players chance to win does not depend on the experiment status at the beginning of the game, the player who starts the game wins it with the probability $\frac{8}{15}$ and the player who takes the second turn tossing the coins wins it with the probability $\frac{7}{15}$.

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REMARKS ON SOLUTIONS OF EQUATIONS WITH ABSOLUTE VALUE FUNCTION

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Abstract. In this paper, we give the proposition of didactic discussion on the existence of solutions of some equations containing expressions with absolute value. We are interested in the possibility of applying the method of conceptual problem-solving. Under this term we understand such a procedure, in which the solver is not limited to the automatic application of the definition of the absolute value, but he can reject some cases based on his mathematical knowledge. To do this, one must make use from various features of this concept – not only from its definition.

1. Introduction

In the paper [1], we included remarks on solving several types of equations containing terms with the absolute value. There we paid attention on the conceptual problem-solving, i.e. the procedure of solving based on the use of theorems which involve the absolute value. We also set this method against the algorithmic method of solving, i.e. the procedure which bases on mechanical use of absolute value definition and considering cases deriving from the range of formulas' applicability.

In the current paper, we present some examples of problems which are generated by the discussion of existence of solutions to the following equation:

$$|f(x)| + |g(x)| = m, \tag{1}$$

where $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$ and m is an arbitrary real number. We are interested in the possibility of applying the conceptual problem-solving method in reference to equations of this type.

Issues listed below may provide a basis for building mathematical tasks and problems which, thanks to the use of conceptual problem-solving method, allow us to intense creative mathematical activity.

2. Exemplary issues

First, let us notice that equation (1) has a solution only if m belongs to the set of values of the function $x \to |f(x)| + |g(x)|$. However, indication of this set sometimes is importunate. The form of equation (1) ensues the following prerequisite for existence of solution to this equation:

Theorem 1. *Let $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ with $D_1 \cap D_2 \neq \emptyset$ be given functions and let m be an arbitrary real number. If equation* (1) *has a solution, then m is a nonnegative number.*

Non-negativity of the number m is not the sufficient condition for existence of the solution to equation (1). This is evidenced by the following example.

Example 1. *Let us consider the equation*

$$|x^2 - 4| + |3x| = m.$$

The analysis of the graph of the function $x \to |x^2 - 4| + |3x|$ for $x \in R$ demonstrates that the values of the function are obviously nonnegative. We can also notice that the discussed equation does not have solutions for $0 \leq m < 4$.

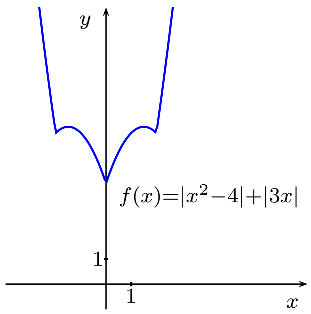

Figure 1.

To find the sufficient conditions for existence of a solution we start with the proof of the following lemma.

Lemma 1. *Let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$. Let A and B designate the following sets:*

$$A = \{x \in D_1 \cap D_2 : f(x)g(x) \geq 0\}, \quad B = \{x \in D_1 \cap D_2 : f(x)g(x) < 0\}.$$

Therefore,

$$\max(|f(x) + g(x)|, |f(x) - g(x)|) = \begin{cases} |f(x) + g(x)|, & x \in A, \\ |f(x) - g(x)|, & x \in B. \end{cases}$$

Proof. Let $x \in A$. Then $|f(x)+g(x)| = f(x)+g(x)$, when $f(x)$ and $g(x)$ are nonnegative, or $|f(x)+g(x)| = -f(x)-g(x)$, when $f(x)$ and $g(x)$ are negative, and $|f(x)-g(x)| = f(x)-g(x)$, when $f(x) \geq g(x)$ or $|f(x)-g(x)| = -f(x)+g(x)$, when $f(x) < g(x)$. Hence, it follows that $\max(|f(x)+g(x)|, |f(x)-g(x)|) = |f(x)+g(x)|$.

For $x \in B$ we have

$$|f(x)+g(x)| = \begin{cases} f(x)+g(x), & x \in B \text{ and } f(x)+g(x) \geq 0, \\ -f(x)-g(x), & x \in B \text{ and } f(x)+g(x) < 0, \end{cases}$$

and

$$|f(x)-g(x)| = \begin{cases} f(x)-g(x), & x \in B \text{ and } f(x)-g(x) \geq 0, \\ -f(x)+g(x), & x \in B \text{ and } f(x)-g(x) < 0. \end{cases}$$

Therefore, it ensues that $\max(|f(x)+g(x)|, |f(x)-g(x)|| = |f(x)-g(x)|$, which ends the proof.

In posterior deliberations we will make use of the known feature of the absolute value of real number.

Lemma 2. *For arbitrary real numbers p, q the following equation is satisfied*

$$|p| + |q| = \max(|p+q|, |p-q|).$$

The following theorem is true.

Theorem 2. *Let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$ be given functions and let m be an arbitrary nonnegative real number. Equation* (1) *has a solution if and only if there exists the number $x_0 \in D_1 \cap D_2$ such that the following conditions hold*

$$|f(x_0)+g(x_0)| = m \quad \textit{and} \quad |f(x_0)-g(x_0)| = m,$$

or

$$|f(x_0)+g(x_0)| = m \quad \textit{and} \quad |f(x_0)-g(x_0)| < m,$$

or

$$|f(x_0)+g(x_0)| < m \quad \textit{and} \quad |f(x_0)-g(x_0)| = m.$$

Proof. If equation (1) has a solution, then there exists the number $x_0 \in D_1 \cap D_2$ such that

$$|f(x_0)| + |g(x_0)| = m.$$

It follows herefrom and from lemma 2 that

$$\max(|f(x_0)+g(x_0)|, |f(x_0)-g(x_0)|) = m, \tag{2}$$

and, in consequence, conditions contained in the thesis of the led theorem.

Let us presume that there exists $x_0 \in D_1 \cap D_2$ which satisfies disjunction of conditions from the led theorem. Thus, equation (2) holds and on account of lemma 2 we receive that x_0 is the solution of equation (1).

In the quoted paper [1], we consider equation (1) in which functions f and g satisfy the condition

$$\exists_{c\in\mathbb{R}}\forall_{x\in D_1\cap D_2}|f(x)-g(x)|=c,$$

and the function $|f(x)+g(x)|$ is boundless from the top in its domain. The below theorem is the generalization of theorem 5 from the mentioned paper.

Theorem 3. *Let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$ be given functions, $h_1(x) = |f(x) + g(x)|$, $h_2(x) = |f(x) - g(x)|$, $x \in D_1 \cap D_2$ and let m be an arbitrary nonnegative real number.*

a) If

$$\exists_{c\in\mathbb{R}^+}\forall_{x\in D_1\cap D_2}|f(x)-g(x)|=c \tag{3}$$

and $m \in h_1(D_1 \cap D_2)$, hence equation (1) *has a solution if and only if*

$$c \leq m. \tag{4}$$

b) If

$$\exists_{c\in\mathbb{R}^+}\forall_{x\in D_1\cap D_2}|f(x)+g(x)|=c \tag{5}$$

and $m \in h_2(D_1 \cap D_2)$, hence equation (1) *has a solution if and only if the inequality (4) holds.*

Proof.

a) Let us assume that equation (1) has the solution $x_0 \in D_1 \cap D_2$. Then, by lemma 2 and condition (3), we have

$$\max(|f(x_0)+g(x_0)|,c)=m,$$

hence inequality (4) holds. Let us assume that inequality (4) holds. From (1), lemmas 2 and (3) we have

$$|f(x)|+|g(x)|=\max(|f(x)+g(x)|,c)=m.$$

Hence, from (4) we receive the inequality $|f(x)+g(x)| \leq m$. The fact that $m \in h_1(D_1 \cap D_2)$ shows that there exists the number $x_0 \in D_1 \cap D_2$ which satisfies equation (1).

b) The proof of this part of the theorem is carried out in analogical fashion.

Assuming (3) or (5), it follows from theorem 3 that equation (1) can be replaced by the equation

$$|f(x) - g(x)| = m \quad \text{or} \quad |f(x) + g(x)| = m.$$

Let us consider the following example.

Example 2.

a) The equation $|ax+b|+|ax+c| = m$, where a, b, c, and m are given real numbers, has a solution on the strength of theorem 3a) if and only if $|b - c| \leq m$. Thus, it can be replaced by the equation $|2ax + b + c| = m$ (compare [1]).

b) The equation $|-x^2| + |-x^2 + 2| = m$ has a solution on the strength of theorem 3b) if and only if $m \geq 2$. Thus, it can be replaced by the equation $|-2x^2 - 2| = m$.

c) Theorem 3 does not determinate the number of solutions of equation 1. For example, the equation $|\sin x| + |\sin x + 2| = m$ has infinitely many solutions for $m = 2$ (see Figure 2).

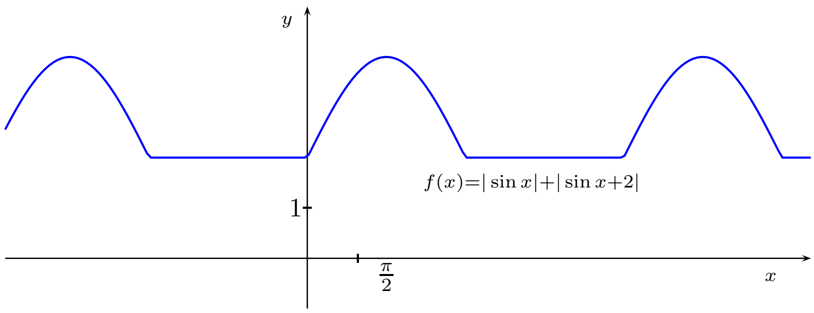

Figure 2.

In Figure 2, the graph of the function $f(x) = |\sin x| + |\sin x + 2|$ is presented. The condition for existence of solution to the considered equation cannot be described by means of theorem 3, because the absolute values of the sum and the difference of the functions are not boundless. Simultaneously, this equation can be replaced by the equation $|2\sin(x) + 2| = m$ if $m \in [2, 4]$.

At the end we will solve the problem of existence of solutions to equation (1) with additional assumption.

Theorem 4. *Let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$ be given functions and let m and k be arbitrary positive real numbers. Equation* (1) *has a solution which satisfies the condition*

$$f^2(x) + g^2(x) = k^2 \tag{6}$$

if and only if $m \in [k, k\sqrt{2}]$.

Proof. From (6) and (1) we obtain the equation

$$|f(x)| + \sqrt{k^2 - f^2(x)} = m,$$

assuming that $|f(x)| \leq k$. Hence we get the equation

$$2f^2(x) - 2m|f(x)| + m^2 - k^2 = 0,$$

which has a solution for $m \leq k\sqrt{2}$. Moreover, if we square equation (1) and take into account condition (6), then we obtain that $m \geq k$, which ends the proof of the theorem.

Let us notice that if $f(x) = \sin x$ and $g(x) = \cos x$, then $k = 1$. The following conclusion is derived from theorem 4.

Conclusion 1. *The equation*

$$|\sin x| + |\cos x| = m$$

has a solutions if and only if $m \in [1, \sqrt{2}]$.

From conslusion 1 it follows that the set $[1, \sqrt{2}]$ is the set of values of the function $x \to |\sin x| + |\cos x|$. This fact may explain the frequent appearance of the following task in many of the tasks collections:

Solve the equation

$$|\sin^2 x| + |\cos^2 x| = \sqrt{2}.$$

3. Summary

Issues presented in this paper and some similar issues were considered at class with students of Mathematics Teaching Faculty. Observations of students' work and researches carried out in other groups (see [1]) indicate that there occur great difficulties of learners in formulation of hypotheses, including the necessary conditions and sufficient conditions for relevant facts. Surveyed students have considered equations mainly by putting particular values into formulas. This type of attitude can be explained by some mathematical immaturity of students in the field of general mathematical reasonings.

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CONVOLUTE AND GEOMETRICAL PROBABILITY SPACES

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Abstract. Let X and Y be two independent random variables, either discrete or continuous. The question is "what is the probability distribution of $Z = X + Y$"? Clearly, the probability distribution of $Z = X + Y$ is some combination of f_X and f_Y which is called the convolution of f_X and f_Y. It is denoted by $*$. We have $f_Z(t) = f_{X+Y}(t) = f_X(t)*f_Y(t)$. In this paper it is shown how we can use geometrical probability spaces to find (without convolution) the distribution of random variable $Z = X + Y$.

1. Introduction

A random variable is one of important notions of the probability calculus. Of particular importance are continuous random variables because they have applications in mathematical statistics, economics, theory of insurance, and physics. Mathematical tools used for examining these random variables are rather complicated (the characteristic function, the Riemann–Stieltjes integral, the notion of the functional convolution). In the works [2], [3] and [4] it is presented how it is possible to use geometrical probability space for examining continuous random variables. In this work we suggest a method of finding the cumulative distribution function and the density function of the sum of independent continuous random variables, with the use of the geometrical probability space.

2. Basic definitions

To begin with, we recall definitions and theorems which are essential for subsequent part of this work.

Definition 1. *Let $(\Omega, \mathcal{Z}, P)$ be any probability space. A random variable in this probability space is defined as any function X from the set Ω in $\mathbb{R}$ that satisfies the condition:*

$$\{\omega \in \Omega : X(\omega) < x\} \in \mathcal{Z} \text{ for any } x \in \mathbb{R}. \tag{1}$$

Theorem 1. *If X is a random variable in the probability space $(\Omega, \mathcal{Z}, P)$ and $\mathcal{B}$ is the set of Borel subsets of a straight line and P_X is a function defined by formula:*

$$P_X(A) = P(\{\omega \in \Omega : X(\omega) \in A\}) \text{ for any } A \in \mathcal{B}, \tag{2}$$

then the triple $(\mathbb{R}, \mathcal{B}, P_X)$ is also the probability space.

Definition 2. *Let X be a random variable in the probability space $(\Omega, \mathcal{Z}, P)$. The function P_X defined by formula (2) on the set of Borel subsets of a straight line is called the probability generated on a straight line by the random variable X or the distribution of the random variable X, and the triple $(\mathbb{R}, \mathcal{B}, P_X)$ is called the probability space generated on the straight line by the random variable X.*

Definition 3. *A function F_X defined on $\mathbb{R}$ by the formula*

$$F_X(x) = P_X((-\infty, x)) \text{ for any } x \in \mathbb{R}$$

is called the cumulative distribution function (also the cumulative density function) or briefly the distribution function of a random variable X.

Definition 4. *A random variable X, for which there exists such a nonnegative and integrable function f_X defined on $\mathbb{R}$ that*

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt,$$

is called continuous, and its distribution P_X is called a continuous distribution. The function f_X is called the density of a random variable X or the density of a distribution P_X.

Definition 5. *Random variables $X_1, X_2, \ldots, X_n$ from the same probability space $(\Omega, \mathcal{Z}, P)$ are called independent if for any Borel sets $B_1, B_2, \ldots, B_n$ on a straight line, the events $A_1, A_2, \ldots, A_n$, where $A_j = \{\omega \in \Omega \colon X_j(\omega) \in B_j\}$ for $j = 1, 2, \ldots, n$, satisfy the following condition:*

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_n).$$

Definition 6. *Let Ω be a subset of k-dimensional Euclidean space ($k = 1, 2, 3, \dots$) having the positive k-dimensional Lebesgue measure, let $\mathcal{Z}$ be a set of subsets of the Ω set having the Lebesgue measure and let P be a function defined on $\mathcal{Z}$ by the formula:*

$$P(A) = \frac{m_l(A)}{m_l(\Omega)}, \text{ where } m_l \text{ denotes the Lebesgue measure.} \tag{3}$$

The triple $(\Omega, \mathcal{Z}, P)$ is called the geometric probability space, and P is called the geometric probability.

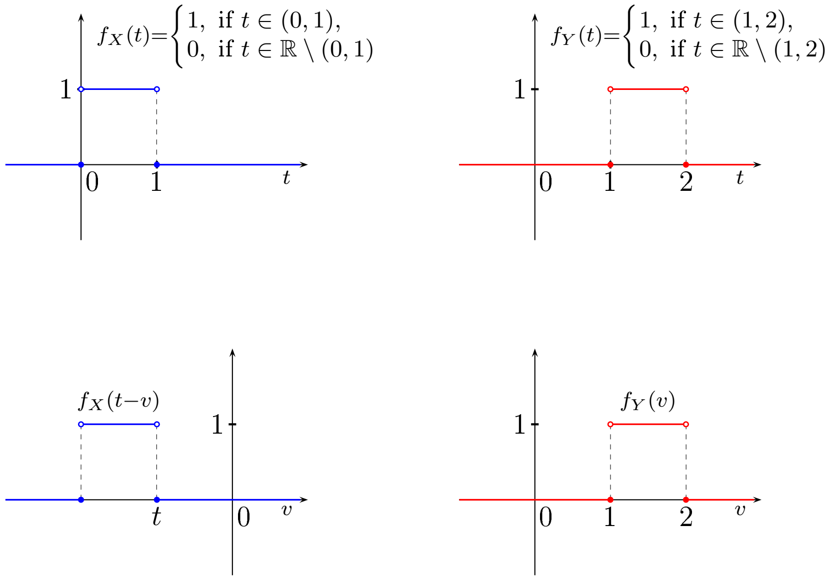
3. The sum of two independent uniform random variables – the classical method

Let X and Y be independent random variables with uniform distributions and let

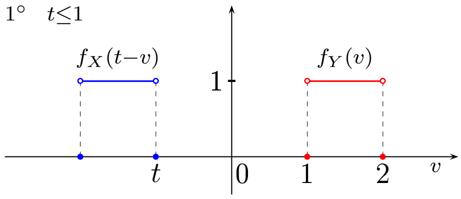
$$f_X(t) = \begin{cases} 1 \text{ if } 0 < t < 1, \\ 0 \text{ otherwise}, \end{cases} \qquad f_Y(t) = \begin{cases} 1 \text{ if } 1 < t < 2, \\ 0 \text{ otherwise}. \end{cases}$$

Let $Z = X + Y$.

The graphs below illustrate the method of determining the density function of the random variable Z.

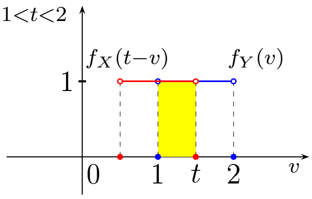


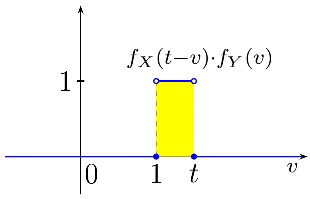
1° $t \leq 1$



$$f_X(t-v)\cdot f_Y(v)=0 \Rightarrow f_X * f_y(t)=\int_{-\infty}^{\infty} f_Y(v) f_X(t-v) dv = 0$$

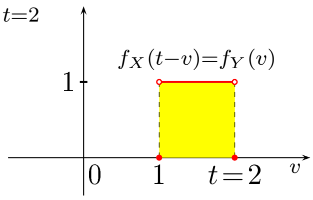
2° $1<t<2$

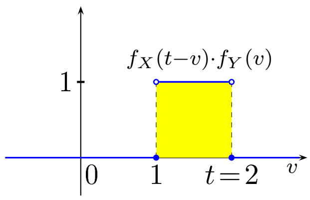




$$f_X * f_Y(t)=\int_{-\infty}^{\infty} f_Y(v) f_X(t-v) dv = (t-1)\cdot 1 = t-1$$

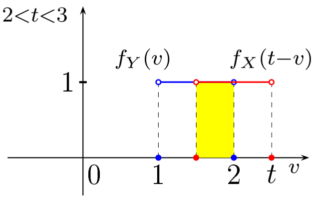
3° $t=2$

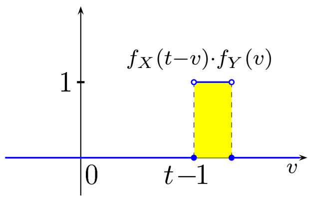




$$f_X * f_Y(t)=\int_{-\infty}^{\infty} f_Y(v) f_X(t-v) dv = (2-1)\cdot 1 = 1$$

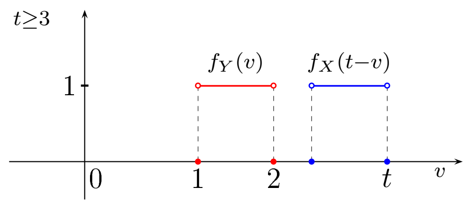
4° $2<t<3$





$$f_X * f_Y(t)=\int_{-\infty}^{\infty} f_Y(v) f_X(t-v) dv = (2-(t-1))\cdot 1 = -t+3$$

5° $t \geq 3$



$$f_X(t-v) \cdot f_Y(v) = 0 \Rightarrow f_X * f_Y(t) = \int_{-\infty}^{\infty} f_Y(v) f_X(t-v) dv = 0$$

We have:

$$f_Z(t) = \begin{cases} 0 & \text{for } t \leq 1 \vee t \geq 3, \\ t-1 & \text{for } 1 < t < 2, \\ -t+3 & \text{for } 2 < t < 3. \end{cases}$$

4. The sum of two independent uniform random variables – the alternative method

Let us consider independent continous random variables X and Y with the density functions f_X and f_Y. Let

$$\Omega^{XY} := \{(x, y, z) \in \mathbb{R}^3 \colon 0 \leq z \leq f_X(x) f_Y(y)\}$$

and let $\mathcal{Z}$ be a family of subset of the set Ω^{XY} having the Lebesque measure. Let us notice that $m_l(\Omega) = 1$. The triple $(\Omega, \mathcal{Z}, P)$, where $P(A) = m_l(A)$, is a geometrical probability space. The probability space $(\Omega, \mathcal{Z}, P)$ will be called the *basic geometrical probability space of independent random variables X and Y.*

Let X and Y be independent random variables with uniform distributions and

$$f_X(t) = \begin{cases} 1 \text{ if } 0 < t < 1, \\ 0 \text{ otherwise}, \end{cases} \qquad f_Y(t) = \begin{cases} 1 \text{ if } 1 < t < 2, \\ 0 \text{ otherwise}. \end{cases}$$

Let $Z = X + Y$,

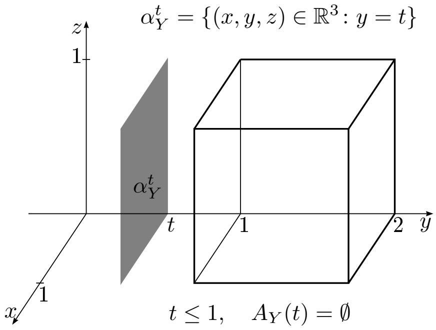
$$\Omega^{XY} = \{(x, y, z) \in \mathbb{R}^3 \colon 0 < x < 1,\ 1 < y < 2,\ 0 \leq z \leq 1\} = (0,1) \times (1,2) \times [0,1],$$

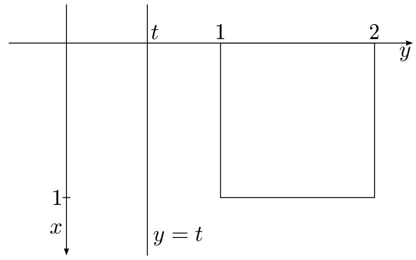
and $X \colon \Omega \to \mathbb{R}$ be given by the formula $X(x, y, z) = x$, whereas $Y \colon \Omega \to \mathbb{R}$ be given by the formula $Y(x, y, z) = y$. Let us define

$$A_X(t) = \{(x, y, z) \in \Omega^{XY} \colon X(\omega) < t\} = \{(x, y, z) \in \Omega^{XY} \colon x < t\}$$

for $t \in \mathbb{R}$.

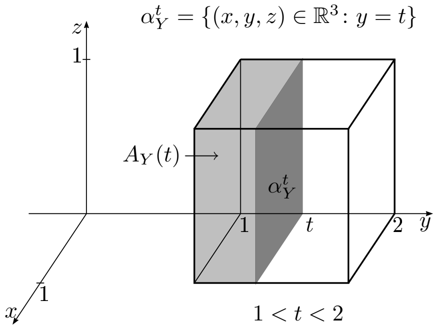
The graphs below illustrate the method of determining the distribution function of the random variable X.

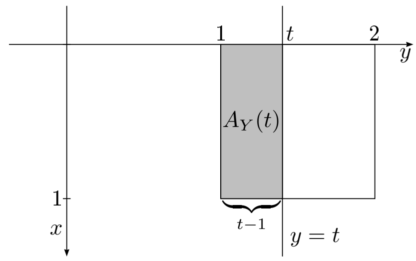




$F_Y(t) = m_L(A_Y(t)) = m_L(\emptyset) = 0$

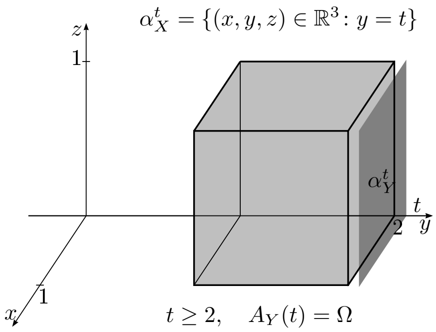
$f_Y(t) = F_Y'(t) = 0$

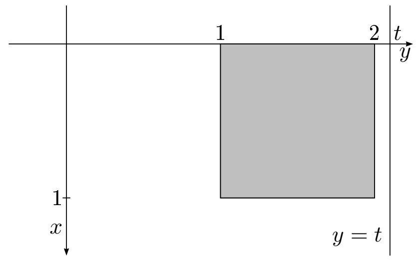




$F_Y(t) = m_L(A_Y(t)) = (t-1) \cdot 1 \cdot 1 = t-1$

$f_Y(t) = F_Y'(t) = (t-1)' = 1$





$F_Y(t) = m_L(A_Y(t)) = m_L(\Omega) = 1$

$f_Y(t) = F_Y'(t) = 0$

The cumulative distribution function of a random variable X is expressed by the formula:

$$F_X(t) = \begin{cases} 0 \text{ if } x \leq 0, \\ t \text{ if } 0 < t < 1, \\ 1 \text{ if } t \geq 1. \end{cases}$$

Hence it follows that the density function of a random variable X is given by the formula:

$$f_X(t) = \begin{cases} 1 \text{ if } 0 < t < 1, \\ 0 \text{ if } t \leq 0 \vee t \geq 1. \end{cases}$$

Reasoning in a similar way, one may state that Y is a continuous random variable for which

$$f_Y(t) = \begin{cases} 1 \text{ if } 1 < t < 2, \\ 0 \text{ if } t \leq 1 \vee t \geq 2. \end{cases}$$

It may be easily shown that random variables X and Y are independent random variables.

Let us now consider a random variable $Z = X + Y$. We have:

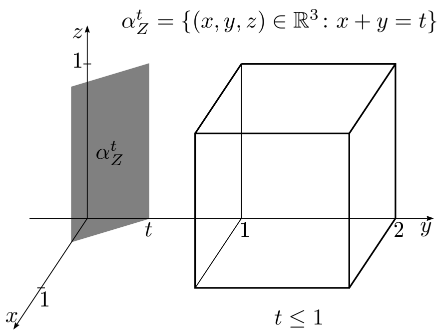
$$Z(x, y, z) = x + y \quad \text{for } (x, y, z) \in (0, 1) \times (1, 2) \times [0, 1].$$

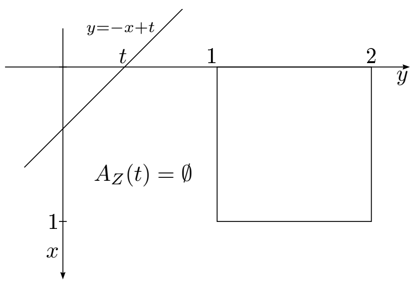
Now we determine the cumulative distribution function of the random variable Z. Let

$$A_Z(t) = \{(x, y, z) \in (0, 1) \times (1, 2) \times [0, 1] \colon x + y < t\}$$

for $t \in \mathbb{R}$.

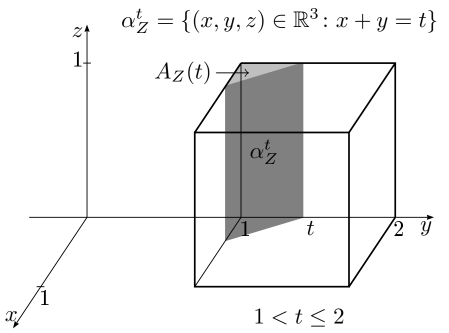
The graphs below illustrate the method of determining the distribution function of the random variable Z.

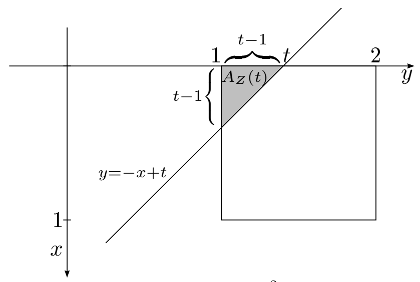




$$F_Z(t) = m_L(A_Z(t)) = m_L(\emptyset) = 0$$

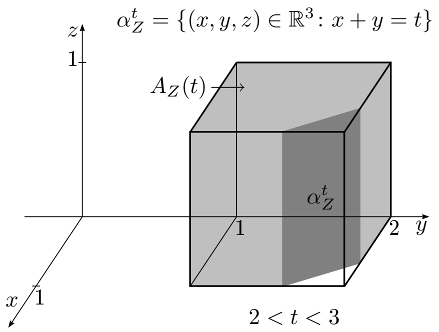
$$f_Z(t) = F_Z'(t) = 0$$

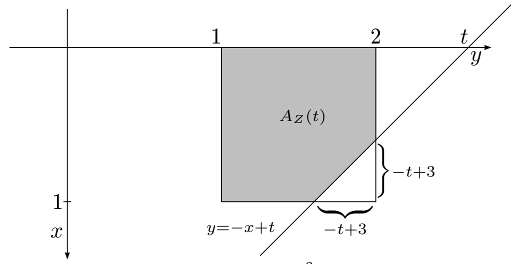




$$F_Z(t) = m_L(A_Z(t)) = \tfrac{(t-1)^2}{2} \cdot 1 = \tfrac{t^2}{2} - t + \tfrac{1}{2}$$

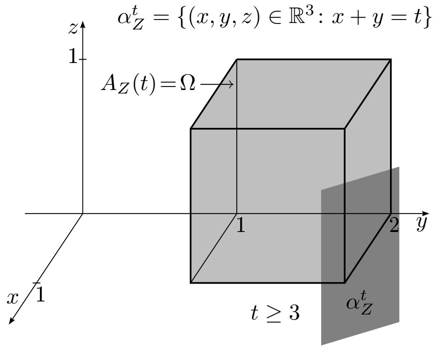
$$f_Z(t) = F_Z'(t) = (\tfrac{t^2}{2} - t + \tfrac{1}{2})' = t - 1$$

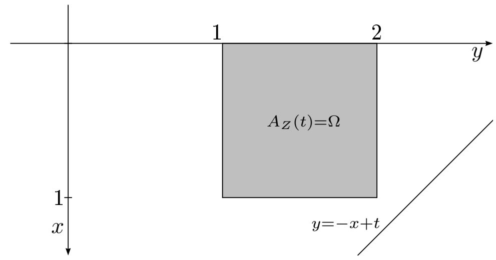




$F_Z(t)=m_L(A_Z(t))=(1-\frac{(-t+3)^2}{2})\cdot 1=\frac{-t^2}{2}+3t-\frac{7}{2}$

$f_Z(t)=F_Z^{'}(t)=(\frac{-t^2}{2}+3t-\frac{7}{2})'=-t+3$





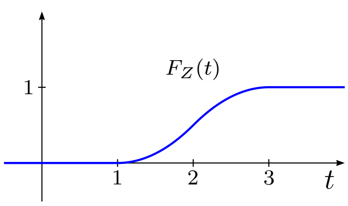
$F_Z(t)=m_L(A_Z(t))=m_L(\Omega)=1$

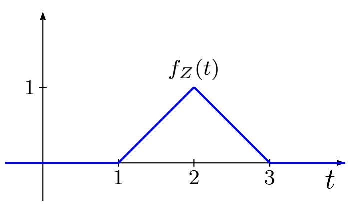
$f_Z(t)=F_Z^{'}(t)=1'=0$

We have:

$$F_Z(x)=\begin{cases} 0 & \text{for } t\leq 1,\\ \frac{t^2}{2}-t+\frac{1}{2} & \text{for } 1<t\leq 2,\\ \frac{-t^2}{2}+3t-\frac{7}{2} & \text{for } 2<t<3,\\ 1 & \text{for } t\geq 3,\end{cases}$$

$$f_Z(t)=\begin{cases} 0 & \text{for } t\leq 1\vee t\geq 3,\\ t-1 & \text{for } 1<t<2,\\ -t+3 & \text{for } 2<t<3.\end{cases}$$





5. Conclusion

It is worthwhile to solve the problems presented above with the students at mathematics teachers training majors. Proving the theorems with elementary methods with the use of mathematical analysis and geometrical methods allows us to consider the elements of probability calculus in a different (than traditional) way. Quite elementary tools make the presented problems simple to understand and to operative use.

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IMPORTANCE OF ELECTRONIC SUPPORT OF MATHEMATICAL EDUCATION IN PROFESSIONAL TRAINING OF PROSPECTIVE TEACHERS OF ELEMENTARY EDUCATION: PART TWO

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Abstract. Electronic support of teaching is a prospect of future education, especially for students of tertiary education. Since 2005/2006 academic year the study at the Faculty of Education, University of Prešov, has been implemented by the combination of methods. Mathematical disciplines have been taught in the form of contact classes and electronic support through LMS Moodle. The article analyzes the importance of the electronic support for mathematical education and professional training of prospective teachers in the elementary stage.

1. Introduction

Current higher education training of future teachers in the primary stage requires incorporation of modern technological instruments into the pedagogical processes. Electronic support becomes an inseparable part of education. The point of departure is, on the one hand, the utilisation of information technologies, and the modification in the conception of didactic system of mathematics, on the other.

According to Scholzova [4], the process of conceptual and curricular transformation of mathematical education in the field of Pre-school and Elementary Education put forth new challenges for modernisation of content, methods and forms of undergraduate training. We think that a need for such a transformation dwells also on a follow up graduate field of study – Teaching in Primary Stage.

Utilisation of e-learning in teacher training has found its way also at the Faculty of Education, University of Prešov. Since the 2005/2006 academic year we have strived for the effective incorporation of e-learning into education in both undergraduate and graduate levels of study programmes. The e-courses are accessible at http://moodle-pf.unipo.sk/. Selected e-courses offered by the Faculty of Education, University of Prešov, are analysed in [2], [3], and [5].

2. Aims and methods of the research

The aim of this research was to obtain information on the employment of electronic support for teaching at the Faculty of Education, University of Prešov. The principal method of the survey was a questionnaire. The students could express, via questionnaire, their views on utilization of electronic support of teaching during their own study. The questionnaire was distributed to full-time students who studied in 2010/2011 academic year in the Teaching in Primary Stage field of study. The research sample included 62 students of the 1st year and 49 students of the 2nd year – full-time students of the graduate degree programme. In creating and analyzing the questionnaire items we departed from Coufalová questionnaire [1].

3. Research findings

We were trying to identify the information sources utilized by the students during their study of mathematical disciplines.

Real utilization of information resources during study of mathematical disciplines are presented in Table 1.

Table 1

Information resource	Average utilization of the information resource	
	1st year	2nd year
Higher education textbook (scriptum)	100.00 %	93.88 %
Notes from contact classes (lectures, seminars)	100.00%	95.92%
Scholarly literature (from library)	75.58%	77.55%
Internet courses delivered by Moodle	95.16%	87.76%
Internet information (www pages)	67.74%	51.02%
Discussions with peers	67.74%	51.02%
Consultations with tutor	19.35%	16.33%
Consultations with teacher of mathematics in primary or secondary school	4.84%	10.20%
Other resources	0.00%	2.04%

The students have utilized primarily traditional information resources. This confirmed the prevailing trend from the undergraduate (Bachelor) degree level in which the dominant position was occupied by the specialized higher education textbooks and the notes from contact classes (either lectures or seminars). Relatively important position was held by the electronic courses designed for the Moodle interface. Another group of information resources includes scholarly literature, information from Internet (www pages) and discussions with peers. We have registered a decrease in interest in the information from Internet, which might be attributed to a more critical approach to consuming unreviewed information due to a higher level of students' expertise.

The second questionnaire item reads: "If I have an access to all information resources offered, in which order would I use them?" Students had to rank the list of resources following the criterion of importance when studying mathematical discipline (the scale ranged from 1 to 9). The offered options with their average order position regarding their usage are presented in the following Table 2.

Table 2

Information resource	Average order position of information resource by students' preference	
	1st year	2nd year
Higher education textbook (scriptum)	2.00	2.33
Notes from contact classes (lectures, seminars)	1.84	1.84
Scholarly literature (from library)	4.02	4.51
Internet courses delivered by Moodle	2.97	3.49
Internet information (www pages)	6.24	6.18
Discussions with peers	5.87	5.86
Consultations with tutor	5.74	5.24
Consultations with teacher of mathematics in primary or secondary school	7.50	7.08
Other resources	8.79	8.73

The presented data indicate that the position of the most preferred information resource is again taken by the contact classes in the form of lecture or seminar and the information obtained from higher education textbooks. Another group representing the information resources of comparable rating (from the aspect of their utilization) are electronic courses designed for the Moodle interface and scholarly literature. Consultations with tutor of the given course, discussions with peers and information from Internet pages form the third category. Consultations with teacher of mathematics in primary or secondary school are the least preferred information element.

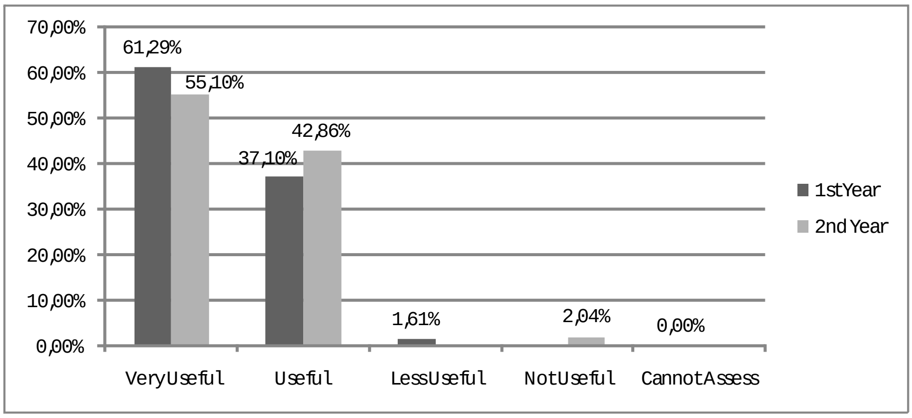

Figure 1

In the third questionnaire item we have focused on finding students' opinion on the usefulness of e-learning in their study. The findings, presented in the form of graph (Figure 1), indicate that almost all the students in the observed years of study consider e-courses during their study as very useful or useful. Based on the given data, we find e-courses designed in the Moodle interface to be a suitable complement for full-time students.

Table 3

Study unit in LMS Moodle	Average choice of unit element	
	1st year	2nd year
Concise outline of a lecture in electronic format (e.g. Word document)	98.39 %	90.91 %
Collection of excercise	88.71 %	54.55 %
Electronic test with feedback	77.42%	68.18%
Complementary study materials (e.g. video sequences)	62.90%	59.09%
Discussion forum for electronic consultation of study problems	17.74%	18.18%
Other	1.61%	4.55%

In the following part of the questionnaire we wanted to establish what the electronic course designed in the Moodle should contain. The data from the analysis of this item are presented in Table 3. Based on the data, we conclude that students' views correspond with the view of Turčáni [6] on the structure of an e-course in which each unit should contain theory (outline of a lecture, in our case), exercises (batteries of task, in our case) and a test (self-corrective on-line tests, in our case).

Based on the above findings, the electronic courses designed for mathematical disciplines in the Teaching in Primary Education programme have the following structure:

– Theoretical points of departure (elements from mathematical domains)
– Exercises (collections of tasks)
– Pedagogical interpretation (follows from mathematical domains)
– Tests (self-correcting on-line tests).

In Table 4 we present students' reasons for preferring the Moodle system.

Table 4

I have used the Moodle system because (you can enter more options)	Average choice of study unit	
	1st year	2nd year
It is a source of new information necessary for study	98.39 %	87.76 %
I have access to new information at any time	90.32 %	83.67 %
It offers a possibility of taking self-corrective tests (immediate feedback)	33.87%	8.16%
It offers an opportunity to consult with tutor (teacher)	1.61 %	8.16 %
It was required by a tutor	6.45%	4.08%
It opens possibilities to consult with other students (e.g. via discussion forum)	0.00%	4.08%

From the given data we conclude that time independence and relevance of study elements are the dominant factors in utilizing an e-course during study. Very important element in e-courses is a possibility of taking self-corrective on-line tests. 33.87 % of the respondents from the 1st year labelled it as a reason for utilizing Moodle. However, taking advantage of the consultations with tutor via the Moodle system did not appear as a strong factor of e-courses preference. We attribute the given finding to the relative amount of time necessary for written electronic communication and inadequate skills of students to formulate their study question.

4. Conclusion

Based on the survey results, it is possible to conclude that mathematical training of the students of the Teaching in Primary Stage programme supported by e-learning is a tool which enables students (and tutors) more efficient use of their contact meetings. From full-time students' point of view, electronic support of mathematical training is a preferred information resource. Thus, in such a case Moodle is a suitable tool enabling the processing, presentation and distribution of electronic study units.

From the aspect of tutor, it is necessary to employ new information technologies, on the one hand, and to adapt individual elements of curriculum to the needs of e-learning, on the other.

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ON SOLVING MATHEMATICAL TASKS BY PUPILS WITH SPECIAL EDUCATIONAL NEEDS

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Abstract. In the contribution we present a specimen of outcomes obtained from a pilot research conducted during a project aimed at verification of the new Czech curriculum for pupils with special educational needs. We include tasks falling into the area of mathematical literacy, which enable us to characterize pupils' competences.

1. Introduction

Mathematics is regarded as an important part of education and of cultural and historical background of a person. It is a tool of orientation in the world and a tool of not only thinking and prediction but also a tool for solving routine and unusual real-life problems and tasks. Mathematical literacy includes (cf. *European Commision: Second report on the activies of the Working Group on Basic Skills, 2003*) a set of knowledge, which a person is able to apply in areas such as family budget, shopping, travelling and free time, of special skills (such as the use of mathematical terminology or units, the use of tools and means – ICT). This means that people are able to recognize and understand mathematical problems, study them and use mathematics in their private life, work or among friends or relatives as constructive, active and pondering citizens [2].

The supplement of Framework Education Programme for Elementary Education on education of pupils with slight mental handicap stresses the fact that "an area of education is based on practical activities, application of mathematical knowledge in real life, strengthens the ability to think logically and enhances space imagination. Pupils acquire basic mathematical concepts and symbols and techniques and their possible usage. They learn to be precise, to apply rules of mathematics, to use calculators and mathematical software. Mathematics penetrates all the basic education, gradually helps pupils to acquire mathematical literacy and teaches them skills which they can use in practice" [6, p. 22].

2. Aims and methodology of research

In our research we give a specimen of outcomes of a pilot research conducted during "Education of Children, Pupils and Students with Special Education Needs", which is the first Czech research aimed at verification of efficiency of changes introduced by the new curriculum for pupils with special needs.

The pilot research was performed in November and December 2008 at two elementary schools in Kyjov and Olomouc. It verified the methodology of data acquisition on a reference sample of 15 pupils chosen from the complete sample of pupils of the last year of elementary school by means of practically performed random sampling. Based on the experience obtained during the pilot research, the test materials, which are going to be used during the main research, were adjusted.

The test materials consist of three main parts characterized by the area of competence studied:

1. socio-personal and work competence,
2. competence of a reader and language competence,
3. mathematical competence.

We tested correctness and completeness of solution. The time allocated for the test was 1 lesson (= 45 minutes). After the lector hands in the worksheets, he or she comments the test (e.g. "read the test carefully, mark correct answers in a certain way – circle them, include or side calculations, etc."). The tested pupils are allowed to use their calculators.

3. Tasks and expected outcomes of the subject matter

1. Multiply 99 by 1,2,3, etc., 8, 9 respectively. Can you say whether (and why) the results are interesting?
   $99 \cdot 1 = \ldots$, $99 \cdot 2 = \ldots$, $99 \cdot 3 = \ldots$, $99 \cdot 4 = \ldots$, $99 \cdot 5 = \ldots$, $99 \cdot 6 = \ldots$, $99 \cdot 7 = \ldots$, $99 \cdot 8 = \ldots$, $99 \cdot 9 = \ldots$,

2. Count and complete the table:

Goods	1 kg	2 kg	3 kg	4 kg
Flour	14.70 Kč			
Rice	23.40 Kč			
Spaghetti	45.90 Kč			

Verify the calculations (use calculator) and round the results to tens.

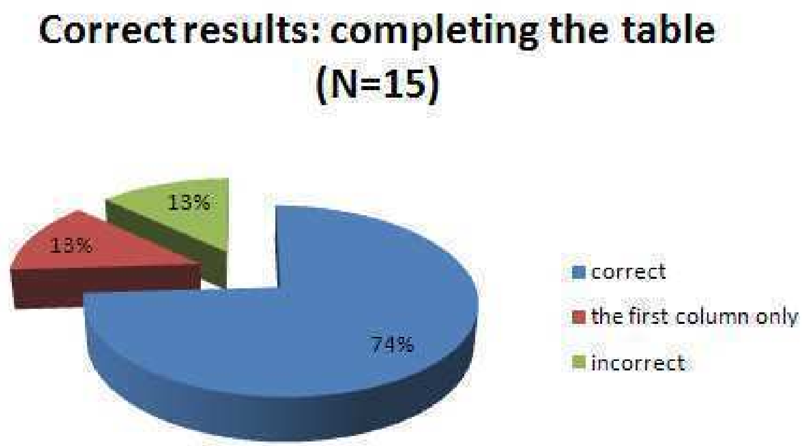

Figure 1: Percentage of students who completed the table correctly.

3. Eve decided to prepare a salad for her mothers' birthday. She will need half a kilo apples, two bunches of radishes, half a kilo of tangerines and kiwis. The prices at the supermarket she went to were:
   Apples (1 kg) ... 22 Kč, Tangerines (1 kg) ... 35 Kč, Radishes (bunch) ... 6 Kč, Kiwi (1) ... 3 Kč
   How much is Eve going to pay for all the salad ingredients?

   Pupils solve real-life situations and simple calculations of financial mathematics (tasks with money – shopping, savings, loans).

4. Complete the table using correct units:

Side of the square	30 mm			6 cm	10 cm
Perimeter of the square		16 cm			
Area of the square			25 cm^2		

Compare the data in the first and second columns:

The side of the square increased by ... mm, the perimeter incresed by ... mm and the area of the square increased by ... cm^2.

5. In the morning Phillip spends 10 minutes washing himself. Then he eats his breakfast, which takes him 15 minutes, brushes his teeth for 5 minutes. His walk to school is 20 minutes long. The lessons start at 8 o'clock. What time does Phillip have to get up in order not to be late for school?

   A) at 8 o'clock B) later than half past eight
   C) before 7 o'clock D) at 7:15

   Pupils manage simple procedures to find length, weight and time. They use the acquired data to describe reality and in simple calculations (change of units of time, weight, time).

4. Research results

We are going to discuss solution of tasks 2 only. Pupils results ($n = 15$) are given in graphs and commented. However, before the data is considered, we must mention that the test in mathematics was the last one to take at both schools. Pupils at both schools were visibly tired and some even demotivated to solve yet another set of tasks. This might have influenced pupils attention and care with which the tasks were worked out. On the other hand, the lector (based on experience with the pupils) suggested that it was not tiredness but lack of mathematical competence that caused the failure of some pupils.

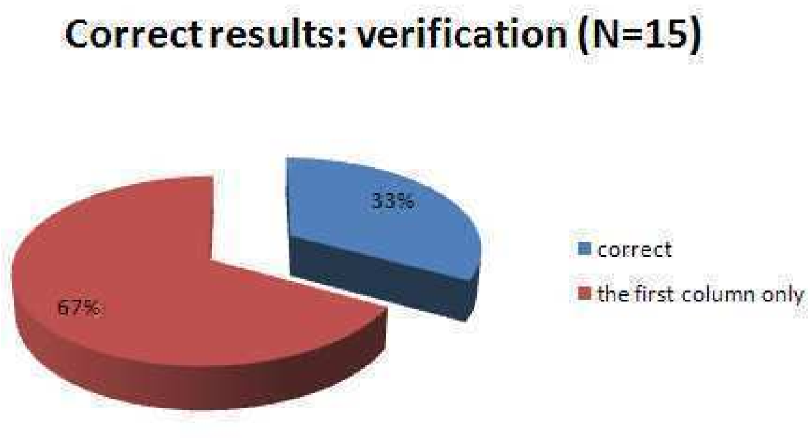

Figure 2: Percentage of students who verified the results correctly.

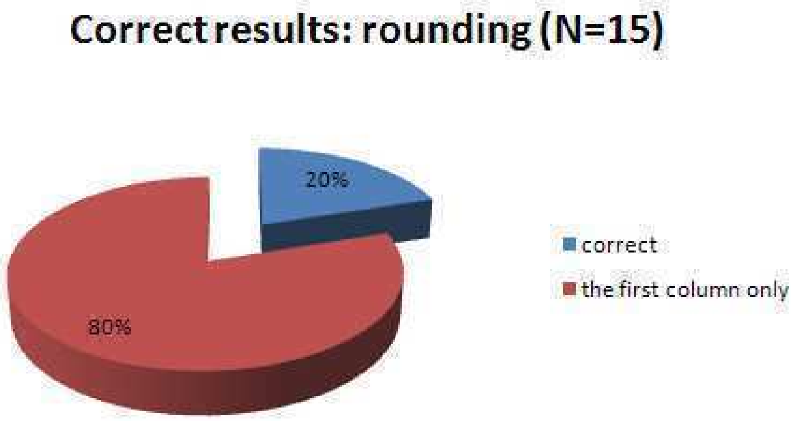

Figure 3: Percentage of students who rounded the results correctly.

We divided the solution of the task into three steps: completing the table (see Fig. 1), verification (see Fig. 2) and rounding (see Fig. 3). This division provedto be optimal during evaluation. As far as completing the table is concerned, three possible scenarios occurred – for details see Fig. 1. The pupils either completed the table correctly or wrote numbers without any sense (or having consulted the lector omitted this part because they did not know the way the task should be solved). Alternatively, they completed the first column only, i.e. dubbled the numbers.

Five pupils verified the solution while the remaining ten either did not perform any verification, or the verification was not correct. It could seem that Figures 1 and 2 do not match as more pupils calculated the task then verified the solution. However, some pupils who used their calculators did not include verification. Solving the task correctly and performing verification are independent and there indeed were pupils unable to deduce the way to verify the results from the correct numbers in the completed table.

Only 3 out of 15 pupils rounded all numbers correctly. For details see Fig. 3. These three pupils passed all three parts of the task. Simultaneously, they are the best solvers of all tested areas. Thus we could conclude that mathematical skills are a certain measure of intellectual abilities of the tested group of pupils.

5. Conclusion

There are significant differences between the pupils as far as solutions of the above tasks are concerned. The comparison of all three areas of the test resulted in the following finding: pupils who were very successful in linguistic and mathematical parts of the test posses more elaborate and visible nonverbal communication means. On the other hand, pupils errors indicate an overall low level of mathematical competences. Pupils had problems with understanding word problems. As it was emphasized in [7], this is often a result of low level of reading comprehension. Even in spite of this fact, pupils were better at solving linguistic tasks. Solutions and results of some pupils might suggest that decimal numbers were a big problem. Some students even computed the results correctly but did not write the decimal point. We treated such solutions as correct if all other computations and solutions were correct. This is also true for solutions with small mistakes which can be obviously attributed to lack of attention instead of insufficient knowledge of mathematics.

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STUDENTS' MATHEMATICAL COMPETENCE AFTER COMPLETING THE FIRST STAGE OF EDUCATION

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Abstract. In the school year 2009/10 a mathematics achievement test was conducted on a sample of 576 students in the 3th grade of school. A mathematical skills test examined the ability to complete both simple and complex tasks. It also tested their aptitude for the application of knowledge to practical and problematic situations. The skills checked were mastered at a level of about 65%. Skills in geometrics were at a lower level - around 37%. Problem solving ability was mastered at a very low level. One of the reasons is that school pupils are usually under teacher guidance while solving simple typical tasks. Rarely do they solve them independently or collectively. The teacher does not allow the student to err in search of a solution, and problematic tasks rarely be found in textbooks.

Poor results in mathematics in the matriculation exam, the final gymnasium exam and the final primary school test continuously concern everyone, especially the people responsible for Mathematical education, including maths teachers. Constant changes to the *Curriculum*, which sometimes give the impression of manipulation of it, are not beneficial to solid mathematical education. The four stages of it, where teachers of one stage often do not know the mathematical content learnt by the students in earlier or later stages, do not support consistent mathematical education, but they often create disparate fragments of knowledge. School textbook publishers tend to "relieve" teachers by the introduction of exercise books in which a child is limited to filling in the "blank spaces". This fact does not encourage student's self-reliance during the solution of tasks. The limitation of the new *Curriculum* for the early school education stage, according to which not more than 1/3 of students' notes can be made in exercise books, has not brought satisfactory changes. The reason is that a teacher has already got used to "easy" work based on "prepared materials".

What are the effects of this situation?

After the first school term 2009/10, the research on students' mathematical achievement was conducted on a sample of 576 students attending the III grade of primary school. A school mathematical achievement test, checking the range of material learnt by a student at school, was conducted [1]. It was a one stage test (without distinguishing tasks for particular requirement levels), however it contained tasks both easy and complex as well as of varied difficulty, which demanded the skill to apply knowledge in practical and atypical problematic situations. The test examined the following skills:

- arithmetic,
- geometric,
- practical,
- solving text-based tasks.

Two tasks, one text-based and the other related to geometric problems were atypical and required the students to notice inter-relationships, skills which are not always developed during classes in mathematical education. They will be analyzed separately. A precise, yet straightforward and clear, answer coding key showing correct and incorrect responses has been developed, which helped to detect the mistakes most commonly made by students. Basic indicators depicting the results of the research are shown in Table 1 as well as in the graph (Figure 1).

Table 1. Collation of basic indicators describing the test results

Number of subjects	n	576
Maximum possible result	X	33
Maximum result achieved	$X_{\max}$	32
Mean result	$\bar{X}$	20.6
Median	X_{me}	22
Mode	X_{mod}	23
Standard deviation	S_x	6.8

Source: Self study

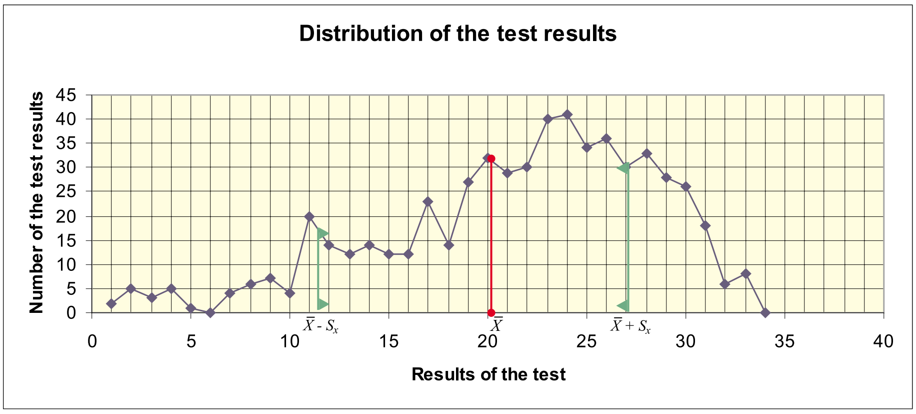


Figure 1. Graph showing the distribution of the test results.

The median value, 22 points – with the mean result being 20.6 – indicates that more than half of the subjects (61.5%) achieved a result higher than the mean. There were 71 poor results – below 12 points ($\bar{X} - S_x$) – constituting 12.3% of all the subjects and there were 86 results higher than 27 points ($\bar{X} + S_x$) constituting 15% of all the subjects. The above distribution of the test results could be considered to be satisfactory, although it is worthwhile analyzing the results achieved in the particular skills categories.

Table 2. Distribution of the results achieved in particular skills categories

Category of the skills being tested	Arithmetic	Geometric	Practical	Text based tasks	Problem based tasks
Maximum possible result	9	4	8	11	5
Mean result	5.72	1.49	5.68	7.60	1.60
Percentage of tasks solved in a category	63.60	37.15	71.03	69.10	31.94

Source: Self study

Skills in the category *the application of mathematics in practical situations* have been the best acquired. The tasks tested the skill to read a thermometer, to read a watch and to convert units of weights and measures. The level of arithmetic skills can be considered as low (63.6% of tasks solved) – the skills to arrange numbers in order and to calculate the results of arithmetical operations up to the number 100 were being tested.

Text based tasks, especially those regarding practical situations, were not too difficult for the students either. An atypical task, requiring students to notice a certain inter-relationship, was an exception. The task originated from research led by Miroslaw Dabrowski in 2005 in the project: **Third grade primary school students' basic skills test** [2]. Let us focus on the task.

Task 14
For 4 chocolate bars and 4 chocolates you have to pay 28 PLN. 3 of the same chocolate bars and 4 of the same chocolates cost in total 23 PLN. How much is a chocolate bar and how much is a chocolate?
Solution ....
Answer ....

Students of elementary education, who were also solving the task, most often started by making this set of equations:

$$\begin{cases} 4c + 4b = 28 \\ 3c + 4b = 23 \end{cases}$$

A third grade pupil is fortunately not familiar with this method and solves the task by applying the available information. In order to solve the task the pupil had to notice the difference between the two purchases. In the second purchase one less chocolate bar was bought and the difference in the cost of the two purchases represents the cost of one chocolate bar.

28 PLN − 23 PLN = 5 PLN	cost of one chocolate bar
5 PLN · 4 chocolate bars = 20 PLN	total cost of chocolate bars from the first purchase
28 PLN − 20 PLN = 8 PLN	total cost of 4 chocolates
8 PLN : 4 = 2 PLN	cost of one chocolate

The cost of one chocolate can also be worked out by proper calculation of the second purchase. Some pupils depicted the described situation which helped them to find the hidden values. A correct solution was the one in which the calculation or the result had been worked out on the basis of the depiction and there was a written answer. In this way 22.9% of the students tested solved the task. Twenty students (3.5%) limited themselves to giving the solution without writing the answer. These solutions can also be considered to be partly correct. A group of students (8.5%) made an attempt to solve the task, however they were unable to complete it.

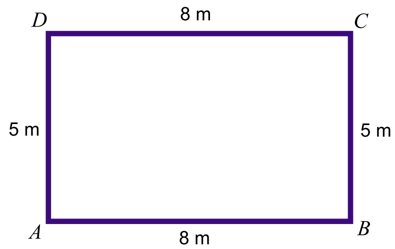
Unfortunately, as many as 65% of the students tested did not solve the task correctly, including 39.4% of the students who did not make an attempt to solve it. For such a large group the task proved to be too difficult. As mentioned above, the difficulty did not lie in complicated calculation, but in noticing a relatively simple inter-relationship. The conclusion may be that the students are not familiar with solving tasks which require independent analysis as well as noticing the inter-relationships between the described variables.

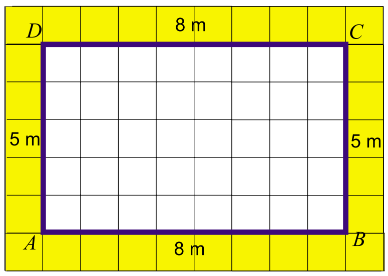
The geometric skills were tested by the use of two tasks. The first one concerned the identification of parallel and perpendicular segments – it was solved by 35% of the students. The second one required the students to notice an inter-relationship in a practical but atypical situation.

Task 5.

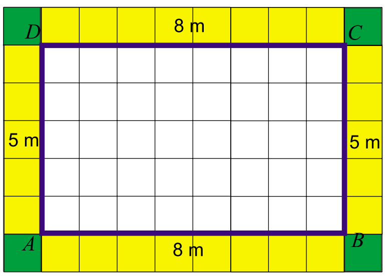
Around the swimming pool of measurements given in the picture, a pavement of square paving stones with the measure of 1 m per side was made. How many paving stones were used?

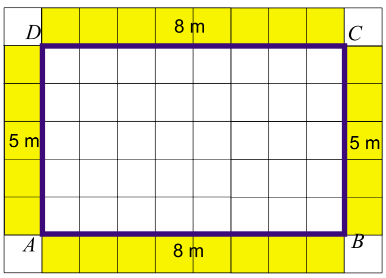
The task seems to be extremely easy as it is enough simply to draw a picture of the pavement surrounding the swimming pool, which helps to count the number of paving stones required. Several students dealt with the task in this manner. They marked the square paving stones in the picture and gave the result: **30 paving stones are needed**.





The second approach is to calculate by any method the perimeter of the swimming pool, which is a rectangle, and then add the number of corner paving stones to the result.





However, it turned out that the number of paving stones required was properly calculated by merely 12 of the students tested, which represents 2.1% of all the participants. Perhaps, if the initial picture had been drawn on graph paper, more students would have noticed the missing corner paving stones.

The vast majority of the students calculated the perimeter of the rectangle by the use of one of the following equations:

$$\text{Per.} = 2 \times a + 2 \times b \qquad \text{or} \qquad \text{Per.} = 2 \times (a + b)$$

or simply by performing the following calculations:

$$2 \times 8 + 2 \times 5 = 26 \quad \text{or} \quad 2 \times (8 + 5) = 26 \quad \text{or even} \quad 5 + 5 + 8 + 8 = 26.$$

71.4% of the tested students completed the calculation of the task with the above result. As many as 14.4% of the students did not make an attempt to solve the task, and 12% tried to solve it, however, they made errors which proved that they had not understood the instructions, or made errors in arithmetic calculation. Both of the presented tasks required the skill to seek the inter-relationships, namely research to discover mutual inter-relationships between the items of the given information. At school children solve simple typical tasks usually with the guidance of a teacher. Rarely do they seek the solution independently or collectively. The problem often first needs to be noticed and then individually formulated into a task. A teacher does not allow a student to err in search of a solution, and tasks with too little, too much or mutually exclusive data, which compelled a student to think, have been removed from school textbooks. Dorota Klus-Stańska has repeatedly been highlighting the necessity to seek and research within the framework of mathematical education in her studies [3].

Can a lack of a particular skill, namely the disability to solve problem tasks – the lack of the ability to notice the inter-relationships between the items of the provided information be pronounced on the basis of poor results in the two chosen tasks? After all, as it has been shown above, the overall results are not poor, they can even be considered to be satisfactory. The only problem is that the majority of the tasks regarded the so-called simple typical skills connected with arithmetic calculations.

However, the acquisition of mathematics is not only learning arithmetic and the skill to apply it to simple situations, but primarily the process of thinking, sometimes also referred to as mathematical thinking. Thinking or reasoning is a complex mental process which consists of seeking the inter-relationships between concepts and deductions. **Mathematical thinking** is unique in its logical thinking based on the defined assumptions, logical rules – definitions, theorems, but at the same time in its need to pose questions – hypotheses, although it is not always possible to answer them. It requires analysis and synthesis. Logical thinking, often considered as mathematical thinking, is necessary in every field of science which requires the skill to associate facts with their mutual inter-relationships [4].

Therefore, a question should be posed: *Are we concerned with students who acquire knowledge at the level of simple schemes or with creative students who have been prepared to solve even uncomplicated problem tasks?* The size of the group allows us to make a certain generalization: **students are unable to solve atypical tasks**. Where does the reason for this situation lie – in the low effectiveness of mathematical education in relation to the skills from higher categories of teaching aims?

In my opinion, the answer is relatively simple. The reason lies in schools. A teacher is limited by the *Curriculum* which "must be completed" by means of textbooks and workbooks provided, which allow a student only to "fill in blank spaces" instead of solving tasks independently.

Additionally, the introduced *Curriculum* reform and the retention of only simple text tasks in elementary mathematical education, as well as the change of system where, at least for the present, there are six and seven year old children in one class of over twenty pupils, do not improve the effectiveness of school education in the first stage of primary school.

The consequences of this situation at the end of the VI grade are reflected in the results of the test which takes place at the end of the second stage of education. They are available on the website of the Central Examination Board.

There is still the issue of the teacher – is he/she definitely prepared for creative work with a child within the framework of mathematical education?

However, this suspicion requires independent research.

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[1] In the research a test conducted by a group of students in the V year of elementary education at the University of Opole was used. It was carried out under the guidance of prof. dr. hab. Gabriela Kapica, with the consultation of Jerzy Nowik.

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THE CUBIC DICE IN THE ELEMENTARY MATHEMATICAL EDUCATION

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Abstract. The main issue of the article is geometry, arithmetic and stochastics of the dice in children's mathematics. It is a well-known object having mathematical features which can be used for developing mathematical skills at an elementary level. We consider not only geometrical and arithmetical aspects of mathematical education but also the combinatorial and stochastic ones.

1. The die and its W_K feature as a mathematical discovery

Today's die was created from a cube by placing different numbers of spots on its faces. The spots go in numbers from 1 to 6 and are placed on the cube in such a way that spots on opposite faces add up to 7. Thanks to this W_K feature the die has a rich mathematical structure. This feature is often unseen in school textbooks (see [8], p. 17). Throwing the dice is a random experiment.

Let us start with a simple trick. A student throws the die (so that the teacher does not see the result) and remembers the number of spots s_1. Then the student turns the die over and adds the number of spots on the opposite side to the previous one. He does not reveal the sum s_2. Now the student rolls the die again and adds the number of spots to the sum s_2. He does not reveal the new sum s_3 either. Now the teacher comes to the die, throws it, thinks for a while and then gives the exact sum s_3. Surprised students start asking questions: How is that possible? How can we explain this? And all of them lead to an important one: Was it just a coincidence or maybe there is a rule behind this? The W_K feature becomes the center of further argumentation.

Task 1. Roll the die and note down the number of spots you see. Turn the die over and check the number of spots on the opposite side of it. Repeat the throwing a couple of times, each time check up the opposite side spots. Can you see anything unusual?

Creating student's own dice from the cubic grid can also be a mathematical activity.

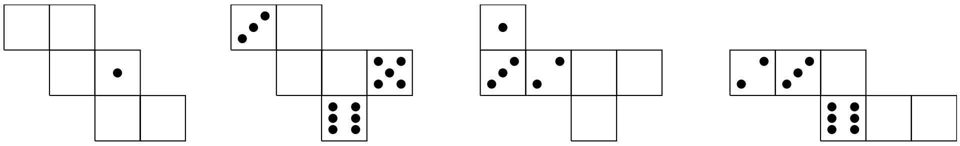

Fig. 1.

Task 2. There are some spots in some squares of each grid in Fig. 1. Fill in the missing spots in such a way that will enable you to get the die out of each grid.

Task 3. Each side of the die has four neighbouring sides and one opposite side. Which squares will become neighbours ith the side with four spots after we glue up the grids from Fig. 1? Which will become the opposite side?

2. The dice arithmetic

The W_K feature of the die enables us to count the spots on it quickly. Each pair of opposite sides has 7 spots together, there are 3 such pairs, so the sum of the spots equals the product of $3 \cdot 7$.

Task 4. Roll the die and guess the number of spots on the side touching the table.

Task 5. Roll two (three) dice. How many spots are there on all the sides touching the table?

Task 6. Place two (three) dice on the table in such a way that the number of spots on their bottom sides is the highest.

Task 7. Every student builds a tower of two dice on the table. The student who gets the biggest number of spots on the side walls of the rower wins the game. Did you manage to show the winner? Is it possible to win such game if the students build their towers of three dice? Why?

Task 8. Explain the fact that there is no winner in the game described above no matter how many dice the students use.

Task 9. Can you place the dice on a table in such a way that the spots on side walls add up to 18? Why?

Task 10. Roll the die and count the spots on the four side walls. Repeat that as many times as you like. Is it a coincidence that the spots on the die side walls always add up to 14? Can you explain this?

Among the four side walls of the dice there are two pairs of opposites, so the sum of spots on them always equals $2 \cdot 7$. The number of spots on the side walls of a tower does not depend on the dice placing, but only on the number of "floors".

Task 11. Some of the dice in Fig. 2 touch three walls of the room. How many sides of the die are invisible in each case? How many spots are there altogether on the invisible sides of the die in Fig. 2a, how many in Fig. 2b and how many in Fig. 2c?

Task 12. Roll four dice and quickly (without turning them over) join them in a 2×2 combination. How many spots are there altogether on the invisible sides of the dice?

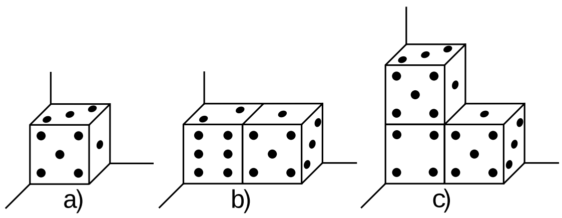

Fig. 2.

We create buildings using many dice placing them in such a way that their sides touch one another. Those touching sides are invisible to us, so they are the back walls and some of the side walls. It is an arithmetic activity to add up all the spots on the invisible sides of all the dice forming a tower.

Task 13. How many spots are there on the invisible sides of dice in the building A in Fig. 3, how many in the building B and how many in the building C? Why is it not possible to state this number in the building D?

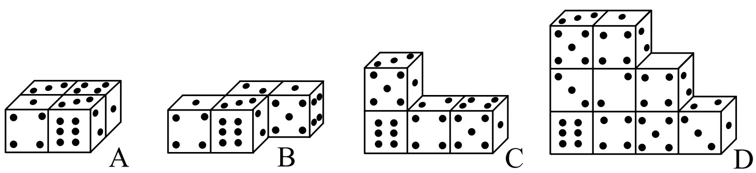

Fig. 3.

3. Odd and even numbers and elements of logic in the dice arithmetic

In the following tasks some elements of logic and deduction appear in the context of the dice and odd and even numbers.

Task 14. How many spots must appear on the die to give an odd number on the bottom side?

From the W_K feature we know that the number of spots on the bottom side of the die is the difference between 7 and the number of spots on the upper side. If the number of spots showing on the die is an odd number, then the number of spots on its bottom side must be – as a difference of two odd numbers – an even one.

Task 15. Casper threw three dice. Each of them showed an odd number of spots. Casper claims that on the bottom sides of all the dice the numbers of spots are even. Is he right? Why?

Task 16. Melchior threw three dice. Each of them showed an odd number of spots. Melchior claims the sum of the spots on the bottom sides of the dice is an even number. Is he correct? How do we explain this?

The W_K feature causes the number of the bottom sides of the dice being a difference of 21 ($21 = 3 \cdot 7$) and the sum of the spots showing on the upper sides of the dice. The number $3 \cdot 7$ is an odd one. The number of spots showing is also odd (being the sum of three odd numbers), so the difference between them (as a difference of two odd numbers) is even. Melchior is right.

Task 17. Baltasar threw three dice and said that the number of spots he got was odd. He also said that the number of spots on the bottom sides of the dice is even. Is he correct?

Task 18. Artie claimed that the tasks 16 and 17 were in fact one task (they both deal with the same problem). Why is he wrong?

Task 19. Bart threw four dice and said that the number of spots he got was odd. He also said that (as in task 17 with three dice) the number of spots on the bottom sides of his dice was even. Is he right?

Task 20. Artie invited Bart to a game: – *On my mark we both roll one die each. Then we count how many spots our dice show together. If the number is odd – I win, if it is even – you win,* said Artie. The boys threw their dice, but they did not reveal the scores.

1. Both dice showed odd numbers. Who won?
2. The number of the spots was more than 6 and dividable by 5. Who won?
3. Is it possible to state the winner if all we know is that the number of spots was dividable by 5? Why?

4. The dice in space geometry

The die will be moved in steps. Each step means turning the die to one of the side walls. Each move directs the die into one of the four sides. Let us implement the compass names to distinguish the sides and assume the movements occur in snow and the spots leave marks in it.

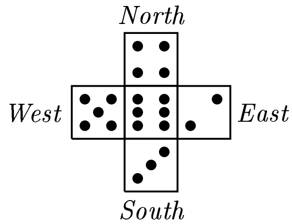

Fig. 4.

If the bottom side of the die has 6 spots, the front side has 3 spots and the right hand side has 2 spots (Fig. 5), then we can move in the following ways:
— to the South, then the bottom side has 3 spots,
— to the North, the bottom side has 4 spots,
— to the West, the bottom side has 5 spots,
— to the East, then the bottom side has 2 spots.
All the possible markings after taking the first step are shown in Fig. 4.

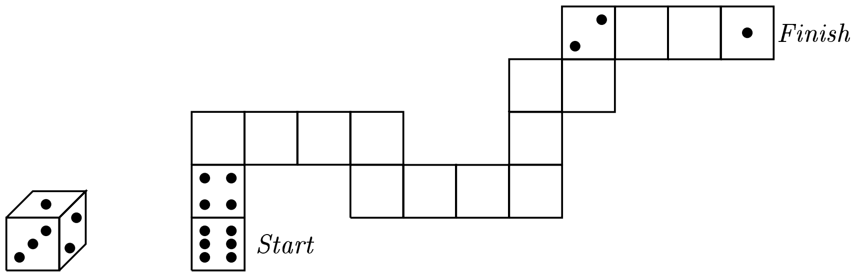

Fig. 5.

Task 21. Place the dice which you see in Fig. 5 on the "start" square. On the bottom side of the die there are 6 spots. You will move it by turning to its sides along the way shown in Fig. 5. Before you actually turn the die, write down the number of spots which will show in the snow after your move in the following squares of the route. Will it be the mark with one spot in the "finish" square, as shown in Fig. 5?

We can write down the markings in the snow left by the die moving along the route from Fig. 5 as a following sequence:

6-4-1-2-6-5-4-1-3-6-5-1-4-2-6-5-1.

Task 22. In the "start" square the bottom side of the die had 6 spots, the front side had 3 spots and the right hand side had 2 spots. The code of the die markings is the sequence: 6-3-2-4-5-3-2-4-1-5-4-2-3. Draw the die route.

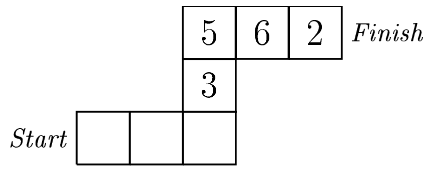

Fig. 6.

Task 23. In the "start" square the upper side of the die has one spot, the front side had 3 spots and the right hand side had 2 spots. Figure 6 shows part of the die movements (we cannot see the first three steps). Reconstruct the obliterated steps.

5. Natural numbers as sums of two other natural numbers

Task 24. Place two dice, the white one and the red one, in a row in such a way, that the numbers of spots showing add up to 7. How many spots are there on the bottom sides of the dice?

Figure 7 shows top views of four of the possible rows.

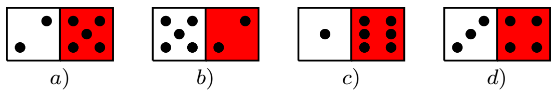

Fig. 7.

Task 25. Place three dice, red one, white one and green one, in a row in such a way that the number of spots showing adds up to 11. How many spots are there on the bottom sides of the dice? Is it possible to arrange three dice in a row in such a way, that the sum of the spots showing equals 19?

Task 26. Paul threw the dice twice, but he did not reveal how many spots he scored in the first or the second roll. He only said that in the first trial he got 5 spots more than in the second one. What was Paul's final score?

Task 27. Gaweł rolled two dice, the white one and the red one, but he did not reveal the scores. He said that on the white die he got 3 spots more than on the red one, and that his final score was 7. How many spots did he get on the white and how many on the red die?

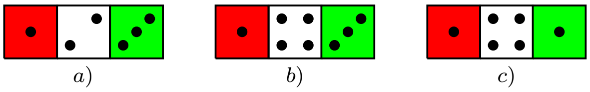

Fig. 8.

Task 28. Place three dice, red one, green one and white one, on a table (the red one first, then the white one and the green one third) in such a way, that:

a) the spots on the white and green dice add up to the number of spots on the red one,

b) on the upper side of the white (middle) die there are more spots than on the upper side of the red (left) one and more than on the green (right) one – is it so in Fig. 8a?

c) on the upper side of the red die there are less spots than on the white one, and on the white die – less spots than on the green one,

d) the number of spots showing on the red die is a sum of the numbers of spots on the white and green dice,

e) on the upper side of the white (middle) die there are twice as many spots as on the red one and on the upper side of the green die there are twice as many spots as on the white one.

Task 29. Paul placed red, white and green dice in a row, but covered them with his hands. He said that the white one shows two spots more than the red one, the green one shows two spots more than the white one, and the total score is 5. How many spots are there on the upper side of each die in this row?

6. The dice combinatorial aspect – different ways of showing the same problem

Task 30. Let us get back to placing two dice (the white and red one) in a row in a way that the total number of spots showing equals 7 (task 24). How many ways of doing so are there?

The point here is to show the number 7 as a sum of two other natural numbers, remembering that the order of the dice matters (we can easily distinguish them), and to state an exact number of ways of doing so. Four possibilities are shown in Fig. 7. The question of how many ways of doing so are there touches the theory of combinations. We need to get 7 spots on the upper sides of two dice. We can see that the real question here is in how many ways we can divide the set of 7 into two unempty sets. We are considering another, but equal, aspect of task 30.

$7 = 2 + 5$ $\quad$ $7 = 5 + 2$ $\quad$ $7 = 1 + 6$ $\quad$ $7 = 3 + 4$

Fig. 9.

Figure 9 shows seven spots in a row. In order to divide this set into two we need to place a line between any two neighbouring spots. The number of spots preceding the line represents the number of spots showing on the white die, and the number of spots following the line represents the number of spots on the red die. There are six possibilities of placing such a line, so there are six ways of placing two dice in a row and getting 7 spots on them both.

The interpretation of placing the dice in a row in a specific way (so that there are seven spots on the top) being the same as dividing the set of seven spots into two parts suggests some actions having to be taken (arranging the spots, placing the line). Those activities are different from placing the dice on a table chaotically as they lead the student to the solution of the problem.

Four examples of placing lines shown in Fig. 9 represent four solutions of the task shown in Fig. 7.

7. Dice tricks, the dice arithmetic and generalization as mathematical activity

Let us get back to the trick with one die. Let x means the number of spots scored in the first roll and y mean the number of spots scored in the second roll of the die. Let us consider the equation:

$$\boxed{x} + \boxed{7 - x} + \boxed{y} = 7 + \boxed{y}.$$

Each of the left-side components represents an action taken: roll of the die (the number s_1), turning it over (the number s_2) and the second roll (the number s_3). After the arithmetic transformation the left-side sum becomes the sum of 7 and the score of the second roll, and the later we can still see, as the die is still unmoved after the second roll. So the number s_3, which was to be guessed, is really the number of spots showing on the die enlarged by 7.

An important argument is the W_K feature of the die, so our argumentation is of arithmetic nature.

But the trick described above may create another mathematical activity, which is a generalization and inferring by analogies. We can modify this trick by complicating it a bit.

Now a student rolls two dice, the red one and the white one, then he:

— counts all the spots (the sum s_1),

— turns the white die over and adds the number of spots showing to the sum s_1 (the sum s_2),

— rolls the white die again and adds the score to the sum s_2 (the sum s_3).

The sum s_3 may be guessed if we know the trick with the single die. If we add 7 to the number of spots scored on the white die (both sides), we will get the number guessed in the one-die trick. The sum s_3 guessed in this trick has one more component – the number of spots scored on the red die (and this number is still visible, as the red die is still on the table). So the sum s_3 is really the sum of all the spots visible on the red and white dice and the number 7.

Task 31. Suggest a similar trick with two dice (the red one and white one) in which both dice will be thrown, turned and thrown again.

Task 32. Prepare a similar trick using three dice in which after the initial roll:
a) only the white die will be turned over and thrown again,
b) the red and green dice will be turned over and thrown again.

In relation to the dice tricks described above an element of surprise appears as an inspiration of mathematical activities.

8. Dice snakes and the dice arithmetic

We will create snakes using one white die and some red ones. The white die will represent the snake head, and the red ones will form its tail. The tails will be of different length. In Fig. 10 we can see four such snakes. The upper, visible side of the white die will be the snake head top, the upper sides of the red dice will form the snake back, and the bottom, invisible sides of the red dice will form the snake belly.

When we form such snakes and check the characteristic features of their "body parts" a number of arithmetical tasks will appear, each of them connected to adding, subtracting and multiplying natural numbers. Some tasks will refer to representing a number as a sum of two other numbers.

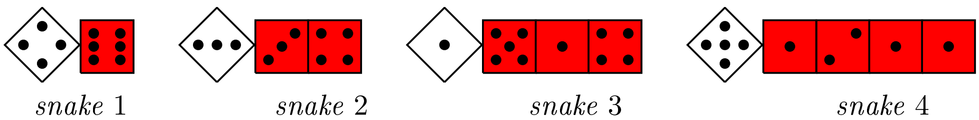

Fig. 10.

Task 33. For each snake from Fig. 10 count the number of spots on the head top, on the back and on the (invisible) belly. How many spots are there on all the dice forming the tail of each snake?

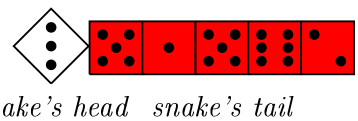

Fig. 11.

Task 34. Figure 11 shows the top view of a snake that was formed using one white die and five red ones. Its tail length equals 5. Count the spots on its back and then on its back and head top. How many spots are there on this snake's belly?

Task 35. Take five red dice and the white one. Form a snake in such a way that on its head top there are 6 spots and on its back there are six spots too. How many different snakes with six spots on their heads have tails with such backs? How many spots are there on such snake's belly?

Task 36. Take five red dice and the white one again. Form a snake that has the total number of spots on the back equal to the number of spots on the head top. Try several such snakes. What tail must a snake have if the number of spots on its head is 5? Is it possible for such a snake to have 4 spots on the head top? What other head tops are not possible for such a snake?

Task 37. Form a snake using five red dice and the white one in such a way that its back has 7 spots more than its head top.

Task 38. The snake tail is five dice long. There are 5 spots on its back. How many spots are there on its belly?

Task 39. The snake tail is six dice long. Is it possible for such a snake to have the same number of spots on the back and on the head top?

Task 40. Can a snake with a five (six) dice long tail have the same number of spots on the back and on the belly?

9. The dice versus certain, impossible and probable random events

A coincidence may also be a creator of "dice snakes".

Task 41. Roll two dice, the white and the red one. Form a snake without turning the dice. The white dice becomes the snake head and the red one – the snake tail.

1. How many different snakes with a tail of one die are there?
2. How many spots can there be on the head and tail of such a snake together? Is it possible to get a snake of 13 spots? What about 2?

The number of snakes we can get is the same as the number of possible scores of a two-dice roll – 36. It is worth to note that all the scores are equally probable. Let us connect the two-dice roll and form a snake with the following events:

$A = \{$*there are 14 spots on all the sides of snake's head*$\}$,
$B = \{$*there are 13 spots on both sides of snake's tail*$\}$,
$C = \{$*there is one spot on snake's head top*$\}$,
$D = \{$*there are 6 spots on one of the sides of the snake's head*$\}$.

It is easy to state that:

— we are sure to get the result A every time, so A is a *certain event*;
— it is not possible to get the result B at all, so B is an *impossible event*;
— getting the result C is not certain but it is not impossible either, so C is a *probable event*;
— it is also probable to get the result D.

Let us notice that whenever we get the event C we do not get the D at the same time, and the other way round. There is no such score of a two-dice roll that would be propitious for both events C and D. So C and D are *disjoint events*.

In the elementary mathematics education we will only decide whether two events are equally probable or not and if not, which of them is more possible to appear. We will also claim some events to be very likely to happen and some to have a slight possibility of happening.

Task 42. In a moment we will roll two dice, the red one and the white one, and form a snake. Before we do so, you can bet on the number of spots which the snake will have on both its head top and its back. If your bet is correct, you score a point. What numbers can you bet on? Does it matter what number you choose? Why?

In the game described above we bet on one of the events:

$$A_j = \{\textit{in a two-dice roll the total number of spots scored is } j\}$$

where $j = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$. Those events are not equally probable. Among them the event A_7 has the best chance of appearing, because there are 6 scores of the two-dice roll propitious for it (each of the remaining sums has less than 6 possible scores). This reasoning about "the chances of certain event happening" is correct as each score of the two-dice roll is equally probable. It is the most profitable to bet on the event A_7.

In the stochastics for everyone, we see two kinds of events as the most important:
— almost certain events which are very likely to happen,
— almost impossible events which are very unlikely to happen (see [4], p. 146).

Let us connect a six-dice roll with two random events:
$A = \{\textit{every number of spots will show}\}$,
$B = \{\textit{one of the possible numbers of spots will not show on any dice}\}$.
The event A is almost impossible, the event B – almost certain. Those stochastic features of the events A and B can be discovered *a posteriori* in the classroom. The scores of many repeated six-dice rolls are the qualitative (not quantitative) estimation of the probability of those events. The organization of the data gathered is discussed in [4] and [7].

Task 43. You have lost your die. What can you replaced it with? How? Can you "make" the die out of four coloured spheres?

The ways of replacing the die with an urn with three or four spheres of different colours are discussed in [4] (pp. 76-79).

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ELEMENTS OF MATHEMATICAL LITERACY IN PRIMARY TEACHER TRAINING

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Abstract. Mathematical literacy is an ability to be developed not just in pupils but also in teachers. It is a constituent part of their undergraduate teacher training. The courses listed under the programmes of study offered by the Faculty of Education, University of Prešov provide a platform for developing mathematical literacy of students – prospective teachers in primary school. The courses syllabi include OECD PISA tasks. They are utilized as means of exploration of theoretical starting points in certain domains of arithmetic and algebra. The article offers the task analysis carried out from the aspect of the present elements of mathematical literacy. We also present some examples of the way how particular primary mathematics topics from the syllabus are interpreted in terms of mathematical literacy.

1. Introduction

Developing students' mathematical literacy is a part of undergraduate training of primary school teachers. Specific subject training in the given area is a precondition for teacher's competence to create and formulate tasks and activities which develop pupil's mathematical literacy. OECD PISA international assessment scheme defines mathematical literacy as *"individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen"* ([2], p. 7).

The team at the Department of Mathematical Education, Faculty of Education, University of Prešov has been researching since 2010 the project aimed at raising standards of subject specific preparedness of primary teachers to develop pupils' mathematical literacy reflecting on the context of curricular reform as well as the OECD PISA and IEA TIMSS schemes. In view of the above context, more tasks on developing mathematical literacy and competence have been included into the modules and courses of teacher training.

Surveys carried out at the Faculty of Education, University of Prešov aimed at determining students' standards of mathematical literacy indicate that *"among the most problematic areas of mathematical literacy of students – prospective teachers in pre-elementary and elementary stage are: interpretation of data from graphs and tables"* ([3], p. 84). The authors of the survey state that some of the students have difficulty to solve the tasks in which it is necessary to apply lower secondary stage mathematics such as reading and writing decimals, roman numerals, percentage calculus, divisibility of natural numbers, elementary geometric and arithmetic terminology, and processing data from tables and graphs. The above results were obtained from the analysis of student's solutions of the test which included the released items from OECD PISA.

2. Arithmetic and algebra with didactics

Arithmetic and Algebra with Didactics is a course included into undergraduate teacher training programmes of study. The aims of the course are set as follows:

– to acquire the essential mathematical knowledge from arithmetic and algebra, followed by didactic transformation of them in conformity with the objectives of teaching mathematics in primary school as stipulated in the State Programme of Education ISCED 1;

– to introduce possible didactic interpretations of the theoretical knowledge in teaching mathematics in primary school;

- to identify relationships between the notions from the theoretical domains of arithmetic and algebra, and concepts developed in the minds of pupils of junior school age.

The course integrates the theoretical knowledge of arithmetic and algebra with didactic interpretation of it in teaching primary mathematics. The course contains the following elements of mathematical literacy:

Competences at the level of reproduction: mastery of mathematical terminology, reproduction of what is learnt, applying learnt algorithms and identifying analogy.

Competences at the level of connections: solving problems with known elements, linking several known methods, and the use of more complex procedures.

Competences at the level of reflection: the use of mathematical symbols and language.

3. Functions and dependence relations

Tasks containing elements of mathematical literacy, in the form of model situations and assignments, are used when presenting theoretical essentials of such notions from arithmetic and algebra which are used in primary mathematics on the level of propaedeutics (introduction). A function is one of the basic notions in algebra. This domain is included in the state mathematics curriculum in the *Sequences, Relations, Functions, Tables and Graphs* thematic area. The given thematic area provides for both finding quantitative relations and presentation of types of their systematic changes. Dependence relations are represented in the form of tables, graphs and diagrams [5].

Thinking in functions is important for resolving real life problems. Its rudiments start to be developed much earlier than the notion of a *function* is exposed in teaching mathematics. Children acquaint with dependence relations as early as in pre-school and primary school. The process of constructing concept of function is long-term and starts with solving tasks of propaedeutic character. In our view, it is important for primary mathematics teachers to be aware of those processes. By introducing the tasks taken from the real-life situations with the elements of mathematical literacy to undergraduate mathematical primary teacher training we strive to develop subject-specific and pedagogical competences of students, thus enhance their preparedness for teaching practice.

The notion of a function is introduced in school mathematics only in lower secondary stage, but the first conceptualisations of elementary functions as linear function and direct proportionality are already developed through some primary school activities.

In view of the above, the activities focusing on didactic processing and interpretation of tasks on functions are included in the *Arithmetic and Algebra with Didactics* course. Students are presented with theoretical points of departure such as: the definition of the notion of a function, the domain of definition of a function, the range of a function and the characteristics of the notions of dependent and independent variable. Elementary functions such as constant function, linear function and direct proportion are defined as well as the ways in which they can be assigned: by algorithm, where algorithm of a function is given by the formula for calculating the values of dependent variable, by table of function values, and by Cartesian graph.

The theoretical part of the given thematic area is made accessible through presenting the above notions in concrete examples. The analysis of task solutions is followed by another stage in which theoretical notes – definitions and characteristics of the notions relevant in the given domain – are explained. The above approach utilizes the elements of constructivist pedagogy in which knowledge is constructed in human mind on the basis of his/her experience with solving task and experimenting [1].

OECD PISA 2003 released items are utilized by us in the thematic areas of *Solving Applied Tasks and Tasks Developing Specific Mathematical Thinking.* In the following part, we will introduce two tasks developed for the purpose of testing mathematical literacy, used in teaching a function. Before that, however, OECD PISA testing tasks will be characterised.

4. Characteristics of tasks from OECD PISA testing

Each PISA task contains three components: situations or context, mathematical content and competences. Situations or contexts refer to real life. Mathematical content is subdivided into four parts: quantity, space and shape, change and relationships, and uncertainty. Relationships are given by a variety of representations, including symbolic, graphic, tabular and geometrical. Uncertainty refers to probabilistic and statistical phenomena and relations. Competences are abilities which could be drawn upon in solving concrete task or problem. They are divided into three levels:

– the level of reproduction: routine linkage, reproduction of practised procedures;
– the level of connection: utilising practised procedures when solving new problems with different domains of mathematics, includes tasks of divergent character;
– the level of reflection: tasks aimed at argumentation, abstraction, creation and utilisation of new algorithms and the use of mathematical apparatus in unknown situations.

5. Tasks on mathematical literacy used in teacher training

The task named Growing Up (M150Q01, M150Q02, M150Q03) ([2], p. 12) is from the Change and Relationships group, recommended for the thematic area of Numerical Operations, Functions and Reading from Graphs. It is aimed at competences at reproduction and connection levels. The task contains the elements of assigning function and dependence relation by graph or table.

Growing Up ([2], p. 12)
Youth grows taller. In 1998 the average height of both young males and young females in the Netherlands is represented in the graph (Figure 1).
Question 1: *Since 1980 the average height of 20-year-old females has increased by 2.3 cm, to 170.6 cm. What was the average height of a 20-year-old female in 1980?*
Question 2: *According to this graph, on average, during which period in their life are females taller than males of the same age?*

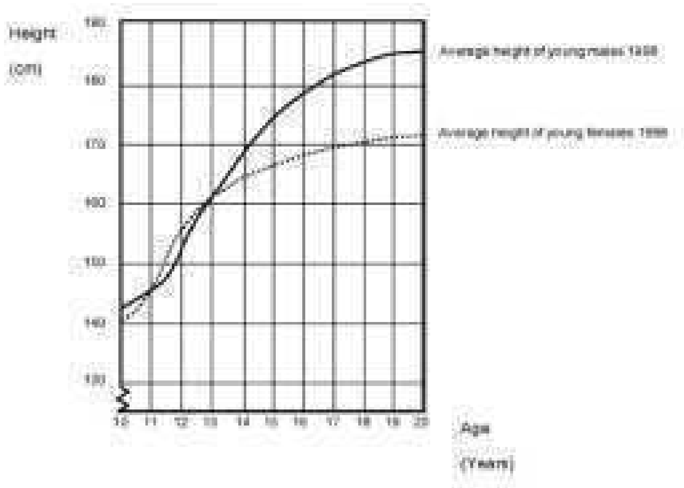

Figure 1

Question 3: *Explain how the graph shows that on average the growth rate for girls slows down after 12 years of age.*
Characteristics of the task:
Question 1: Solving the task does not require reading the data from the graph. It is indirectly formulated verbal task which relies on subtraction of decimal numbers.
This part of the task is solved by students without problems.
Question 2: This part of the assignment requires reading and interpreting the data from the graph. It is important to identify a graph which represents the dependence of the female average height on age and the dependence of male average height on age. The data from both graphs are compared, which leads to the correct answer.
On the background of the graph, the notions of the domain of definition of a function (the values of age), the range of a function (the values of average height), dependent and independent variable (the average height is dependent on the age), a graph of a function, data on the x-axis and y-axis are explained to students. The function in the task is given by a graph.
Question 3: This part of the item requires reading and interpretation of the data from the graph as well as the ability to argue and give the rationale for his/her idea. Since the properties of a function (monotonicity: increasing and decreasing) are not defined yet, the students are expected to provide answers using imprecise mathematic terminology. The response should refer to the change of the gradient of the graph for the dependence of female height on age.
The reasoning may be based on comparing the given values in the graph, in the period after 12 years of age in the female group. Transcription of the data from the graph can also be used:
10 years – 140 cm
12 years – 155 cm. In the period preceding the 12th year of age (two years), the increase is by 15 cm. The yearly growth is 7.5 cm.
20 years – 172 cm. The period after the 12th year (by the 20th year, i.e. in 8 years) the growth is only at about 17 cm. The yearly growth is only 2 cm.

Age intervals can be arbitrary, but the result or the reasoning for the given fact are important. From the given data it is obvious that the acceleration of female growth has decreasing tendency after the 12th year of age.

Alternatively, it is possible to include transcription of the data from the graph into a table and interpret the rationale on the basis of the information from Table 1.

Table 1

Age Interval	Height in the Given Period	Growth
10. – 11.	140 – 145 cm	5 cm
11. – 12.	145 – 155 cm	10 cm

In a two-year period there is 15 cm growth, i.e. 7.5 cm a year in average.

Table 2

Age Interval	Height in the Given Period	Growth
12. – 13.	155 – 160 cm	5 cm
13. – 14.	160 – 165 cm	5 cm
14. – 15.	165 – 167 cm	2 cm
15. – 16.	167 – 169 cm	2 cm
16. – 17.	169 – 170 cm	1 cm
17. – 18.	170 – 170.5 cm	0.5 cm
18. – 19.	170.5 – 171 cm	0.5 cm
19. – 20.	171 – 171.5 cm	0.5 cm

The growth in eight-year period is 16.5 cm, i.e. 2 cm a year in average. From the above data it follows that acceleration of female growth after the 12th year of age is decreasing.

Building upon the presented contextual situation, it is possible to formulate another task aimed at the third level of competences – reflection, for example, to compare the acceleration of female and male growth in the given periods and give reasons to one's answers. The task is of divergent character due to the possibility of different solutions. A student can use the data represented in graphs or transform them into a table, and the answer will follow from the data presented in the table.

The task Exchange Rate (M413Q01, M413Q02, M413Q03) includes a Quantity domain. It develops competences on reproduction and reflection levels and could be classified under Direct Proportionality thematic area. Therefore, it is included in the Arithmetic and Algebra with Didactics syllabus in the part which deals with the theoretical fundaments of an elementary function - direct proportion.

Exchange rate ([2], p. 14)

Mei-Ling from Singapore was preparing to go to South Africa for 3 months as an exchange student. She needed to change some Singapore dollars (SGD) into South African rands (ZAR).

Question 1: *Mei-Ling found out that the exchange rate between Singapore dollars and South African rands was: 1 SGD = 4.2 ZAR*

Mei-Ling changed 3000 Singapore dollars into South African rands at this exchange rate. How much money in South African rands did Mei-Ling get?

Question 2: *On returning to Singapore after 3 months, Mei-Ling had 3 900 ZAR left. She changed this back to Singapore dollars, noting that the exchange rate had changed to:*

1 SGD = 4.0 ZAR

How much money in Singapore dollars did Mei-Ling get?

Question 3: *During these 3 months the exchange rate had changed from 4.2 to 4.0 ZAR per SGD. Was it in Mei-Ling's favour that the exchange rate now was 4.0 ZAR instead of 4.2 ZAR, when she changed her South African rands back to Singapore dollars? Give an explanation to support your answer.*

Characteristics of the task:

The task is aimed at propaedeutics of direct proportion and at assigning a function by algorithm. We have devised a task for students in which they were expected to create an algorithm for a function on the basis of the given information.

Question 1 is aimed at the interpretation of a simple mathematical model, and the solution requires applying multiplication of decimal numbers.

1 SGD = 4.2 ZAR

3 000 SGD = x ZAR

3 000 × 4.2=12 600 (ZAR)

Direct proportionality is used, i.e. the amount of SGD will increase as many times more as will the amount of ZAR: increase in the value of one variable is in direct proportion to the increasing value of another variable.

Question 2 requires the interpretation of a simple quantitative model (direct proportionality) and applying it together with basic decimal numbers numeric operation. It is important to identify the operation to be applied.

3 900 ZAR = x SGD

4.0 ZAR = 1 SGD

3 900:4=975 (SGD)

It is a direct proportionality relation when the rule of three is proper: the ratio 3 900:4 equals to the ratio x:1.

The students create algorithm of the function from the task: $y = 4.2x$; $y = 4x$. On the background of the algorithm structure it is easy to identify direct proportionality.

Question 3: this part of the task is about explanation and reasoning of one's own thinking procedure. For example, it suffices to verify how many Singapore Dollars one gets at the exchange rate of 4.2 ZAR = 1 SGD

3 900 ZAR = x SGD

3 900:4.2 = 928.57 SGD

From the given, it is obvious that at this exchange rate one get less SGD for 3 900 ZAR than at the exchange rate which was current to the date (she got 975 SGD). Lower exchange rate is more profitable. It is not necessary to use numeric operations and state exact sum which she would get at the given exchange rates. Reasoning, though, must be precise and clear. There is also an option to draw the graphs of both functions with the table of domain and range values, and to present own reasoning based on the given data from the graphs and the table.

5. Conclusion

We have outlined one possibility to develop mathematical literacy of prospective primary school teachers. Teachers should realize that in teaching mathematics they can take advantage of real situations which correspond with the inner world of junior school age children [4]. In the future, more tasks which context reflects real life situation will be included into the curriculum of the programmes taught at the Faculty of Education, University of Prešov. The tasks of the given character are instrumental in presenting not only pedagogical approaches to the particular domains of mathematics but also theoretical points of departure of the given problem.

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DEVELOPMENT OF COMBINATORIAL THINKING OF ELEMENTARY SCHOOL STUDENTS

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Abstract. In the contribution, the development of combinatorial thinking of elementary school students is discussed. A set of problems including their elaboration with different solving strategies is also presented.

1. Introduction

Combinatorics plays a very important role in the development of mathematical thinking. Its importance is primarily in the development of logical thinking and common combination skills. But it can also be considered as the basis for subsequent probabilistic problem solving.

In particular, it is a part of the high school (secondary school) curriculum, where it is restricted to classical problems of forming groups of subjects and determining the number of all groups that meet specific conditions. Usually, it is not a part of the elementary school curriculum, only in some math-focused schools their students get in touch with it.

Some elementary school students solve combinatorial problems already in the 5th grade when they take tests to get into a grammar school ("gymnázium"). Apart from the tests, they also solve them during mathematical competitions or while taking various quizzes. On the second level of elementary schools, students also solve combinatorial problems but only intuitively by using their common sense or by substituting random figures without using formulas or general combinatorial principles.

We focus on the development of combinatorial thinking on the basis of problem solving. The attention is paid to various solving strategies without the use of given combinatorial relations with progressive problems and building up the theory of combinations.

It is important to realize that combinatorics is tightly linked to other mathematical branches. We need to solve several problems dealing with combination of letters, numbers, etc. And that is why we believe that the development of combinatorial thinking is very important and should be initiated as soon as possible.

2. Basic concepts of combinatorics

In daily life we need to solve problems in which we compile certain groups of objects. We want to know the number of such groups and the order of particular elements. For example, an ice hockey coach who forms groups of players needs to form all possible groups plus to know who is going to play a certain position. Therefore the order of players in every group is very important.

When a school principal builds the schedule of classes, he does this it the same manner as the hockey coach forms groups of players. He forms groups of classes in which their order is important. Combinatorics can help us solve such problems. Combinatorics examines groups (subsets) of elements which are taken out of a certain basic group (set). Depending on whether the elements in the subsets may or may not be repeated, we differentiate several kinds of groups of elements – subsets with repetition and without repetition. We also recognize ordered and disordered subsets.

We choose k from n given elements of a finite set $N\,(k \in N, n \in N)$ of all natural numbers and form (dis)ordered k-tuples. To find all the possibilities we can apply basic combinatorial rules (the rule of sum and the rule of product), basic defined concepts (permutation, variation, and combination), or a list of all possible solutions (a table chart, a logical tree, etc.). We can also use some solving strategies based on the graph theory (graphical illustration of the problem, etc.).

Combinatorics can be also used in many other mathematical branches, particularly in algebra (the representation theory of groups), in the number theory and the game theory, in geometry, in topology, and mathematical analysis. Now we will focus on the application of basic combinatorial rules in the development of combinatorial ideas and students' solving strategies.

2.1. The rule of sum (addition principle)

The rule of sum is the first rule of combinatorics. We use it when we need to find all possible solutions of a problem or to set a rule while creating those solutions. It is good to divide the examined object (the given numbers, the number of contestants, etc.) into a few mutually disjoint groups $A_1, A_2, \ldots, A_k$ and look into each group/set separately. Let $A_1, A_2, \ldots, A_k$ be finite sets. If a set A_1 includes n_1 elements, a set A_2 includes n_2 elements,..., A_k includes n_k elements and if every two from these sets are disjoint $(A_i \cap A_j = \emptyset)$ for $i \neq j$; $i, j = 1, 2, \ldots, k$, then the number of all elements of the union of sets

$$A_1 \cup A_2 \cup \ldots \cup A_k = \bigcup_{i=1}^{k} A_i \tag{1}$$

is equal to

$$n_1 + n_2 + \ldots + n_k = \sum_{i=1}^{k} n_i. \tag{2}$$

At the end, we count the numbers of obtained solutions in each group.

2.2. The rule of product (multiplication principle)

The second combinatorial rule, the rule of product, is a bit more complicated than the first one. When forming groups of two elements, we often know how many ways are there to select the first and the second element, while the number of options of selecting the second element does not depend on the way the first element was selected. Let the first element can be selected by m ways and the second element by n ways. Then the couple of those elements (m, n) may be selected by $m \cdot n$ ways. This characteristic may be generalized for the selection of k-tuples. If a set A_1 consists of n_1 elements, a set A_2 of n_2 elements, and a set A_k of n_k elements, then the number of all possible ordered k-tuples is equal to

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k = \prod_{i=1}^{k} n_i \tag{3}$$

2.3. The concept of order and duplication of elements

Understanding the concept of a dis-/ordered k-tuple consisting of certain n elements with or without the possibility of repetition of the elements is very important. The correct comprehension may be further developed on the basis of problem solving without the use of combinatorial rules and principles.

For elementary school students, working with numbers is the most natural activity – formation of numbers according to given rules. While doing that, they practice terminology (a number, a digit, a single/double digit number, etc.). We can use tasks focusing on formation of numbers due to their divisibility, parity, the order of digits, etc.

We must not forget the students motivation – the way we set problems, their relevance and application in daily life, and the whole elaboration of them. There is a set of problems below in which there are some examples of problems. To solve them, basic combinatorial rules may be used.

3. Set of problems

U1 Form all possible double digit numbers by using the digits 1, 2, 3, 4. The digits must not be repeated.

U2 How many even natural numbers can be formed by using the digits given below? Each digit can be used only once.

a) 1, 2, 3, 4, 5

b) 0, 1, 2, 3, 5

U3 How many six-digit natural numbers can we get by using the digits 1 and 2 if no twos can be next to each other?

U4 There are five tickets available. There is the digit "1" written on three of them, the digit "2" on one of them, and the digit "3" on another one. How many five-digit numbers can we get using the tickets?

U5 Determine how many three-digit natural numbers are there:

a) if each digit in their decimal record occurs only once

b) if some digit in their decimal record occurs at least twice

U6 How many natural numbers bigger than 15 can be formed using the digits 1, 2, 3, 4, 5? Each digit can be used only once (must not be repeated).

U7 Vašek forgot his schoolmate's phone number. All he remembers is that it is a nine-digit number and starts with 23. No digits can be repeated and it is divisible by 25. Determine how many possible phone numbers are there?

When solving given problems, we focus on illustration. Pupils are given e.g. playing cards or cards with numbers and are led to systematically record all possibilities.

4. Examples of problem solving

U1

Students are given multi-colored cards with digits 1, 2, 3, and 4 on them. While they are forming pairs, they need to realize that the digits must not be repeated (i.e. no two pairs can have the same color, Figure 1 (right)).

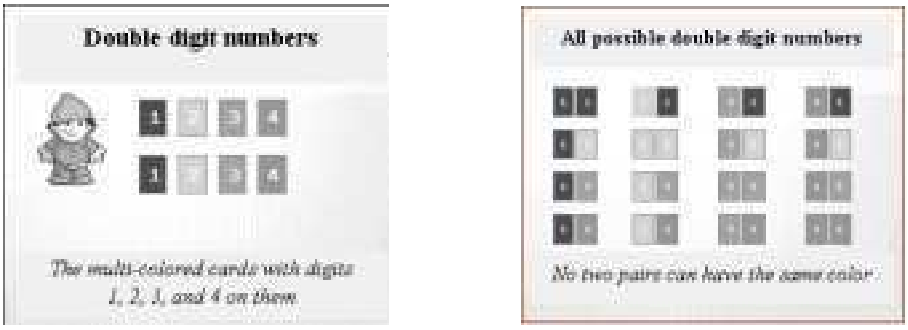

Figure 1: Double digit numbers – U1

Another possibility is to use a table diagram. This method is very well-arranged. In an arranged diagram a student crosses out numbers in which the same digit is used twice.

U5

In the following example, the emphasis is put on the clarification of the basic concepts which are used by teachers and students. At first, we read the task and then we take a closer look at it to prevent potential misunderstanding of the task. Everything is done illustratively to encourage the kids to ask questions. They then use logic to solve the task.

A teacher asks questions such as:

- How many digits does a three-digit number have?
- What does "the decimal notation" mean?
- What is in the first, second, and third place of a three-digit number?
- Read the given number and tell me how many units, tens, and hundreds it has.
- What is the smallest/largest three-digit number?

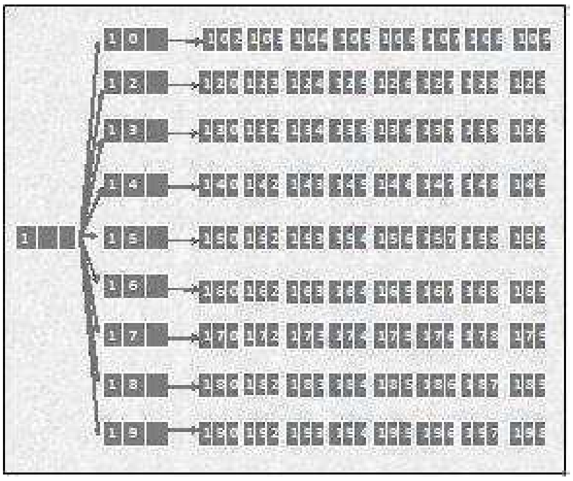

Figure 2: Logical tree – U5

<u>Note:</u> *We explain to the kids that between these double digit numbers there are all three-digit numbers and that we will take only these ones that meet the conditions of the task.*

The assignment: **Determine the number of all three-digit natural numbers such that in their decimal notation each digit appears only once.**

Procedure of problem solving:

- We determine how many different digits can be in the place of hundreds, tens, and units.
- We tell the kids that in the first place there must not be the digit 0 because if it was there, then it would not be a three-digit but a double digit number.
- In the second place there may be all the digits 0-9 except for the digit which is already in the first place. That way we meet the assignment that says that each digit can appear only once.

Another procedure:
We make a logic tree which will help the kids to find all possible solutions (Figure 2) and to understand the principle of the given task. We ask them questions, so they use logic to find solutions. We ask them questions such as:

- What digit can be in the first place?

- If the digit 1 is in the first place, what digits can be in the second place?
- Can the digit 1 be both in the first and in the second place? Why/why not?
- Can the digit 0 be in the first place? Why/why not?
- What digits can be in the third place if there is the digit 1 in the first place and the digit 2 in the second place?
- How many three-digit numbers which meet the conditions are there?

The kids must keep in mind that each digit can appear only once, i.e. it must not be repeated. In the picture (Figure 2), possible steps how to record the creation of numbers are shown.

The kids will notice that there are 9 possibilities of the record of the first place of a three-digit number (digits 1–9). There are also 9 possibilities in the second place (digits 0–9 except for the digit which is already in the first place, i.e. $10 - 1 = 9$). And in the third place, there are 8 possibilities of the record (digits 0–9 except for the digits in the first and in the second place, i.e. $10 - 2 = 8$). In every chain oriented by the first digit, there is $9 \cdot 8 = 72$ possible solutions. In the end they will count the total of all solutions which is $9 \cdot 72 = 648$ solutions.

U7

In this last example of problem solving, we will pay attention to students' motivation – there is a real situation which they are familiar with. The use of combinatorial rules and principles is fairly obvious. As another element of motivation, a colored picture is used. Different possibilities are in different colors (Figure 3).

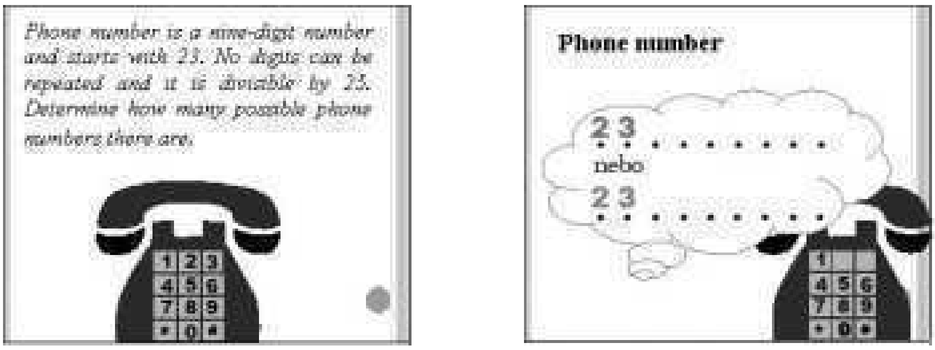

Figure 3: Phone number – U7

5. Conclusion

In the contribution, a possible procedure how to develop combinatorial thinking of primary school students and to build their knowledge structure without the understanding of basic combinatorics is outlined. The emphasis is put on the development of solving strategies, creative approach of a teacher and a student, motivation, and practical application of gained knowledge.

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EXAMPLES OF USING NEW INTERACTIVE TECHNOLOGIES FOR FILLING THE GAPS IN STUDENTS KNOWLEDGE

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Abstract. Many changes in the curriculum of mathematics have been made during a few last years. Frequent changes carry certain consequences. Pupils which finish high school may possess different knowledge in different years. This is because they may have various ranges of educational material (different from those of their older or younger colleagues). Additionally, some parts of school mathematical material are more difficult to learn than others. Unfortunately, sometimes teachers tend to treat them cursorily. The planimetry is such a specific field of mathematics which requires specific thinking and analysis. It is necessary to reduce such differences in knowledge and skills, to supplement lacks of knowledge of students of the first year mathematics study. It is necessary to use the suitable tools to do this quickly and effective. Utilization of interactive GeoGebra based simulations and visualizations may be helpful in such a situation. Perfect co-operation with the interactive whiteboard and possibility of delivering didactic materials by Internet are their additional advantage.

We will show examples of such materials relating to similarities in our presentation. They are a part of developing project – the course of geometrical transformations on the plane. It is addressed to students of the first year of mathematics study. However, these materials can be used at high school level during additional activities according to pupils interests.

1. Introduction

During several recent years we observe many changes made in Polish curriculum of mathematics. Frequent changes bring certain consequences. Pupils finishing the secondary school next years can possess different knowledge. Pupils finishing high school in successive years can possess different knowledge, because they are obliged by somewhat different range of the material (different from the range of their older or younger colleagues) [1]. Teachers at high schools at various stage realize particular portions of the material depending on the various quantity of hours. On the other hand, the beginning of studies at university is a serious challenge for a young man. Appearing in a completely new environment, coming across a new system of work and education, operating in new realities with different requirements – this is the problem for high school student at the beginning of the studies.

Universities try to undertake various actions which ought to make easier student's entry in the new reality and to help match new requirements. They have also noticed the program differences and very often organized equalizing lessons for new students. Such help is organized using various ways, forms and knowledge depending on needs and possibility. Focusing attention on technological development of society, accessibility to various sources of knowledge exchanged by Internet etc., we should use available resources for such lessons. in this situation, using of suitable ICT tools can support the process of learning, improve it, can make problems, terms and questions more accessible for students. Such a didactic material can be partly realized in traditional way, during study at the university, and partly by Internet, e.g. by e-learning platform. It certainty allows to save time and finance, and also it makes possible individualization of the process of learning. The project, which started up in 2002 under direction of dr. T. Ratusiński in Institute of Mathematics at Pedagogical University of Cracow, is an example of such a material [2]. It is an answer to the abovementioned lacks, a trial of solution of the appeared problem.

2. Course of geometry

A course of elementary geometry is one of subjects realized in first year of studies since years. It is perceived as very useful for young students starting not easy mathematical studies. The subject area and the way of realization of its curriculum can help students to understand what higher mathematics is, and also to become acquainted with its special language. In the curriculum there is a large part of material concentrated on transformations (e.g., the properties of isometry, similarity of figures or geometrical constructions).

Young students beginning studies at a high school should fill lacks for the short time. In view of the range of changes in the curriculum, this is a very serious problem. That is why the e-learning course of elementary geometry was created as a supplement to studies realized in traditional way (Figure 1).[1]

Figure 1. The main page of the course.

Contents of teaching were organized in thematic blocks. The respective parts of the course are concentrated on concrete transformations, for example, such as main isometries, central symmetry, axial symmetry, translation symmetry, translation, rotation, similitude, similitarity, conchoid transformation, the power of a point, inversion, some kind of "glued" transformation and many others.

All units consistituting the whole course are closely related with each other. Materials relating to respective transformations contain links to essential contents at the actual stage. All pages of the whole project have standardized graphic and cardinal components and navigation. Students often have problems with understanding the idea of a new acquainting term. Working with abstract terms, it is often hard to assimilate them or imagine a suitable model. That is why the principal part of presented materials include theoretical knowledge which students ought to assimilate during study.

[1] The legends in Figures being screen images are in Polish.

Students can become familiar with terms, their definitions and properties in easy way thanks to utilization of interactive constructions made with DGS (Dynamic Geometry Systems) like GeoGebra (Figure 2). The huge advantage of such a solution is the possibility of their multi-regeneration. Utilization of interactive constructions helps to analyze deeper mathematics problems step by step. Students can also modify geometrical objects used in the construction and observe positions or others properties of constructed figures. It is possible, for example, to discover terms, to observe properties of figures, to notice how a figure changes in transformation depending on its shape or initial bearing. Such an approach makes it possible to look at the whole contents and also facilitates investigation of special cases.

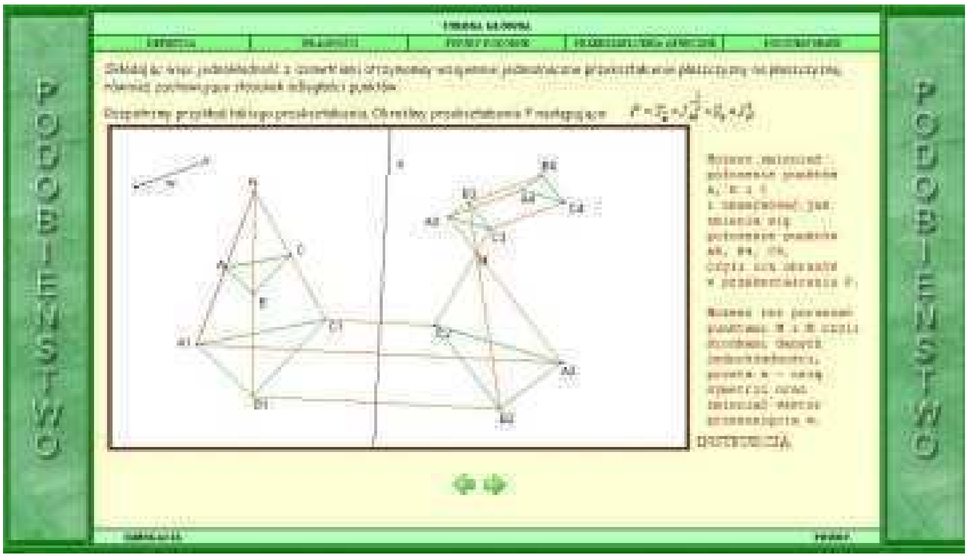

Figure 2. Interactive definition of similarity.

Proofs are essential complements to the theoretical part of the course. Applying the formal notation, saving logical sequence of reasoning, the proofs are next activities which are not easy for students. In the course, proofs are often introduced in interactive way.

Students can analyze the presented notation and observe illustrating figures in the part of materials. Sometimes a proof is important to learn itself, and in such a situation interactive didactic material can manage students to became an active author of them (Figure 3). For example, the text of the proof is displayed on a screen step by step and contains certain gaps which should be filled in with correct statements. The only way to pass each step of proof is to choose the correct answer. Interesting solutions used in the course are tips which appear after incorrect answers. They ought to help students to correct their mistakes. In such a way a computer becomes a helper in the process of getting knowledge, and a student is not only a reproducer of skilled theory, but becomes its creator.

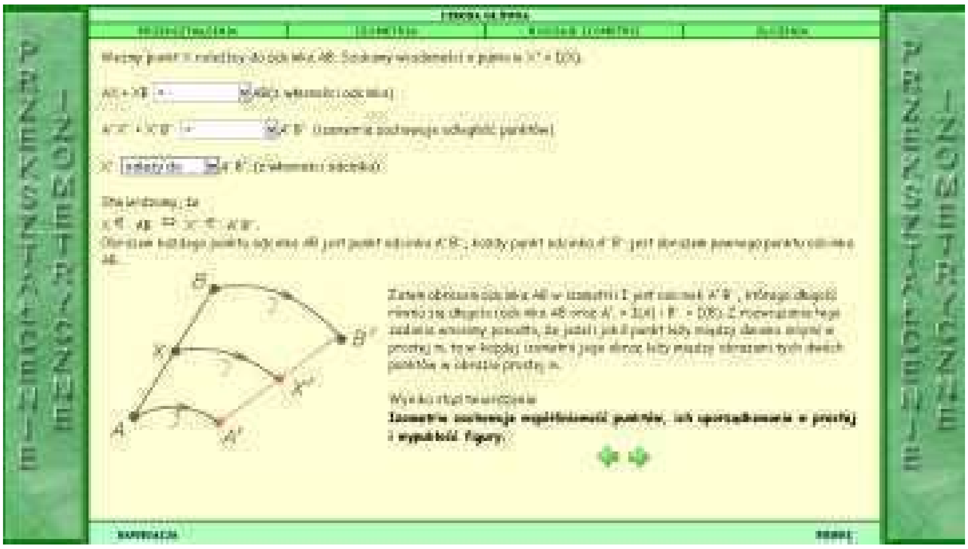

Figure 3. Example of short active proof.

The course also contains various types of tasks and problems. Using the course, solving the problems, students can evaluate not only their knowledge and acquired skills, but also the stage of understanding the properties of transformations. A part of problems should be conceived so that their solution allows us to make certain observations, notice properties and make correct notations. There are many types of tests, for example, a single and multiple choice, a task to make opinion about truth of a statement (Figure 4), a problem of the type TRUTH/FALSE, a complementing text with gaps and others.

Every time when students display each page of the course, all the problems are "new", they regenerate. This allows to avoid mechanical approach to solved problems. Order of problems or answers changes each time automatically (numerical data also change). Each time a computer shows only a part (chosen problems) from the large base of questions and tasks. Additionally, the course has lists of thematic tasks in text files which students can use for independent repeating the composed material.

The huge advantage of material prepared in such a way is individualization of learning process. A student himself decides how he can work with the course. The accessibility of material, the way and form of work with it allows students to decide how much they are engaged and how many time they spend on it. Students can pass the chosen part of materials many times for repetition or for supplement knowledge.

Figure 4. Example of self evaluation task.

Some fragments of the course also contain additional tips and information which students can display if they recognize that they need them. There are also some curiosity facts in the course such as, for example, physical look at mathematical transformations, materials relating to utilization of geometry in physics, and also some popular mathematical theorems and figures of their discoverers.

3. Conclusions

The described course can be used by e-learning platform or by Internet website (what is more and more popular solutions). Such an approach to process of teaching / learning based on traditional method supported by online materials (blended learning) allows us to approach mathematics in the functional way. The bases of this conception were founded by A.Z. Krygowska, and she said that it is unusually essential conception because "*the efficient character of mathematics comes out in the pupil mathematical language in every situation*" [3].

The described conception concentrate on solving problems through executing consciously actions. Pupils can name and rank them. The search of the solution can be active: the use of the method of tests and mistakes, or through utilization of already known schemes. The use of such e-learning course allows us to pass every stage of the solution step by step. A teacher is able to organize sensible problematic situations and lead pupils from concrete actions through conceivable ones to mathematical abstractions.

The course based on blended learning method has one more advantage – the phenomenon of social facilitation [4]. It consists in such a fact that new material is assimilated more easily without observers. The stress does not generate, a pupil does not act under the pressure. He works in the own pace, in conditions convenient for him. A teacher ought to turn the attention at this fact when planning work for students. The first contact with some new problems can be held by e-learning platform.

The described didactic material was prepared, used and verified in classes with students of the first years of mathematics studies during several last years. The presented course turned out be also helpful for pupils of high schools interested in mathematics. Utilization of units of the course is possible as the whole course and as respective fragments. The course is also perfectly suitable as a material for use at extended lessons at high school (during additional activities) as well as a material for filling lacks of students knowledge at the beginning of mathematics studies.

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EXAMPLES OF REASONS OF USING ICT IN CLASSROOM FOR CREATIVE PUPILS ACTIVITY

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Abstract. We can create many activities when we place geometrical problems to solve in class. For example: analyzing questions, making observation, discovering property of geometrical objects, searching the dependences among them, making hypotheses, making argumentation, proving, investigation of special cases and many others. Using traditional didactic resources causes that the abovementioned activities may be difficult for pupils at every level of education. It is also hard to explore the considered problems. That is why it is legitimate to search how to use ICT to support pupils in this process. Using GeoGebra based materials is an interesting idea for this purpose. Well made interactive materials can be used during normal lessons as well as during additional activities. Such a solution may increase understanding of the problem, and also allows a teacher to discuss with pupils extracurricular mathematics contents.

1. Introduction

The process of teaching mathematics is very difficult one. However, when we look at it globally, we can distinguish, with a considerable simplification, four components [1]:

- forming mathematical terms,
- mathematical reasoning,
- task solving,
- establishing mathematical language.

The main aim of this paper is not to show examples of using ICT at lessons of mathematics, but to show reasons of such utilization in almost each component. Our research is based on the course of geometrical properties of triangles. It consist of interactive GeoGebra based didactic materials. The investigation took place in the 1st and 2nd classes of high school (64 pupils being 16-18 years old). The pupils who approached investigation are from university classes with mathematical - physical - computer science profile. During everyday work they apply ICT in the wide meaning of this word. Utilization of supplies of e-learning platform as well as Internet or graphic calculators and computer programmes for these pupils are as natural as in traditional classes handiwork of auxiliary figure on sheet of paper. So as the figure is not the proof for pupils in the typical class, so for these classes simulations and inspecting tens of examples for given problem is not sufficient. However, they use GeoGebra for realization of illustration, for visualization their hypotheses willingly. Let us see what kind of didactic material they work on preferably.

2. Interactive definition

The interactive material made in GeoGebra programme is able to show an object of the definition in any but fixed instance and also to illustrate individual properties of this term. Such a help leads pupil across the term in dynamic way. Suitably prepared didactic material can turn pupils' attention to all conditions of the definition in the visual way. The traditional definition causes that very often pupils have large problems in retrieval of the essence of the definition. As a result, it is hard to them to show an example, instance or contrary instance for defined object. They have also problems to decide whether the given object fulfils the definition or not. Our interactive definition turns pupils special attention to necessity of fulfilment of all conditions described in definiens (so that definiendum sets) (Figure 1).

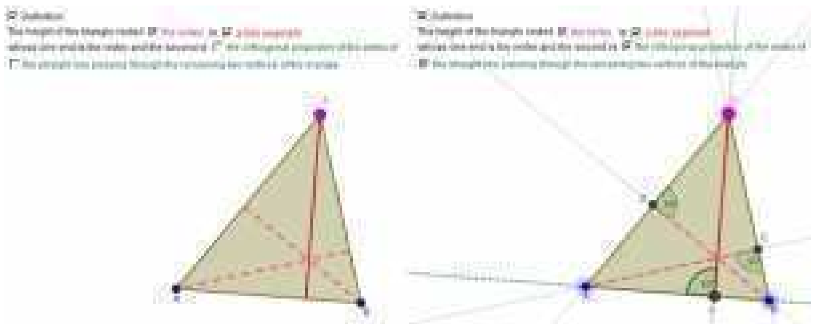

Figure 1. Dynamic definition of the height of a triangular.

With such interactive definition, pupils can easily observe what is essential in the definition. Not only colors (which can be used in the traditional definition) takes important part in this material. The possibility of testing any combination of conditions, which the defined object has to fulfil, is essential for this didactic help. The "transparent clear figure" describing the object, on which elements (properties) chosen by a pupil appear, allows for more solid understanding of the definition. Such a form of the performance definiendum facilitates deciding whether the given mathematical object is an instance, contrary instance or extreme case of the defined object. The possibility of discovering a property of terms directly appearing from the definition is another important aspect of this material. The example of observation of such a dependence is the fact that the midperpendicular marks the midpoint of the segment as the central symmetry figure.

3. Discovering properties of terms and formulating hypotheses

The observation is the most frequent aspect of ICT used by teachers at mathematics lessons. Simulations provoke formulating hypotheses and can help with their verification. Pupils during the work with presented materials also formulate hypotheses which they prove during the learning of new material. Pupils encounter the possibility of observing property of mathematical object and then formulate hypothesis (for example, when studying midperpendicular). The aim of pupils' investigation is to explore that points from midperpendicular are evenly distant from both ends of the segment (Figure 2).

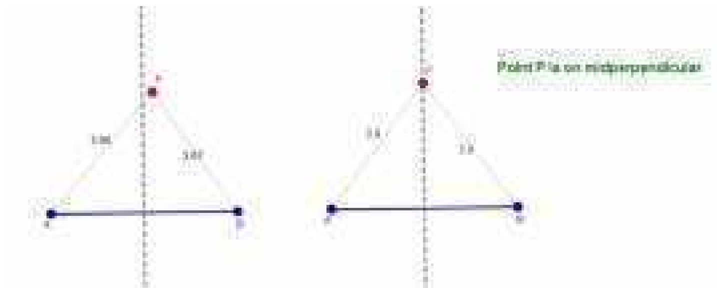

Figure 2. Simulation provokes formulating hypotheses.

During investigation of properties of mathematical objects, the main aim is not to concentrate on the possibility of observation of very wide spectrum of examples, which lead to deep visualization and assurance of the noticed hypothesis, but to prove it in the formal way. GeoGebra can help to create such a tool which will lead pupils across entire formal process of prove.

4. Proof (proving of discovered properties)

The traditional proof represents entire progress of reasoning at ones (as you can see in Figure 3).

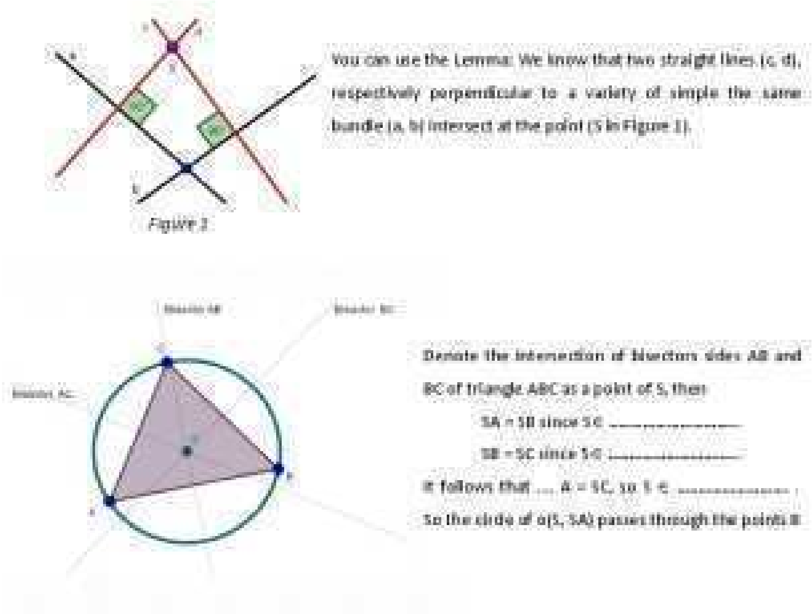

Figure 3. Example of "traditional proof" card to supply.

In such tasks pupils focus on supplement of lacking symbols, often making this "thoughtlessly" because, e.g. one of the steps is inexplicable (very often). We can also present traditional proofs using the interactive help made with GeoGebra (Figure 4).

The active card presenting the proof leads pupils by certain deductive progress of reasoning step by step, illustrating the geometrical interpretation on dynamic figure simultaneously. Pupils cannot see the entire proof at once, so it does not distract their attention. They can focus on the essential elements of the proof at the moment. Pupils have to pass all the steps (earlier stages) to see the last one. Such a dynamic proof can block next steps if there is, for example, inappropriate position of one of mathematical objects (Figure 5).

The next difficulty is the fact that the majority of proofs in mathematics begins from words, e.g.: Let ABC be any fixed triangle ... This formulation "any fixed" very often causes (even for students) feeling of internal contradiction. We establish any case for which we guide the proof (in normal proofs). Establishing any mathematical object, we consider the concrete instance but without concrete properties, except for those from the foundations. Fixed,

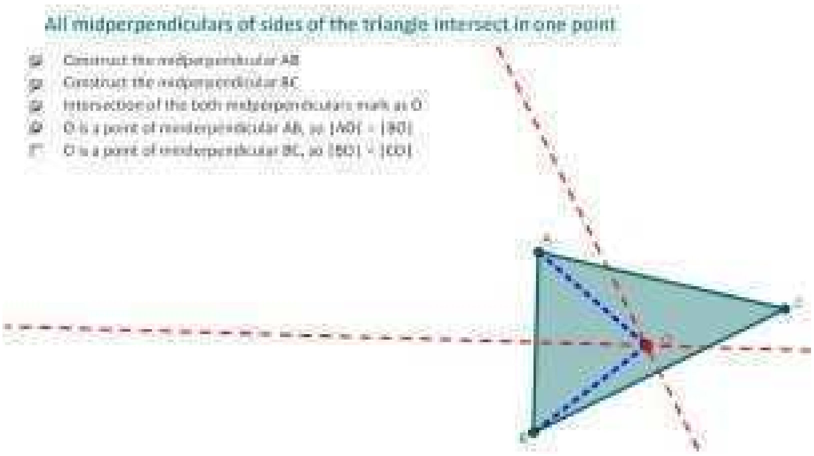

Figure 4. Example of GeoGebra's "interactive proof".

because we have to guarantee invariability of all the parameters during the whole proof (nothing can be changed during the proof). For example, we cannot accept the position of the point A at one moment and then alter its position in the midst of the proof. This will cause so many changesin objects depending on the point A that you really should begin the proof from the beginning. This is a difficult element of the proof even for students. It is inexplicable why the proof is correct for every (any) object but the proof is guided only for the fixed one. Pupils often feel the need of verifying the theorem for other cases (special cases) even if they are convinced of the truth of the theorem. Using the dynamic card made in GeoGebra, it makes possible the proof appears step by step and is equipped by suitable comments. It allows to turn attention on essentials of the given dynamic figure. In GeoGebra a figure is always dynamic, and it can be changed at any time without losing the idea of the proof. Pupils can test (during the proof) how the proof will behave for the special cases (e.g. for rectangular triangle or equilateral one). Such an approach allows schoolchildren to avoid fear that the proof is guided for one fixed (concrete) triangle, but not for all ones with the given properties. GeoGebra can help teaching argumentations and proving in such a way.

5. Tasks and mathematics problem solving

Pupils during their mathematical education meet various types of tasks. Teachers spend few time for constructional problems because of long time for their realization and often because of manual pupils skill (mainly during the construction itself). A pupil creates the range of mathematics through the prism of solved problems [2, p. 3], so it is important to dedicate them a little more

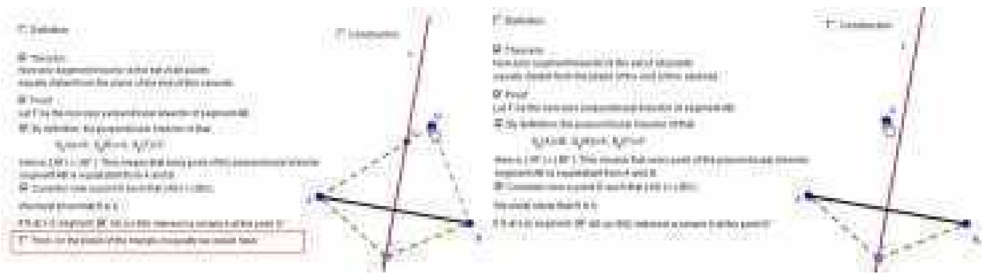

Figure 5. Example of GeoGebra's "interactive proof".

time. Pupils get to know what is mathematics? and what can it be? when they solve actively and suitably the well-chosen mathematical problems [2]. GeoGebra can be a generator of some tasks for practice, for testing, verifying and evaluating the assimilated material. We will show such an example of the constructional problem.

The solution of a constructional problem consists in the following stages: analysis of construction, description of construction, proof of correctness, discussion of existence of solution and the numbers of solutions, possibility of realization, construction [3, p. 78]. While reminding pupils the terms necessary to expand their knowledge of geometry of triangle using application representing the definition of mathematical objects, they replenish knowledge with construction of introduced definiendum. The construction can be described with dynamic commentary which will turn pupils' attention at stages of construction or any possible kind of simulations.

Struggling with the constructional problem, pupils first have to pass its analysis. In "traditional conditions" we find the existence of mathematical object described in content of the problem and approach to realization of auxiliary figure. Such a help is fixed for one concrete case, while the computer programme allows us to change quickly the initial foundations and to observe the consequences (changes in objects related with them). GeoGebra can show dynamic figures and allows to verify observations related with the positions of objects depending from each other. It is more easily to choose the proper necessary statements for proving hypothesis in this way. The active figure allows to verify quickly whether the chosen statement, which we want to use in a proof, is correct or not for actual situation. During change of initial points (or others properties of a figure) the program changes objects related with it automatically, but it never changes mathematical rules (based on the objects dependence and mathematics statements). Let us look at an example – the problem of the heights in a triangle.

Pupils usually identify the height of a triangle as the segment which is perpendicular to the base (at the "bottom" of the triangle) having the end at the opposite apex. When we make such a construction in an acute triangle, we usually do not perceivethe problem. The remaining heights give rise to the problem when a triangle is rectangular or obtuse.

When we make wrong construction of the perpendicular to a segment (but not to a line enclosed the base), the program displays that in this situation an obtuse triangle does not have three heights, but only one (in Figure 6 the segments marked as dotted just disappear). This fact should induce pupils to reflection and search of the essence of the mistake.

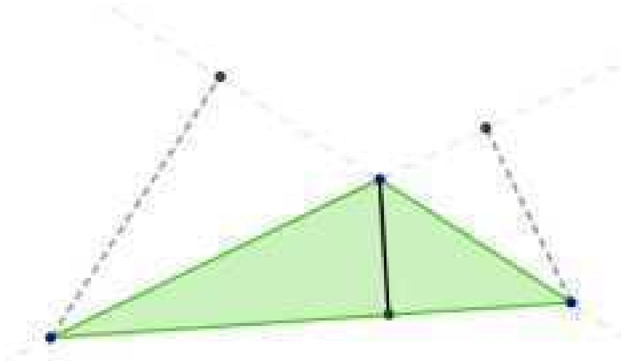

Figure 6. Heights in an obtuse triangle.

The description of construction is the next step of solving a constructional task. Usually pupils make many mistakes at this stage. GeoGebra based didactic material can also help with it. Let us look at another example – the circumcircle and the incircle of a triangle.

Teachers teaching circumscribed or inscribed polygons begin from the subject of the circumcircle and the incircle of a triangle. Very often pupils simplify both problems to a question of finding the position of the centers of the sought circles. But such an approach to the task can cause next mistakes.

At first construction pupils find the point of intersection of midperpendiculars of sides of a triangle and then plot the circumscribed circle. The construction is correct, because it suffices to guarantee that the circle crosses through vertices of a triangle (it is not necessary to know the radius of the circle). It is harder in the second situation, but pupils do the same. They find the point of intersection of interior angle bisectors of a triangle and try to plot the incircle immediately. The new problem appears because there is no point which the sought circle must enclose. Usually they forget to construct the tangency point. Our investigation shows that pupils often construct the incircle by:

- intersection the interior angle bisector and opposite side of a triangle;
- a point not belonging to a triangle, just to make the circle inside the triangle;
- a point belonging to the side of a triangle (Figure 7).

When we have an equilateral acute triangle, it seems to be the correct construction. However, when we change the position of vertices of a triangle, we will see our mistake. If we would proceed this wrong procedure working with a sheet of paper, we get the "correct looking" incircle in the incorrect way. We would not have consciousness that the solution of the constructional task had serious lacks.

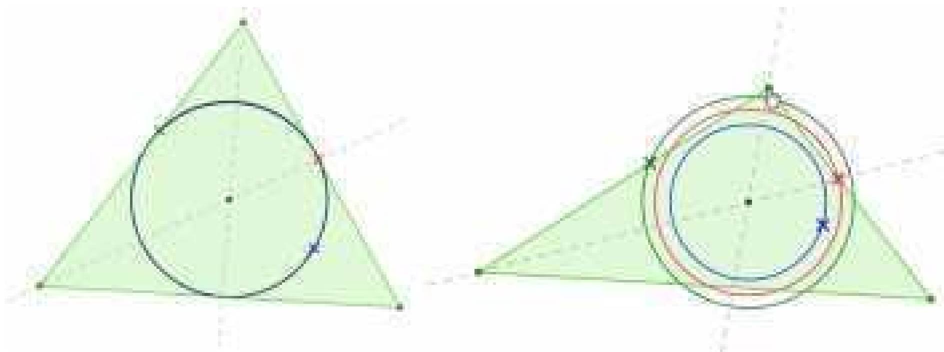

Figure 7. "Incircles" in a triangle.

GeoGebra (like other DGS programs) is ruthless and unfeeling in such situations. It shows all the shortcomings in the construction, informs about all the inaccuracies. Such an experience is particularly valuable for future teachers who will teach such mathematics which they know themselves. Next essential aspect of using DGS programs is the possibility to analyze the protocol of the construction (Figure 8).

Every object from description of the construction has exact connection with the object from protocol of the construction. Every mathematical object is registered in the protocol in such a way that it shows all the connections with other objects. Next steps of solution consist in the search of answers to questions relating quantities of solutions and feasibility of the construction. The properties of GeoGebra described above help in detecting conditions for both of them. A teacher should prepare suitably the didactic material to make it easy.

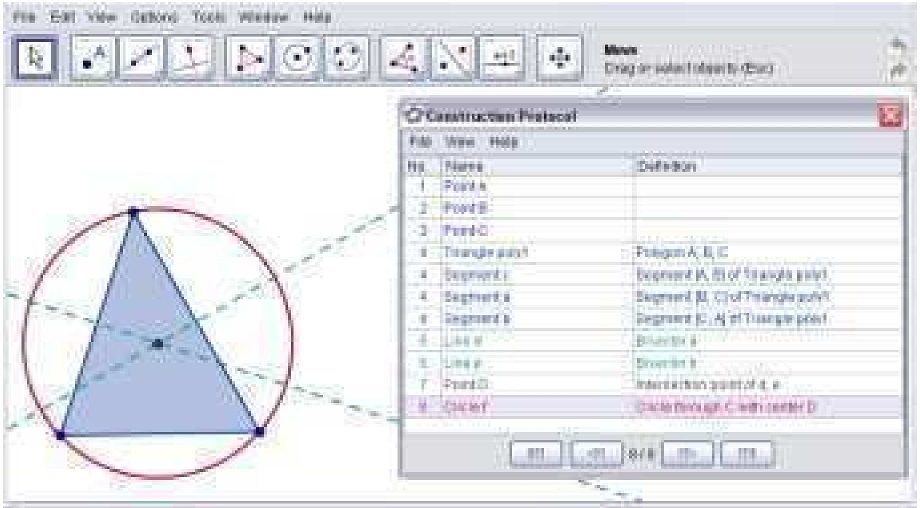

Figure 8. Protocol of the construction.

GeoGebra makes possible the limitation of access available tools (removing temporarily needless icons from interface). The program also allows us to create new tools more adapted to the concrete situation. Simulations with such a program also make easier proving correctness of solution that is the last element of constructional task. We teach perception of properties of mathematical objects and dependence between them with help of ICT in such a way.

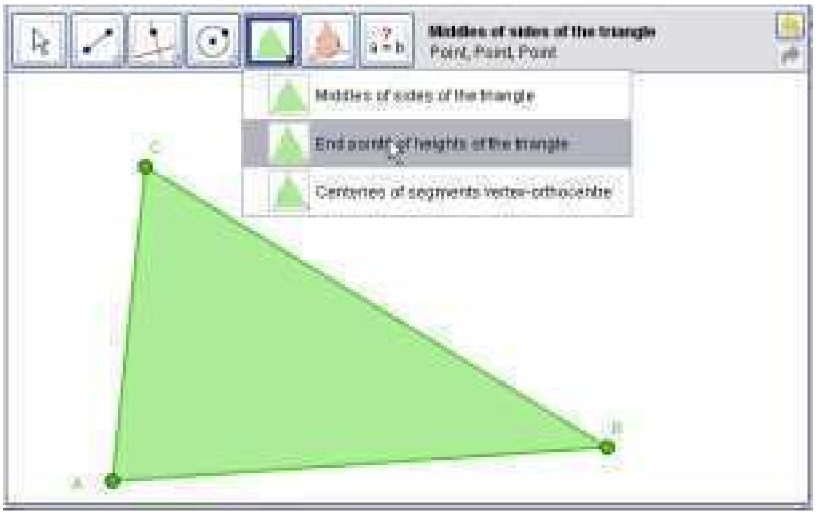

Figure 9. New interface of GeoGebra.

6. Conclusions

Our investigation showed that interactive didactic materials can be successful use for:

- illustration of new mathematical terms considering the necessary and sufficient conditions (for better understanding of definition);
- observation of properties of mathematical objects (to expand and intensify mathematics knowledge and to make reasoning);
- formulation of the noticed hypotheses and their verification (to teach proving);
- solution of tasks and mathematical problems (to make mathematics reasoning).

Didactic material based on Interactive GeoGebra can shape mathematical terms while formulating definitions and during the simulations. It can lead pupils via mathematical reasoning or provoke them to solve mathematics problems during simulation, proofs and constructions. Introduced materials interlace all the components of the process of teaching mathematics. Preparation of such applications is very time-consuming, but well prepared materials represent valid mathematical questions in the alternative, creative way for pupils. This kind of didactic material (as ours investigations showed) obtained the recognition of students because it enlarges their chances of understanding mathematics.

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DIDACTIC REMARKS ON THE POWER SET

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Abstract. The paper is devoted to correct understanding of the notation for the power set. Often this notation is mistaken with a power of the number 2. The correct definition of the power set is presented as well as several task which can serve for strengthening the understanding of this notion.

1. Introduction

In the paper [3] the author paid attention to students' erroneous understanding of the notation 2^X. For example, there appear the following understanding of this notation:

$$2^N = \{2^0, 2^1, 2^2, \ldots, 2^n, \ldots\},$$

$$2^N = \{\{2^0\}, \{2^1\}, \{2^2\}, \ldots, \{2^n\}, \ldots\},$$

if $X = \{a_1, \ldots, a_n\}$, then $2^X = \{\{2^{a_1}\}, \ldots, \{2^{a_n}\}\}$.

It seems to us that these mistakes can be easily rectified referring to the following definition:

Definition 1.

$$2^X = \{Y : Y \subseteq X\}.$$

In this case there is no doubt that it is necessary to determine a set (or, more precisely, a family) of all the subsets of the set X.

It is obvious that

$$2^{\emptyset} = \{Y : Y \subseteq \emptyset\} = \{\emptyset\},$$
$$2^{\{a\}} = \{Y : Y \subseteq \{a\}\} = \{\emptyset, \{a\}\},$$
$$2^{\{a,b\}} = \{Y : Y \subseteq \{a, b\}\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

A family $2^{\{a,b\}}$, i.e. a family $\{Y : Y \subseteq 2^{\{a,b\}}\}$, consists of 16 sets.

It should be noted that appearance of the number 2 in the notation 2^X can suggest to somebody that this notation concerns the power of the number 2 or some set of powers of the number 2.

Appearance of the number 2 can be clarified referring to the notion of the set of transformations of one set into another or to the notion of the characteristic function of a given set as well as to the notion of set power or set cardinal number.

A set of all transformations of a set X into a set Y is denoted by the symbol Y^X. The following notations are equivalent:

$$f \in Y^X, \quad f : X \to Y, \quad X \xrightarrow[f]{} Y.$$

If V denotes a universe, then the characteristic function f_Z of a set Z $(Z \subseteq V))$ is defined as follows:

Definition 2.

$$f_Z(a) = \begin{cases} 1 & \text{if} \quad a \in Z, \\ 0 & \text{if} \quad a \in V - Z. \end{cases}$$

From the power theory (the theory of cardinal numbers) it is known [4] that if $\overline{\overline{X}} = m$, $\overline{\overline{Y}} = n$, where m and n are arbitrary cardinal numbers, then

$$\overline{\overline{Y^X}} = (\overline{\overline{Y}})^{\overline{\overline{X}}} = n^m.$$

If X is an arbitrary set and $Y = \{0, 1\}$, then the power set 2^X is equipotent to the set $\{0, 1\}^X$. Therefore

$$\overline{\overline{2^X}} = 2^{\overline{\overline{X}}} = \left(\left\{\overline{\overline{0, 1}}\right\}\right)^{\overline{\overline{X}}} = \overline{\overline{\{0, 1\}^X}}.$$

If we suppose that $V = X$, then the set $\{0, 1\}^X$ is a set of all the characteristic functions of a set Z contained in X.

Example. If $X = \{a, b\}$, $Y = \{0, 1\}$, then $Y^X = \{0, 1\}^{\{a,b\}} = \{f_1, f_2, f_3, f_4\}$, where

$$\left.\begin{array}{l} f_1(x) = 0 \\ f_2(x) = 1 \end{array}\right\} \text{ for arbitrary } x \in \{a, b\},$$

$$f_3(x) = \begin{cases} 1 & \text{if} \quad x = a, \\ 0 & \text{if} \quad x = b, \end{cases}$$

$$f_4(x) = \begin{cases} 0 & \text{if} \quad x = a, \\ 1 & \text{if} \quad x = b. \end{cases}$$

Then $\overline{\overline{Y^X}} = \overline{\overline{\{0,1\}^{\{a,b\}}}} = \left(\{\overline{\overline{0,1}}\}\right)^{\overline{\overline{\{a,b\}}}} = 2^2 = 4^{\overline{\overline{X}}}$.

The functions f_1, f_2, f_3, f_4 are the characteristic functions of the sets $\emptyset$, $\{a,b\}$, $\{a\}$, $\{b\}$, respectively.

Suppose that a set X is finite and contains n elements ($\overline{\overline{X}} = n$). Calculating the number of all the subsets of a set X, we obtain:

$\binom{n}{0}$ – the number of empty sets $\left(\binom{n}{0} = 1\right)$,

$\binom{n}{1}$ – the number of one-element sets,

$\binom{n}{2}$ – the number of two-element sets,

....

$\binom{n}{n}$ – the number of n-element sets $\left(\binom{n}{n} = 1\right)$.

From the Newton formula $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for $x = 1$, $y = 1$ we get $\sum\limits_{k=0}^{n} \binom{n}{k} = 2^n$. Therefore, the number $2^{\overline{\overline{X}}}$ is equal to the number of all the subsets of a set X.

It can be assumed that some troubles related to set theory appear if we speak about family of sets or about family of family of sets. This is connected with the fact that in the consciousness of many people there are two conceptions of a set [1, 2]:

– in distributive sense (set-theoretic),

– in collective sense (mereological).

For example, in distributive understanding of the notion of a set a one-elemnt set is identified with its element: $(\{a\} = a,\ \{\{a\}\} = \{a\})$.

Sometimes, the features of operations on numbers are extended to set-theory operations. For example, there appear the following equalities:

$$2^A \cap 2^B = 2^{A \cup B}, \quad (A \cup B) - B = A, \quad B \cup (A - B) = A,$$

but the first of these equalities is valid when $A = B$, the second one when $A \cap B = \emptyset$, and the third when $B \subseteq A$.

The task on finding the following sets can serve for strengthening the understanding of the notion of power set:

1. $\inf(2^X, 2^Y)$, $\sup(2^X, 2^Y)$,
2. $2^X \cap 2^Y$, $2^X \cup 2^Y$, $2^X - 2^Y$, $2^X \dot{-} 2^Y$,
   where $X = \{a, b, c\}$, $Y = \{b, c, d\}$, and " $\dot{-}$ " is the operation of symmetric difference: $(A \dot{-} B = (A - B) \cup (B - A))$,
3. $2^{\{a\}} - \{a\}$, $\quad 2^{2^{\{a\}}} - 2^{\{a\}}$,
4. $2^X \cap 2^{V-X}$, where $V = \{a_1, a_2, \ldots, a_n\}$, $X = \{a_1, a_2, \ldots, a_{n-1}\}$.

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ARITHMETIC SEQUENCES OF HIGHER DEGREES CHARACTERIZING FIGURATE NUMBERS

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Abstract. Figurate numbers have simple geometric illustration: polygonal numbers can be represented by polygons, pyramidal numbers by pyramids, prismatic numbers by prisms, and trapezoidal numbers by trapezoids. The numbers mentioned above can be defined by formulae[1] or can be characterized by some arithmetic sequences of higher degrees which allow to obtain the corresponding formulae [3]. Figurate numbers due to their geometrical illustration and intersting properties can be of interest for school pupils.

1. Arithmetic sequences of higher degrees

For arbitrary sequence $\{a_n\} : a_1, a_2, a_3, \ldots$ we can calculate the sequence of the first finite differences $\{\Delta^1 a_n\}$

$$\Delta^1 a_1 = a_2 - a_1, \quad \Delta^1 a_2 = a_3 - a_2, \quad \Delta^1 a_3 = a_4 - a_3, \ldots \tag{1}$$

and the sequences of successive differences

$$\Delta^{k+1} a_i = \Delta^k a_{i+1} - \Delta^k a_i, \quad i = 1, 2, 3, \ldots \quad k = 1, 2, 3, \ldots \tag{2}$$

Using the method of complete mathematical induction it can be proved that an arbitrary term of a sequence $\{a_n\}$ can be described by the following formula (see [1, 4]):

[1] This method was used by W. Sierpiński in the book [5] in definitions of triangle and tetrahedral numbers.

$$a_n = \binom{n-1}{0} a_1 + \binom{n-1}{1} \Delta^1 a_1 + \binom{n-1}{2} \Delta^2 a_1 + \ldots + \binom{n-1}{n-1} \Delta^{n-1} a_1. \quad (3)$$

It is evident that to define the nth term of a sequence $\{a_n\}$ it is sufficient to (know) have the first term and the differences: $\Delta^1 a_1,\ \Delta^2 a_1,\ \ldots,\ \Delta^{n-1} a_1$.

A sequence $\{a_n\}$ is called *an arithmetic sequence of the degree m* ($m = 1, 2, \ldots$) if and only if the sequence $\{\Delta^m a_n\}$ is constant and differs from the zero sequence. A constant sequence will be called an arithmetic sequence of the zero degree.

It follows from Eq. (3) that an arbitrary term of an arithmetic sequence of the degree m is expressed as:

$$a_n = \binom{n-1}{0} a_1 + \binom{n-1}{1} \Delta^1 a_1 + \binom{n-1}{2} \Delta^2 a_1 + \ldots + \binom{n-1}{m} \Delta^m a_1. \quad (4)$$

To determine the differences $\Delta^1 a_1,\ \Delta^2 a_1, \ldots, \Delta^m a_1$ we draw up the following table

$$\begin{array}{lllll}
a_1 & & & & \\
 & \Delta^1 a_1 & & & \\
a_2 & & \Delta^2 a_1 & & \\
 & \Delta^1 a_2 & & \Delta^3 a_1 & \\
a_3 & & \Delta^2 a_2 & & \ddots \\
 & \Delta^1 a_3 & & \Delta^3 a_2 & \\
a_4 & & \Delta^2 a_3 & & \\
 & \Delta^1 a_4 & \vdots & \vdots & \\
a_5 & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots &
\end{array} \quad (5)$$

2. Polygonal numbers

For sequences with general terms

(a) $a_n = n$,

(b) $a_n = 2n - 1$,

(c) $a_n = 3n - 2$,

(d) $a_n = 4n - 3$,

(e) $a_n = 5n - 4$,

$\dots\ \dots\ \dots\ \dots$

(f) $a_n = (s-2)n - (s-3), \qquad s = 3, 4, 5, \dots$

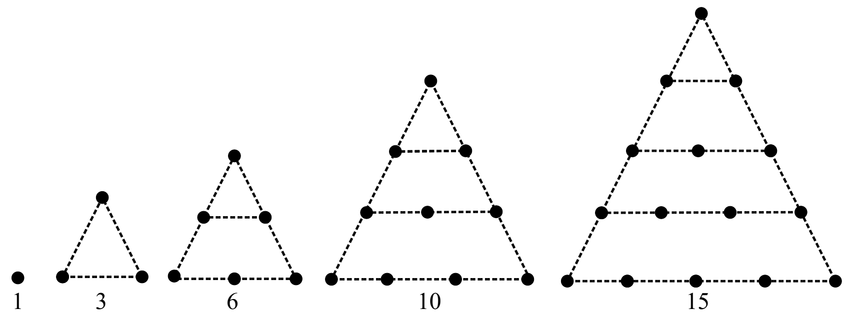
the sequences of partial sums[2] have the following form, respectively:

$\{t^{(3)}(n)\}:$	1	3	6	10	15	21	28	...
$\{t^{(4)}(n)\}:$	1	4	9	16	25	36	49	...
$\{t^{(5)}(n)\}:$	1	5	12	22	35	51	70	...
$\{t^{(6)}(n)\}:$	1	6	15	28	45	66	91	...
$\{t^{(7)}(n)\}:$	1	7	18	34	55	81	112	...
....	...	...	...	...	...	...	...	...
$\{t^{(s)}(n)\}:$	1	s	$3s-3$	$6s-8$	$10s-15$	$15s-24$	$21s-35$	...

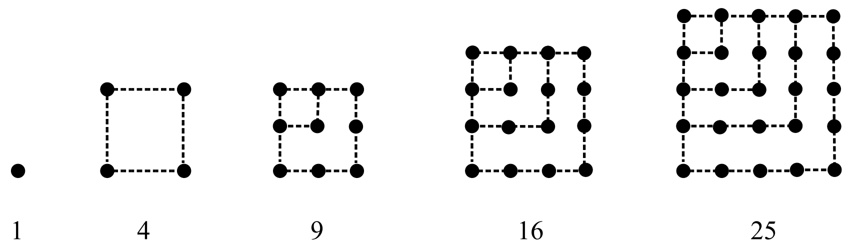
Elements (terms) of a sequence $\{t^{(s)}(n)\}$ are called s-gonal numbers. Hence, there are trigonal numbers, square (quaternal) numbers, pentagonal numbers, hexagonal numbers, etc.

Geometrical illustration of these numbers is as follows:

Trigonal numbers:

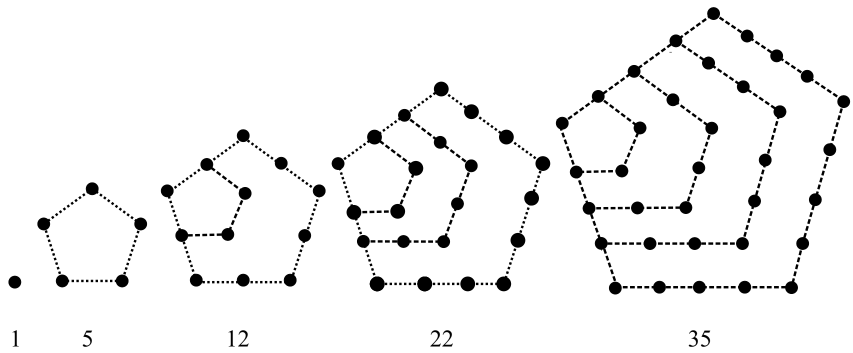


Square numbers:

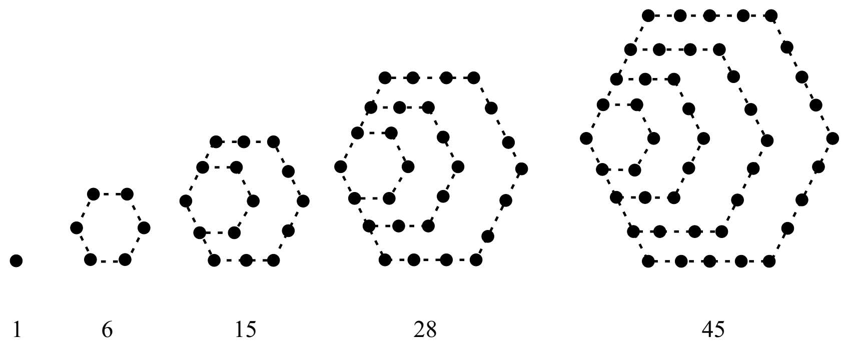


[2]The nth partial sum for a sequence $\{a_n\}$ is $a_1 + a_2 + \dots + a_n$

Pentagonal numbers:



Hexagonal numbers:



The sequences $\{t^{(s)}(n)\}$ $(s = 3, 4, \dots)$ are arithmetic sequences of the second degree. The general term of a sequence $\{t^{(s)}(n)\}$ can be determined by the method of successive differences using table (5) and equation (4). The table of differences for this sequence has the form:

$$
\begin{array}{cccc}
1 & & & \\
 & s-1 & & \\
s & & s-2 & \\
 & 2s-3 & & 0 \\
3s-3 & & s-2 & \\
 & 3s-5 & & 0 \\
6s-8 & & s-2 & \\
 & 4s-7 & \vdots & \vdots \\
10s-15 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

As $t^{(s)}(1) = 1$, $\Delta^1 t^{(s)}(1) = s-1$, $\Delta^2 t^{(s)}(1) = s-2$, then, using Eq. (4), we obtain:

$$t^{(s)}(n) = \binom{n-1}{0} \cdot 1 + \binom{n-1}{1}(s-1) + \binom{n-1}{2}(s-2),$$

i.e.

$$t^{(s)}(n) = \frac{n}{2}[n(s-2) - s + 4]. \tag{6}$$

Therefore trigonal, square, pentagonal, hexagonal, and heptagonal numbers can be defined using the following equations:

$$t^{(3)}(n) = \frac{n}{2}(n+1), \qquad t^{(4)}(n) = n^2,$$

$$t^{(5)}(n) = \frac{n}{2}(3n-1), \qquad t^{(6)}(n) = \frac{n}{2}(4n-2),$$

$$t^{(7)}(n) = \frac{n}{2}(5n-3).$$

Many properties of s-gonal numbers can be found in books [2, 5].

3. Pyramidal numbers

If for a sequence $\{t^{(s)}(n)\}$ the sequence of partial sums is created, then a sequence $\{T^{(s)}(n)\}$ is obtained being an arithmetic sequence of the third degree. Elements (terms) of this sequence are called *s-gonal pyramidal numbers.*

The sequences of trigonal, square, pentagonal, hexagonal, heptagonal, $\ldots$, s-gonal pyramidal numbers have the following form

$\{T^{(3)}(n)\}$:	1	4	10	20	35	$\ldots$
$\{T^{(4)}(n)\}$:	1	5	14	30	55	$\ldots$
$\{T^{(5)}(n)\}$:	1	6	18	40	75	$\ldots$
$\{T^{(6)}(n)\}$:	1	7	22	50	95	$\ldots$
$\{T^{(7)}(n)\}$:	1	8	26	60	105	$\ldots$
$\ldots\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\{T^{(s)}(n)\}$:	1	$s+1$	$4s-2$	$10s-10$	$20s-25$	$\ldots$

respectively.

To determine the general term of a sequence $\{T^{(s)}(n)\}$ draw up a table of successive differences:

$$
\begin{array}{ccccc}
1 & & & & \\
 & s & & & \\
s+1 & & 2s-3 & & \\
 & 3s-3 & & s-2 & \\
4s-2 & & 3s-5 & & 0 \\
 & 6s-8 & & s-2 & \\
10s-10 & & 4s-7 & \vdots & \vdots \\
 & 10s-15 & \vdots & \vdots & \vdots \\
20s-25 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

For the sequences under consideration we have

$$T^{(s)}(1) = 1, \qquad \Delta^1 T^{(s)}(1) = s$$

$$\Delta^2 T^{(s)}(1) = 2s-3, \qquad \Delta^3 T^{(s)}(1) = s-2.$$

Using Eq. (4), we obtain

$$T^{(s)}(n) = \binom{n-1}{0} \cdot 1 + \binom{n-1}{1} s + \binom{n-1}{2} \cdot (2s-3) + \binom{n-1}{3} (s-2),$$

i.e.

$$T^{(s)}(n) = \frac{n}{6}[n^2(s-2) + 3n - (s-5)]. \tag{7}$$

This eqiuation can be written as

$$T^{(s)}(n) = \frac{n}{6}(n+1)[n(s-2) - (s-5)]. \tag{8}$$

Indeed:

$$n^2(s-2) + 3n - (s-5) = n^2(s-2) + 3n + (s-2) + 3 =$$

$$(n^2-1)(s-2) + 3(n+1) = (n+1)[(n-1)(s-2)+3] =$$

$$= (n+1)[n(s-2) - (s-5)].$$

For sequences $\{T^{(3)}(n)\}, \ldots, \{T^{(7)}(n)\}$, according to Eq. (8), the general terms have the form:

$T^{(3)}(n) = \frac{n}{6}(n+1)(n+2), \qquad T^{(4)}(n) = \frac{n}{6}(n+1)(2n+1),$

$T^{(5)}(n) = \frac{n^2}{2}(n+1), \qquad T^{(6)}(n) = \frac{n}{6}(n+1)(4n-1),$
$T^{(7)}(n) = \frac{n}{6} \cdot (n+1)(5n-2).$

In the book [5] many properties of trigonal pyramidal numbers being elements (terms) of the sequence $T^{(3)}(n)$ can be found.

4. Prismatic numbers

Let m be an arbitrary natural number distinct from 1. Consider the sequence:

$$m, \quad 2(m+1), \quad 3(m+2), \quad 4(m+3), \quad 5(m+4), \ldots$$

and create for it the sequence $\{P^{(m)}(n)\}$ being the sequence of partial sums:

$$\{P^{(m)}(n)\}: \quad m, \quad 3m+2, \quad 6m+8, \quad 10m+20, \quad 15m+40, \ldots$$

It is easy to see that the above sequence is an arithmetic sequence of the third degree. The general term of this sequence cab be found drawing up a table of successive differences

$$\begin{array}{ccccc}
m & & & & \\
 & 2m+2 & & & \\
3m+2 & & m+4 & & \\
 & 3m+6 & & 2 & \\
6m+8 & & m+6 & & 0 \\
 & 4m+12 & & 2 & \\
10m+20 & & m+8 & & 0 \\
 & 5m+20 & & 2 & \vdots \\
15m+40 & & m+10 & & \vdots \\
 & 6m+30 & \vdots & \vdots & \vdots \\
21m+70 & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Then we have

$$P^{(m)}(1) = m, \qquad \Delta^1 P^{(m)}(1) = 2m+2,$$
$$\Delta^2 P^{(m)}(1) = m+4, \qquad \Delta^3 P^{(m)}(1) = 2.$$

Using Eq. (4), we obtain

$$P^{(m)}(n) = \binom{n-1}{0} m + \binom{n-1}{1} 2(m+1) + \binom{n-1}{2}(m+4) + \binom{n-1}{3} \cdot 2$$

or

$$P^{(m)}(n) = \frac{n}{6}(n+1)[3m + 2(n-1)]. \tag{9}$$

Elements (terms) of the sequence $\{P^{(m)}(n)\}$ are called *prismatic numbers of the range* m $(m \geq 2.)$

The sequences of prismatic numbers of the second, third, and forth range have the form:

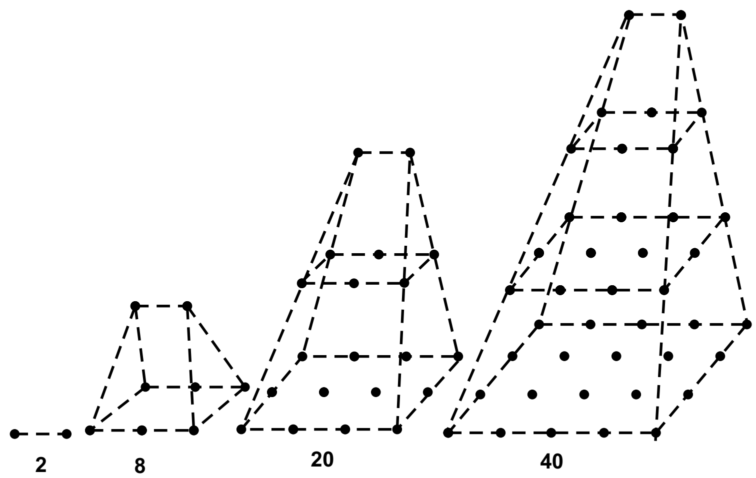
$\{P^{(2)}(n)\}$: 2 8 20 40 70 ... $P^{(2)}(n) = \frac{1}{3}(n+1)(n+2)$

$\{P^{(3)}(n)\}$: 3 11 26 50 85 ... $P^{(3)}(n) = \frac{n}{6}(n+1)(2n+7)$

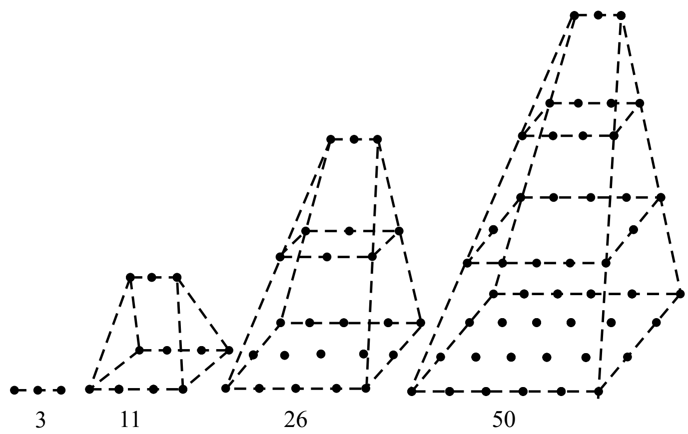
$\{P^{(4)}(n)\}$: 4 16 32 60 100 ... $P^{(4)}(n) = \frac{n}{3}(n+1)(n+5)$

Using prism, the prismatic numbers have the following geometrical illustration:

$m = 2$



$m = 3$



It should be noted that faces of a prism illustrating prismatic numbers being triangles can be interpreted as corresponding trigoanl numbers, whereas lateral faces of a prism being trapezes as *trapezoid numbers*.

For numbers $m = 2,\ m = 3,\ m = 4, \dots, m = s$ we obtain the following sequences of trapezoid numbers:

$\{R^{(2)}(n)\}$:	2	5	9	14	20	$\dots$
$\{R^{(3)}(n)\}$:	3	7	12	18	25	$\dots$
$\{R^{(4)}(n)\}$:	4	9	15	22	30	$\dots$
....	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\{R^{(s)}(n)\}$:	s	$2s+1$	$3s+3$	$4s+6$	$5s+10$	$\dots$

The general term of the sequence $\{R^{(s)}(n)\}$ can be found based on a table of successive differences:

$$
\begin{array}{cccc}
s & & & \\
 & s+1 & & \\
2s+1 & & 1 & \\
 & s+2 & & 0 \\
3s+3 & & 1 & \\
 & s+3 & & 0 \\
4s+6 & & 1 & \\
 & s+4 & \vdots & \vdots \\
5s+10 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

Hence, we have $R^{(s)}(1) = s$, $\Delta^1 R^{(s)}(1) = s+1$, $\Delta^2 R^{(s)}(1) = 1$. Using Eq. (4), we obtain

$$R^{(s)}(n) = \binom{n-1}{0}s + \binom{n-1}{1}(s+1) + \binom{n-1}{2}$$

or

$$R^{(s)}(n) = \frac{n}{2}(n-1+2s). \tag{10}$$

The sequence $\{R^{(s)}(n)\}$ is an arithmetic sequemce of the second degree.

According to (10), the sequences $\{R^{(2)}(n)\}$, $\{R^{(3)}(n)\}$, $\{R^{(4)}(n)\}$ have the following general terms:

$$R^{(2)}(n) = \frac{n}{2}(n+3), \qquad R^{(3)}(n) = \frac{n}{2}(n+5), \qquad R^{(4)}(n) = \frac{n}{2}(n+7).$$

Summarizing we state that all the types of the abovementioned figurate numbers can be characterized by some arithmetic sequences of the second or third degree.

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TESTING OF GEOMETRICAL IMAGINATION

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Abstract. The article deals with the geometric imagination in relation to intelligence tests. During an exploratory investigation of geometric imagination of pupils aged 15-18 years, a non-standardized test was created and evaluated, testing the partial and combinative abilities of students of this age group. The test consists of 40 tasks, and its evaluation process also contains a comparison of the results based on gender and mathematics mark.

1. Prologue

For movement in our world we should have an aptitude, which allows us to orient in the space, to be aware of location of our body and its parts in the space, to perceive the interrelation in the space. Varied names are used for this aptitude, e.g. Visual Thinking, Spatial Ability, Visualization, etc.

Gardner [2] talks about the spatial intelligence. He says: "Prime is aptitude, which secures the accurate perception the visual world. It allows to transform and to modify original percepts and it makes notions from own visual experience without further outward stimulus."

We define Geometrical Spatial Imagination as a "set of abilities, which relate reproduction and anticipation, static and dynamic ideas about shapes, about attributes and about relations between geometrical figures in space" [4].

Restructuring school mathematics often caused a considerable diversion from traditional parts of geometry. More time was given to more modern, more attractive parts of mathematics which are more practical. The limitation of geometry was justified by lack of time and inapplicability of traditional geometry. These remarks can be considered as tangible. The total contribution of geometry is important in a balanced education system. It should not be omitted.

2. Test of triangles

The geometrical spatial imagination is tested e.g. by standard Test of squares, which is a part of Amthauer I-S-T tests of universal intelligence and it comes out from Rybakov figures. We created a didactical test based on the similar principle. We divide an irregular plane figure into two parts only with one cut. Then we put together these two parts to create an equilateral triangle. The test, its administration and results are the components of Jana Slezáková's dissertation [7]. This dissertation was suggested at the Faculty of Science of Palacký University in Olomouc. The test was created and used in the ESF project called "The spotting of talents for the competitiveness and work with them", the area of assistance "The tantamount opportunities for children and pupils, including the pupils with a special educational needs", the registration number CZ.1.07/1.2.08/02.0017.

The test was created so as:

- it was interesting for pupils and it increases the interest in geometry,
- the teachers can easily apply it in teaching,
- it is used for the age category 15 – 18 years,
- it is focused on the geometrical spatial imagination.

The author created a coordinate grid of equilateral triangles and looked up various irregular figures, which can be divided into two parts with only one cut and put together into the equilateral triangle (only in our fantasy). The author created 40 plane figures in the first stage. These figures were tested by a small number of students and then the test was adapted. Two groups about 40 problems arised. The first group of problems – The geometrical spatial imagination (TP1) is for the age category up to 15 years, the second group of problems – The geometrical spatial imagination (TP2) is for the age category from 15 years. In both cases it is an unstandard test of geometrical spatial imagination, which is easily usable for a mathematics teacher.

The task of research was to find out whether the mark in maths and the result in the test are related, whether there exists a closeness of boys results and girls results. It also should order the problems by difficulty.

The test was carried out in June in school year 2009/2010, and 1690 pupils of a grammar school took part in this test. 548 of them (234 boys and 314 girls) were up to 15 years old (the second class, the fourth form) and 1142 of them (421 boys and 721 girls) were older than 15 years (the fifth form, the sixth form, the first class, the second class). It was realized at the faculty grammar school, which is binded by contract with Faculty of Science, Palacký

University in Olomouc. We tried to find out the quality of our surveying and we compared the validity and reliability with values of standard IQ test – Test of squares.

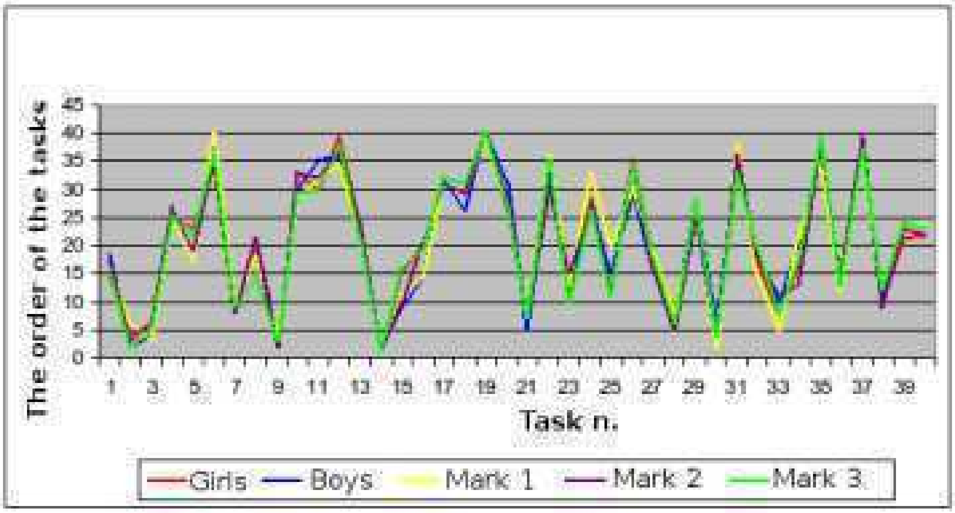

Figure 1: The order of the tasks according to correct answers in the test TP2.

3. Results

Here we present only the results of the test of the imagination – TP2 for the age group over 15 years.

Table 1 shows the relationship between success in the test TP2 and the mark in mathematics (represents the average score for each group of pupils according to marks in mathematics, including their average point difference in %).

The table shows that pupils, which have the mark 1, are clearly better than pupils with the marks 2 or 3. An interesting result is that pupils with the mark 1 were better than pupils with the marks 2 or 3 in each task. This result was not confirmed in the test of lower grammar school pupils. Next testing [7] demonstrated the correlation between success in the TP2 test and the mark in mathematics.

We also attempted to illustrate the sequence of tasks in the TP2 test with their evaluation by the number of correct answers. Now we can see how tasks were difficult for each group and how they would be sorted. Table 2 shows the average score for all the 1142 pupils in various tasks in the test TP2.

Figure 1 shows how tasks are sorted on the basis of test results of each group (particularly for girls, boys, pupils with the marks 1, 2, and 3).

Our results shows how to sort tasks according to increasing difficulty. The task numbers are as follows: 14, 9, 2, 30, 3, 28, 21, 7, 33, 38, 23, 15, 25, 36, 1, 34, 27, 32, 16, 8, 5, 40, 39, 13, 4, 29, 20, 24, 18, 17, 10, 11, 22, 26, 31, 6, 12, 35, 37, 19.

Figure 2 shows the dependence between a gender and the test results.

Task	Overall results	Mark 1	Mark 2	Mark 3	Difference between 1 and 3	Difference between 2 and 3	Difference between 1 and 2
1	81.0	89.2	80.8	76.6	12.6	4.2	8.4
2	91.6	93.7	93.2	89.7	4.0	3.5	0.5
3	89.8	94.9	90.4	87.2	7.7	3.2	4.5
4	70.8	82.9	71.2	67.2	15.7	4.0	11.7
5	76.0	88.6	79.1	68.4	20.2	10.7	9.5
6	55.4	60.1	59.3	52.3	7.8	7.0	0.8
7	86.2	91.8	88.1	82.7	9.1	5.4	3.7
8	78.2	88.6	78.8	76.9	11.7	1.9	9.8
9	92.4	94.9	94.6	89.4	5.5	5.2	0.3
10	62.3	78.5	63.3	58.1	20.4	5.2	15.2
11	62.2	75.9	64.7	58.1	17.8	6.6	11.2
12	54.5	72.2	55.4	48.0	24.2	7.4	16.8
13	74.3	86.7	75.1	69.6	17.1	5.5	11.6
14	93.3	96.8	95.5	91.5	5.3	4.0	1.3
15	82.9	91.8	86.4	76.6	15.2	9.8	5.4
16	78.3	89.9	79.1	72.6	17.3	6.5	10.8
17	62.6	76.6	65.3	55.3	21.3	10.0	11.3
18	66.8	77.2	68.4	56.8	20.4	11.6	8.8
19	43.8	60.8	43.8	39.5	21.3	4.3	17.0
20	68.0	81.0	70.3	59.6	21.4	10.7	10.7
21	88.4	93.0	89.8	86.0	7.0	3.8	3.2
22	61.3	69.0	66.7	54.4	14.6	12.3	2.3
23	83.8	91.1	83.6	81.2	9.9	2.4	7.5
24	68.0	75.9	71.5	63.2	12.7	8.3	4.4
25	82.8	88.0	85.9	81.2	6.8	4.7	2.1
26	60.9	77.2	61.6	55.0	22.2	6.6	15.6
27	79.7	88.6	82.5	73.3	15.3	9.2	6.1
28	89.8	93.0	91.2	86.9	6.1	4.3	1.8
29	68.3	82.3	72.9	59.0	23.3	13.9	9.4
30	91.0	95.6	92.9	89.1	6.5	3.8	2.7
31	57.4	67.1	57.6	55.3	11.8	2.3	9.5
32	78.8	89.9	80.5	72.0	17.9	8.5	9.4
33	86.1	94.3	86.4	83.3	11.0	3.1	7.9
34	80.0	87.3	84.2	75.1	12.2	9.1	3.1
35	53.4	69.6	54.8	47.7	21.9	7.1	14.8
36	82.7	91.8	84.2	77.8	14.0	6.4	7.6
37	52.1	69.0	50.0	48.3	20.7	1.7	19.0
38	84.4	93.0	87.3	78.4	14.6	8.9	5.7
39	74.6	86.7	76.0	67.8	18.9	8.2	10.7
40	75.2	86.1	78.2	68.4	17.7	9.8	7.9
					14.5	6.5	8.0

Table 1: The relationship between success in the test TP2 and the mark in mathematics.

TP2 (1142)	Solved:	Correct		Wrong		Did not solve	
	n	n	%	n	%	n	%
u1	1119	925	81.0	194	17.0	23	2.0
u2	1068	1046	91.6	22	1.9	74	6.5
u3	1089	1025	89.8	64	5.6	53	4.6
u4	980	809	70.8	171	15.0	162	14.2
u5	1088	868	76.0	220	19.3	54	4.7
u6	1091	633	55.4	458	40.1	51	4.5
u7	1040	984	86.2	56	4.9	102	8.9
u8	1009	893	78.2	116	10.2	133	11.6
u9	1092	1055	92.4	37	3.2	50	4.4
u10	900	711	62.3	189	16.5	242	21.2
u11	828	710	62.2	118	10.3	314	27.5
u12	754	622	54.5	132	11.6	388	34.0
u13	975	848	74.3	127	11.1	167	14.6
u14	1104	1065	93.3	39	3.4	38	3.3
u15	1037	947	82.9	90	7.9	105	9.2
u16	1069	894	78.3	175	15.3	73	6.4
u17	849	715	62.6	134	11.7	293	25.7
u18	945	763	66.8	182	15.9	197	17.3
u19	696	500	43.8	196	17.2	446	39.1
u20	997	776	68.0	221	19.4	145	12.7
u21	1073	1010	88.4	63	5.5	69	6.0
u22	799	700	61.3	99	8.7	343	30.0
u23	1023	957	83.8	66	5.8	119	10.4
u24	881	776	68.0	105	9.2	261	22.9
u25	1051	946	82.8	105	9.2	91	8.0
u26	912	695	60.9	217	19.0	230	20.1
u27	991	910	79.7	81	7.1	151	13.2
u28	1058	1025	89.8	33	2.9	84	7.4
u29	910	780	68.3	130	11.4	232	20.3
u30	1065	1039	91.0	26	2.3	77	6.7
u31	953	655	57.4	298	26.1	189	16.5
u32	963	900	78.8	63	5.5	179	15.7
u33	1027	983	86.1	44	3.9	115	10.1
u34	977	914	80.0	63	5.5	165	14.4
u35	821	610	53.4	211	18.5	321	28.1
u36	1013	944	82.7	69	6.0	129	11.3
u37	713	595	52.1	118	10.3	429	37.6
u38	1032	964	84.4	68	6.0	110	9.6
u39	905	852	74.6	53	4.6	237	20.8
u40	953	859	75.2	94	8.2	189	16.5

Table 2: The average score for all pupils in various tasks.

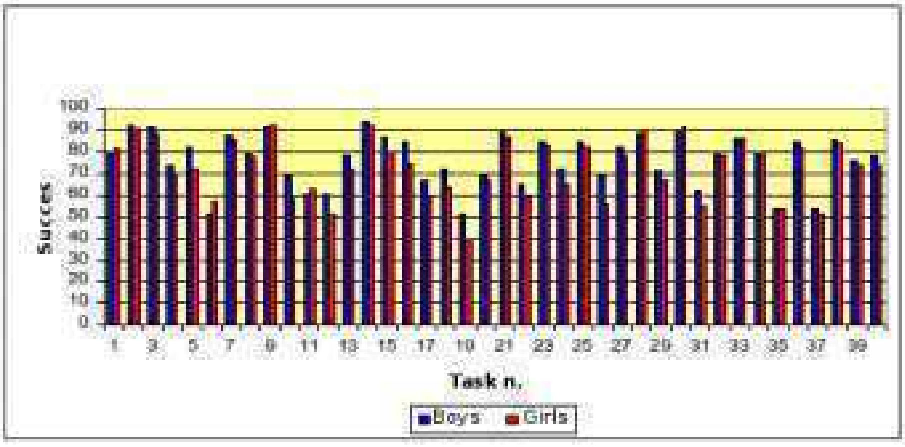

Figure 2: The dependence between a gender and the test results.

The investigation (average points from test of boys and girls) of grammar school pupils shows a big difference between the results of boys and girls. Girls were better than boys only in seven tasks (1, 6, 9, 11, 28, 30 and 34). The largest difference of overall average points was for task 28, it was 11.5% for girls. Also next investigation [7] shows that there is a correlation between success in the test solution TP2 and a gender of pupils.

4. Conclusion

Another goal was to determine the quality of our measurements and to compare the values of validity and reliability with a standardized IQ test – squares. The values of reliability, validity of measurement are in the following tables.

Statistical procedure of SPSS program which determines the value of the Cronbach alpha and the coefficient for the split-half method was used for reliability. Validity was verified by using the correlation between the mark in mathematics and test results.

Spearman's correlation coefficient was used for finding the right relations between tests. It allows to determinate quantitatively how close is the connection between variables which were used for creating orders.

These tables show how high is the grade of reliability for the test TP2. This value is higher than reliability IQ test - Test of squares. (When reliability is higher (close to $+1$), then precision is higher too). Reliabity is $r = 0,837$ for test TP2 and reliability for IQ–test of squares is $r = 0,812$ [9].

The mark in mathematics was chosen as a criterion to assess statistic validity. Predictive validity was used as well as in the case of square test.

Reliability Statistics

Cronbach's Alpha	Part 1	Value	.831
		N of Items	20(a)
	Part 2	Value	.80
		N of Items	20(b)
	Total N of Items		40
Correlation Between Forms			.727
Spearman–Brown Coefficient	Equal Lenght		.842
	Unequal Lenght		.842
Guttman Split-Half Coefficient			.837

a The items are: u1, u2, u3, u4, u5, u6, u7, u8, u9, u10, u11, u12, u13, u14, u15, u16, u17, u18, u19, u20.

b The items are: u21, u22, u23, u24, u25, u26, u27, u28, u29, u30, u31, u32, u33, u34, u35, u36, u37, u38, u39, u40.

Case Processing Summary

		N	%
Cases	Valid	324	28.4
	Excluded(a)	818	71.6
	Total	1142	100.0

a Listwise deletion based on all variables in the procedure.

Reliability Statistics

Cronbach's Alpha	N of Items
.902	40

Table 3: Values of reliability for split-half method in test TP2.

Correlation(research1d)
Correlation are on significance level $p < .05$
Summarize the condition: TP="TP2"
and research="JS"

Variable	Mark	Points	Correct (%)
Mark	1.0000	$-.2162$	$-.1981$
	$N = 973$	$N = 973$	$N = 973$
	$p =$ —	$p = .000$	$p = .000$
Points	$-.2162$	1.0000	.7823
	$N = 973$	$N = 1142$	$N = 1142$
	$p = .000$	$p =$ —	$p = .000$
Correct (%)	$-.1981$	.7823	1.0000
	$N = 973$	$N = 1142$	$N = 1142$
	$p = .000$	$p = .000$	$p =$ —

Table 4: Values of predictive validity for test TP2.

Based on the results and from the tables, we can state that our measurements on the significance level of 0,05 can be considered valid.

It can be said that the test TP2 is suitable for verifying the level of geometric imagination of pupils of grammar schools.

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Appendix. Test TP2. Divide the polygon using only one section so that the transfer of one part to another (only in the imagination) creates an equilateral triangle.

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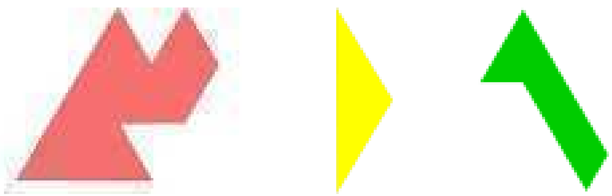

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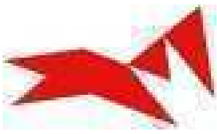
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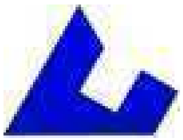
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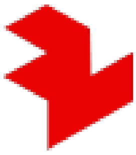
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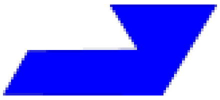

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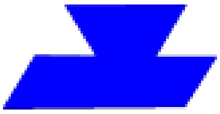
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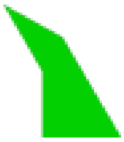
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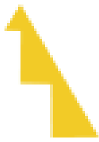

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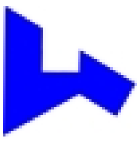
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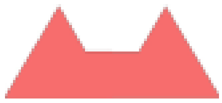

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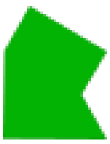
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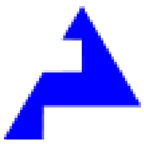

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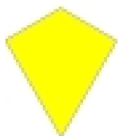

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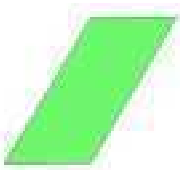

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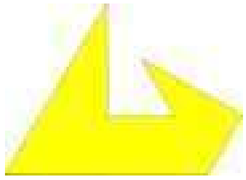

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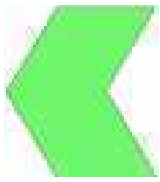

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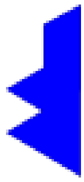

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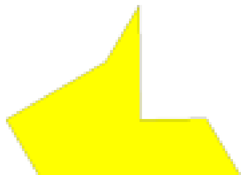

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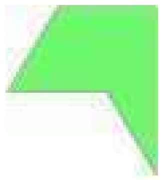

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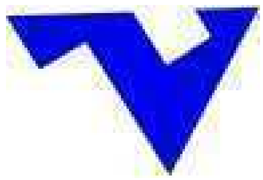

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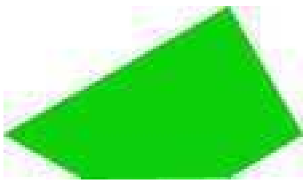

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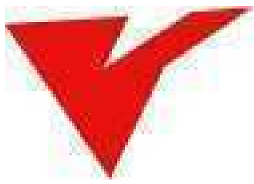

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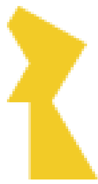

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Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XVI, Częstochowa 2011

MATHEMATICS COMPETENCES OF PEDAGOGY STUDENTS AT THE BEGINNING OF THEIR UNIVERSITY STUDIES

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Abstract. In the contribution the results of the same set of mathematics tasks given to two different pedagogy students groups are compared. It deals with comparison of these two examinations' results. The first group of students was represented by applicants for mathematics pedagogy studies, and the test was applied as an entrance exam in 1985 and 2001. The second group was represented by students of the same field of study at the beginning of their studies in 2010. These students did not sit for an entrance exam because of low number of applicants at that time. The comparison shows a significant decline of current students' competences which are necessary for solving mathematical tasks.

1. Introduction

Mathematics – in the sense of field of study – has a basis character at the universities preparing future secondary school teachers. This basic knowledge is usually considered essential when speaking about effective teaching. To teach mathematics well and with pleasure it is necessary to acquire a specific way of working in mathematics. Next to mastering amount of basic knowledge it requires creating ones own math world in which crystallize your experience with mathematics concepts, algorithms and mathematics tasks solutions in various contexts. "Mathematics cannot be mastered just by memorizing definitions, concepts and theorems, but we have to work with them at the same time" (see Brincková [1, p. 6]).

University teaching of mathematics subjects has gone through a lot of organizational and contextual changes due to an altered philosophy of preparing future teachers. Intuitive experience hand in hand with some researches (e.g. PISA OECD) indicate decreasing level of secondary school graduates' mathematics competences, their knowledge is superficial, episodic and formal. Universities need some inventory showing the level of their preliminary knowledge in the most objective way.

Convenient instrument for measuring the level of their mathematical knowledge is considered to be the analysis of solving tasks. This analysis could become a tool for getting an image of future students.

2. Aims, methods and investigation tools – didactic test and its analysis

The faculties on education in the Czech Republic have no entrance exams at the moment. Their results would show applicants' level of mathematics knowledge. Departments of mathematics obtain only the information about the final marks in mathematics, which is a very distorted information due to different ways of teaching at different types of secondary schools, the amount of mathematics lessons and other factors. Those students, who start to study teaching mathematics at the faculty of education, come from various types of secondary schools – from training institutions and grammar schools at the same time. It was necessary to start solving this situation and to find out how deep is this lack of knowledge, what could be the way to solve this negative phenomenon.

Within the project obtained in Students' grant competition in 2010 the research was done. It took place at four universities: Faculties of Education at Palacky University in Olomouc, Masaryk University in Brno, University of Ostrava and University of Constantin Philosopher in Nitra in Slovakia. The aim of this research was to compare the current state of first grade students' mathematics competences at the beginning of their university studies to wider research done in 1985.

The aims were

- to compare the results of entrance exams by applicants for studies in 1985 with results of the same test written by first year students in 2010,
- to analyze the collection of tasks,
- to find out whether these results might have been influenced by some external or internal factors.

In June 2010 we did the pre-research at the Department of Mathematics at the Palacky University in Olomouc. We wanted to compare the results in the similar test given to applicants in 1985 and first year students. This test consisted of 6 various tasks, the same for both groups of students. These tasks are based on the evaluation standards for grammar schools.

Thematically it corresponds to main area of secondary school mathematics:

- Linear and quadratic equations and their systems
- Rational functions, powers and roots
- Analytics geometry

Changes in curriculum led to content modification of comparative test, for this reason some topics such as trigonometry, combinatorics, sequences and others were removed from the test. Table 1 shows the results of chosen groups of students (S_1 = applicants for studies in 1985, S_2 = chosen group of the first year students at Faculty of Education in Olomouc). Solutions are divided into A, B and C groups with A = correct solution, B = partial solution, C = incorrect solution.

Task number	Relative frequency of results in S_1 and S_2 (%)					
	A (Group S_1)	A (Group S_2)	B (Group S_1)	B (Group S_2)	C (Group S_1)	C (Group S_2)
1	68.7	45	12.5	10	18.7	45
2	37.5	7.5	25	35	37.5	57.5
3	37.5	0	6.2	30	56.3	70
4	25	2.5	25	2.5	50	95
5	21.4	2.5	42.9	27.5	35.7	70
6	33.3	17.5	46.7	60	20	22.5

Table 1: Proportional results in both tests

The test results showed such a huge descent of mathematics knowledge that we decided to create a brand new test with tasks corresponding to collection of exercises from PISA research. It requires the most of all elementary school knowledge, three-dimensional imagination, ability to use mathematics in everyday life and other tasks showing the level of students' mathematics knowledge.

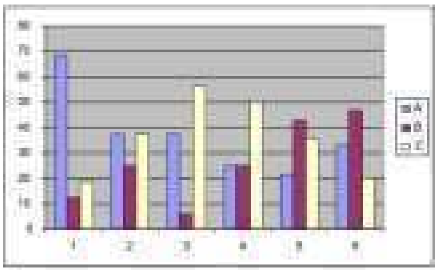

Figure 1: Results of group S_1

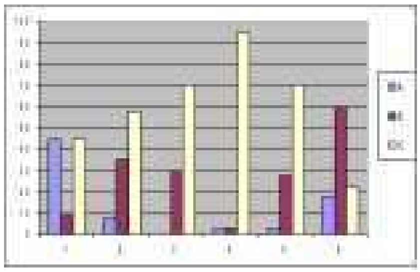

Figure 2: Results of group S_2

It consisted of 10 tasks (some of them had two parts corresponding with educational framework), 8 of which were open tasks and 1 task closed due to demanding interpretation of results. Tasks were put together this way:

Task 1 – student modifies mixed number and solves addition and multiplication of fractions

Task 2 – student uses a rule of three

Task 3 – student forms and solves situations in divisibility of natural numbers

Task 4 – student solves percentage task, uses other mathematics relations

Task 5 – student solves the situations interpreted by proportion

Task 6 – student solves real situations, uses mathematics tools

Task 7 – open task – student sketches and constructs orthocenter and the inscribed circle of a triangle

Task 8 – student uses three–dimensional imagination

Task 9 – student solves goniometrical functions, sketches graphs of sine and cosine

Task 10 – student solves a quadratic equation

Tasks are corresponding with individual wholes of Educational Framework for Elementary and Secondary Schools and are based on Fuchs' evaluation standards from 1994.

Altogether 203 students participated on the research done at the universities in the Czech and Slovak Republics. Graph 3 represents a histogram of test results frequency.

Graph shows better results in comparison to the first testing lap. But realizing the level of tasks (elementary school exercises) it is surprising that a lot of students reached less than average marks. At the same time the international research PISA OECD monitored the biggest deterioration of our students' knowledge since 2003.

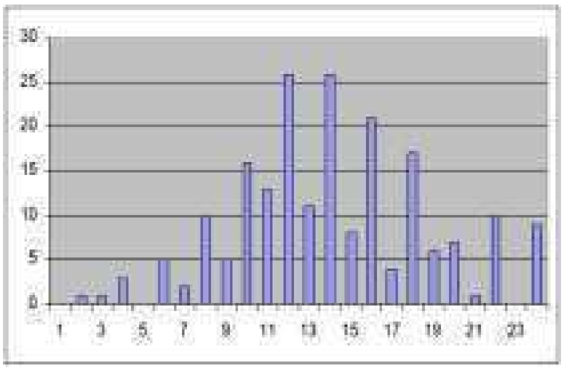

Figure 3: Histogram of test results frequency

In our research we were discussing possible reasons for this knowledge decline. Besides possible social and psychological factors, we were looking for some other we could support by numbers.

The aim of a short questionnaire was to find out the relation between the result in the test and the type of secondary school of particular student. Another factor may be a gender of probed students.

Two hypotheses were created:

H01 – There is no relation between the test result and a gender of probed students

H11 – There is a relation between the test reset and a gender of probed students

H02 – There is no relation between the test result and the type of secondary school

H12 – There is a relation between the test results and the type of secondary school

Statistics confirmed the dependence on the type of school, on the other hand, it showed that a gender of a student does not influence mathematics knowledge.

3. Conclusion

Research enquiry provides us with rich data which will allow not only description, but also causal analysis, relating to the third research aim. Using these results, we have confirmed our expectations about the decreasing level of mathematics knowledge of students at the beginning of their university studies. The results support us in a long-term trend that signalizes low permanence of mathematics knowledge and points out the difficulties connected with secondary school mathematics teaching when considering final state exams. Standard, traditional mathematics curriculum has not become a steady "equipment" of future mathematics teachers.

Problems with insufficient mathematics tools, when considering students at the beginning of their studies, are being discussed and are still current. The level of knowledge reached in the test is being compared with educational curriculum. But it has a prognostic aspect at the same time – it allows us to judge the preconditions for good studies in a proper field of study, preconditions for success later at work, and it can become an inspiration for subjects' innovation at the faculties preparing future mathematics teachers.

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KNOWLEDGE OF QUADRILATERAL AT ELEMENTARY SCHOOL

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Abstract. Quadrilaterals are more difficult than triangles for some students of elementary schools. This pilot study compares knowledge of quadrilateral of pupils of elementary schools and pupils of the same age at grammar schools. Some students prefer algebra to geometry, because it includes practical problems. The aim of this pre-research is to find out differences in knowledge of quadrilaterals between pupils of elementary and grammar schools.

1. Prologue

This pre-research decided to compare the knowledge of the quadrilaterals of the elementary school pupils with pupils of the same age at a grammar school. Knowledge was tested by a didactic test, which is less common form of testing knowledge of mathematics and very unusual form of testing of knowledge of geometry. Pupils usually meet with the tasks, where they construct some object but the test to understand the definitions of objects is unusual. Good understanding and knowledge of these terms may help pupils to solve common types of problems.

2. The objectives of the research

The research was carried out on pupils in the eighth and ninth grade of elementary school and compared with existing research on pupils in the same age of a grammar school [4]. The aim was to compare how the curriculum of quadrilaterals managed pupils at an elementary school and a grammar school. Pupils got an unstandardized didactic test (see Appendix) which contains 9 questions with multiple choice form of answers.

3. Respondents

Testing was carried out in school year 2010/2011 at an elementary school in Letohrad (34 pupils from the eighth grade, 18 pupils from the ninth grade) and compared it with the research conducted in school year 2008/2009 at a grammar school with 159 pupils [4].

4. Analysis of data

The relative frequency of responses was summarized in the following Table, and we compared it with data already obtained from testing pupils at a grammar school.

The frequency of correct answers is comparable for pupils of both school for many questions.

Almost the same ratio of pupils of both schools correctly answered the question No. 1. Pupils of the elementary school know that three elements are not enough for the construction of a quadrilateral. But only 21% of pupils know that they need five elements.

The question No. 2 was about the basic terminology and the majority of pupils answered it correctly.

The number of correct answers for the question No. 3 was comparable for pupils of both types of schools.

The question No. 4 was the major success of grammar school pupils. The number of response B of elementary school pupils was interesting. The pupils meet squares and rectangles very often in school and life. It is very surprising ignorance of a rectangle.

Elementary school pupils are a bit better in question No. 5. The question No. 6 is similar to the previous question. The balanced answers of primary school pupils could be caused because the most pupils only guessed. When we compare this question with the previous one, we can see that a rhombus is more understood than a parallelogram.

Results of question No. 7 were balanced, and pupils mastered a rectangular trapezoid similarly.

Deplorable results of question No. 8 for pupils of elementary schools show scant knowledge of trapezium and its properties. The most surprising answer is D (of the primary school pupils).

The question No. 9 was balanced again. We can only mention the frequency of response C.

Question No.	Answer	Pupils of elementary school	Pupils of grammar school
1	A	15 %	42 %
	B	45 %	34 %
	C	**21 %**	**21 %**
	D	19 %	7 %
2	A	11 %	13 %
	B	**81 %**	**85 %**
	C	4 %	2 %
	D	2 %	0 %
	Didn't solve	2 %	0 %
3	A	9 %	0 %
	B	17 %	23 %
	C	15 %	8 %
	D	**59 %**	**69 %**
4	A	4 %	6 %
	B	68 %	36 %
	C	**26 %**	**51 %**
	D	2 %	7 %
5	A	17 %	17 %
	B	9 %	25 %
	C	**47 %**	**32 %**
	D	25 %	26 %
	Didn't solve	2 %	0 %
6	**A**	**23 %**	**34 %**
	B	32 %	38 %
	C	21 %	4 %
	D	24 %	24 %
7	A	19 %	32 %
	B	**38 %**	**40 %**
	C	17 %	9 %
	D	26 %	19 %
8	**A**	**11 %**	**26 %**
	B	19 %	17 %
	C	23 %	34 %
	D	43 %	23 %
	Didn't solve	4 %	0 %
9	**A**	**40 %**	**32 %**
	B	17 %	21 %
	C	24 %	32 %
	D	15 %	15 %
	Didn't solve	4 %	0 %

Table 1: The relative frequency of responses of elementary and grammar school pupils.

5. Conclusion

The test proved that the knowledge of pupils from both types of schools is for the most part the same. Some errors could be caused be inattention. Some pupils could be surprised with an unusual form of testing. Moreover, the test was carried out at a substitute lesson and not immediately after discussing the subject matter. The results of both types of school show that pupils are not sure in definitions and relations of trapezoids and common quadrilaterals. After studying triangles, the problematics of quadrilaterals may be less discussed, and some concepts may not be adequately explained.

Acknowledgements

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Appendix

Question 1: How many components are needed to construct a general quadrilateral?

a) 3

b) 4

c) 5

d) 6

Question 2: Convex quadrilaterals can be divided into common quadrilaterals, parallelograms and

a) rhombuses
b) trapezoids
c) squares
d) rectangles

Question 3: The diagonals of a rhombus are:

a) of the same length but do not bisect each other
b) of the same length and bisect each other
c) perpendicular but do not bisect each other
d) perpendicular and bisect each other

Question 4: In which quadrilaterals are the diagonals the axes of the internal angles at the same time?

a) only in a square and in a rhomboid
b) only in a square and in a rectangle
c) only in a square and in a rhombus
c) only in a square and in a trapezoid

Question 5: Two diagonals divide a rhombus into:

a) four equilateral triangles
b) four identical isosceles triangles
c) four identical right triangles
d) two isosceles triangles and two identical equilateral triangles

Question 6: Diagonals always divide a rhomboid into:

a) two pairs of identical triangles
b) two pairs of isosceles triangles
c) four identical triangles
d) a pair of right triangles and a pair of isosceles triangles

Question 7: A right–angle trapezoid has four interior angles:

a) two acute angles, one obtuse angle, one right angle

b) one acute angle, one obtuse angle, two right angles

c) one acute angle, two obtuse angles, one right angle

d) two acute angles, two right angles

Question 8: Select a claim for any isosceles trapezoid:

a) we can draw the circumscribed circle

b) we can inscribe the circle

c) we can draw the circumscribed circle and also inscribe the circle

d) each of its diagonals divides it into two isosceles triangles

Question 9: Which of the following statements do not apply to any rhombus:

a) we can draw the circumscribed circle and also inscribe the circle

b) the intersection of the diagonals has the same distance from all the sides

c) its content is less than a square of the side length

d) the total size of every two adjacent interior angles has a size of 180°

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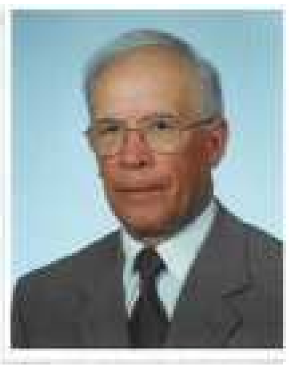
PERSONALIA

Profesor Grzegorz Bryll

Miałam ogromne szczęście pracować w Instytucie Matematyki i Informatyki Akademii im. Jana Długosza z wieloma wybitnymi profesorami, wspaniałymi dydaktykami a zarazem ludźmi o niezwykłych osobowościach. Tym tekstem chcę zainaugurować cykl artykułów poświęconych tym postaciom. Wielu z nich jeszcze pracuje, a część jest na emeryturze, pracując naukowo i wspierając młodsze pokolenie.

Profesor zwyczajny Grzegorz Bryll pracował w Instytucie Matematyki i Informatyki w latach 1991 - 2004. Przez cały ten czas pełnił funkcję kierownika Zakładu Dydaktyki Matematyki.



Grzegorz Franciszek Bryll urodził się 2 stycznia 1935 roku w Gostyniu Wielkopolskim, w 1953 roku ukończył Liceum Pedagogiczne w Lesznie. W roku 1957 uzyskał stopień magistra matematyki w Wyższej Szkole Pedagogicznej w Opolu będąc w grupie pierwszych absolwentów tej uczelni. Po ukończeniu studiów przez rok pracował w Liceum Ogólnokształcącym w Gostyniu jako nauczyciel matematyki i fizyki. W roku 1958 podjął pracę na WSP w Opolu jako asystent a potem starszy asystent.

W roku 1968 uzyskał stopień doktora nauk matematycznych na Wydziale Matematyki, Fizyki i Chemii WSP w Katowicach (obecnie Uniwersytet Śląski).

W latach 1968–1982 profesor Bryll pracował w Wyższej Szkole Inżynierskiej (obecnie Politechnika Opolska), pełniąc tam wiele odpowiedzialnych funkcji od kierownika Zakładu Matematyki poprzez dziekana aż do prorektora ds. nauki i współpracy z przemysłem.

W roku 1982 uzyskał stopień doktora habilitowanego nauk humanistycznych w zakresie logiki matematycznej na Wydziale Filozoficzno-Historycznym Uniwersytetu Wrocławskiego.

W latach 1982–1985 pracował ponownie w Wyższej Szkole Pedagogicznej w Opolu na stanowisku docenta. W roku 1985 prof. G. Bryll rozpoczyna 4-letnią pracę dydaktyczną w Afryce w Centrum Uniwersyteckim w Blidzie (Algieria), wykładając matematykę w instytutach elektroniki i mechaniki. Od roku 1990 aż do dnia 8 czerwca 2006 roku, kiedy to przeszedł na emeryturę, pracował w WSP w Opolu (dziś Uniwersytet Opolski), gdzie przez dwie kadencje pełnił funkcję prorektora ds. nauki i współpracy z zagranicą.

Dnia 8 lipca 1997 roku otrzymał z rąk prezydenta RP tytuł naukowy profesora nauk humanistycznych w zakresie logiki matematycznej.

Profesor Grzegorz Bryll posiada duży dorobek naukowy, w tym 7 monografii i prac zwartych, 103 artykuły naukowe, 5 skryptów, 8 artykułów naukowo-informacyjnych oraz 6 prac naukowo-badawczych na zlecenie przemysłu. Oto niektóre z nich.

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Profesor Grzegorz Bryll wypromował czterech doktorów: Zofię Kostrzycką, Roberta Sochackiego, Anettę Górnicką, Zdzisława Kosztołowicza.

Za swoją działalność naukową, dydaktyczną i organizacyjną uzyskał wiele nagród i odznaczeń, w tym Krzyż Kawalerski OOP, Złoty Krzyż Zasługi, Medal Komisji Edukacji Narodowej oraz nagrody Ministra Nauki i Szkolnictwa Wyższego.

Profesor Grzegorz Bryll zawsze interesował się matematyką, jednak jego pasją zawodową była logika i, jak sam wspomniał w jednym z wywiadów, wydała mu się ona piękniejsza. Grzegorz Bryll jest współtwórcą nowej dziedziny w logice – teorii odrzucania wyrażeń.

W życiu naukowym Profesora ogromną rolę odegrał profesor Jerzy Słupecki, o którym mówił "mój mistrz".

Współpracując z profesorem Grzegorzem Bryllem przez cały okres jego pracy w naszym Instytucie, lubiłam słuchać wykładów prowadzonych przez profesora. Sposób w jaki tłumaczył najtrudniejsze zagadnienia powodował, że wszystko wydawało się proste. Bardzo starannnie pisał na tablicy, zawsze wszystko wyjaśniając i komentując. Nie korzystał z notatek, choć zawsze miał je bardzo starannie przygotowane. Profesor zawsze miał czas dla studentów, chętnie z nimi dyskutował i rozjaśniał wątpliwości. Jako promotor prac licencjackich i magisterskich spędzał wiele godzin ze swymi seminarzystami. Podobnie opiekował się swymi doktorantami.

Dla mnie profesor Grzegorz Bryll jest też mistrzem i ogrmomnym autorytetem zawsze podziwianym, człowiekiem o nieskazitelnej kulturze osobistej. W swej pracy zawodowej staram się postępować podobnie.

Grażyna Rygał
Akademia im. Jana Długosza w Częstochowie

Życzymy Panu Profesorowi Grzegorzowi Bryllowi dużo zdrowia i siły do kontynuowania aktywności naukowej i wydawniczej.

Redakcja czasopisma "Prace Naukowe Akademii im. Jana Długosza w Czestochowie, Matematyka"