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CONTENTS

Part I. Mathematics and its applications

Part II. Theory of training teachers of mathematics

PART II

THEORY OF TRAINING TEACHERS OF MATHEMATICS

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COMMUNICATION IN MATHEMATICS SUPPORTED BY INFORMATION TECHNOLOGIES

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Abstract. I deal with possibilities how to apply Information Technologies in teaching of mathematics. I depict a contemporary state of IT use in teaching in European countries. I further acquaint the reader with the results of the experiment, which goal was to use the Equation Grapher program in teaching of linear and quadratic functions. I state the subsequent hypothesis: Information technologies, or more precisely Equation Grapher, can be effectively used in the phase of exercise and revision of taught theme.

1. The possibilities of IT using in the teaching

There are lots of possibilities how to apply IT in teaching of mathematics. In this chapter I try to summarize the accessible Technologies usable in teaching.

The didactic software offers the students the interface, tool for cognition, investigation and modeling. It is developed to support learning, cognition and progress of information literacy. Its fundamental properties and criterions are: didactical goal, suitable user interface, development of information literacy and information culture, it provides feedback, develops associatively learning, has the form of a game, applies visualization and multimedia, it is interactive, open, oriented, concentrated, provides severity levels, offers individual approach to the person who learns (i.e. respects his tempo, provides interface adjustability, various ways to work with, the order of activities [10].

Applets are programs created in the Java programming language and are appointed only for inclusion in web pages. Their animation is most frequently caused by movement of some point or bigger figure. In many of these web pages is included subsidiary text as mathematical theoretical base, formulas, derivations, guidance tasks, repetitive questions, exercises ... [4].

Applet on the web page [12] can serve us as an example. Author explains on the example of inflowing water the principle of direct proportion and thus the linear function graph too. The amount of water in the vessel before turning the tap on is the y-intersection. The faster the water flows in, the bigger is the angle of the line with x-axis. Applet on the [11] web page is also suitable for understanding the quadratic functions drawing problems. After applying the function expression the function graph $y = a \cdot (x + h)^2 - k$ whose parameters a, h and k are changing by moving a button. Program calculates coordinates of the vertex.

Graphic calculators belong to contemporary society. They can be used in different mathematical activities. They can help by solving of mathematical problems in the cases when numerical and graphical solutions of students are not completely successful. In education process can be usefully used also projection graphic calculators. They have video-output or LCD-output which can be projected through beamer [4].

Graphic calculator is a combination of advanced graphics, tables and data analysis with basic mathematical functions. They are perfect for mathematically oriented high schools. They offer unbelievable speed and versatility [2, 3].

Teleproject is a distance project on which two or more schools are co-operating. Together they solve some problem in given theme. Rich source of teleprojects of Slovak School is on the web page [8].

EBeam belongs to the latest possibilities of using IT. It is an interactive board which solders the tradition of board writing with information technologies. With the assistance of this system not only prepared lectures can be presented, but there is also the possibility of writing own comments into the projected matter, all teaching programs may be controlled from the board and the like [6].

Other new ways of teaching includes also e-learning. This method is based on computer network using in making the self-study educational courses accessible. These courses provide interactive teaching environment using multimedia operation of information as well as continuous evaluation of the teaching approach by means of dialogues, exercises, tasks [6].

2. The contemporary state of IT using in foreign countries

European Commission Information Society and Media released on the web page [9] results of the research of IT using in teaching in year 2006. In the following graph Europe union countries are aligned by the extent of IT using in teaching.

Percentage of teachers who have used computers in class in the last 12 months (2006)

Source: LearnInd CTS 2006; Base: All teachers: Q7. See questionnaire for exact wording

Among the countries of the European Union the best results were achieved in the United Kingdom, where 96% of teachers used computers in class in the last 12 months. All British schools use IT in classes and have Internet access. British teachers are very frequent and intensive ICT users. A majority (65%) of teachers using computers use them in more than a fourth of their lessons, of which there are 21% using it in more than half of their lessons.

Best place among the neighboring countries attained Austria where 88% of teachers used computers in class in the last 12 months. Austria is followed by the Czech Republic with 78% and Slovak Republic with 70%. Poland tails away with 61%. The worst results from Slovak neighboring countries achieved Hungary with 43%. The last but one place took Greece with 36% closely followed by Latvia with 35%. It is about a half in comparison with the 74% European Union average.

3. Equation Grapher

We can distinguish between few programs that make the function graphs drawing possible. Equation Grapher belongs to one of them. It is a 15-days shareware from MFSoft International. Further information about this program can be found on the company's web page [7].

One may take advantage of this program at mathematics lesson in teaching linear and quadratic functions. Apart from quick drawing of random functions, it has also an additional big plus in confrontation with the blackboard or exercise book - the possibility to enlarge or shrink the screen where the function graph is plotted on. Through this we can focus only on selected part of the screen. This program enables to draw up to 12 function graphs into one picture.

Only a few teachers' skills are sufficient for effective work with the PC and it only depends to his creativity how he will take use of this program to diversify the mathematics lessons... [4].

In this chapter I will try to describe the work with this program a little bit closer. There is a main menu with following offer in the program:

In the **main menu** one can find the whole commands overview. There is no need to use it because the program can be manipulated only by means of windows and icons.

Icons serve to hasten the work with Equation Grapher. Simple click executes for example these functions: swell, decrease, determinate the intersection with coordinate axis, maximum, minimum, two graphs intersection... The command line serves to type the expressions. We shouldn't forget to type correct parentheses.

The Equation Grapher program works on windows base:

- Graph window with coordinate system selected by us,
- Function Pad window with buttons for typing of functions,
- Range window for changing the range values,
- Log window with mathematical and numerical buttons.

In the Range window we can change an interval represented in the Graph window namely on the x-axis and y-axis. By plotting of goniometric functions special device through the upper marked icon must be chosen and the lower marked icon Że"has to be pushed. The graph as a whole or its marked part can be increased or decreased by using zoom. In the Range window the current interval setting visible on x-axis and y-axis always turns up.

By typing expression the icon **Functions** or directly the Function Pad window can be used. There are buttons like sinus, cosine, tangent, power, root, unknown x, number pi, absolute value. From numerical buttons are presented classic operations +, -, ., : and DEL for deleting of incorrect typed expression and EXE for submitting the plot of the function graph.

By means of these buttons we can determinate the intersection with x-axis, y-axis, maximum, minimum, intersection between two graphs, or calculate second coordinate, value of which will picture in the Log window [1].

4. Experiment

The realization of the experiment

I executed the experiment during repetition of the curriculum about linear and quadratic functions in two fifth classes of 8-yearlong grammar school. I will call these classes X and Y for better orientation. In both classes I applied one lesson curriculum about linear functions and linear functions with absolute value as well as curriculum of quadratic functions. I prepared a Power Point presentation containing questions and exercises for each lesson. The exercises were compiled by [5]. Every student had a worksheet at his disposal. On the third lesson students have written a test and have filled the questionnaire. I fulfilled the analysis a priori and analysis a posteriori. [1]

Analysis a-priori of didactic situation

The exercises in test were similar to the exercises on repetition lessons. Consequently it was old knowledge. The exercises were clearly formulated. I have not expected any subsidiary questions. The students had 30 minutes to solve 4 exercises. Every student got a paper with entered tasks:

1. Draw two linear function graphs that crosses the point [0,3]. Write relevant expressions.
2. Draw a graph of function y = x2 - 2x - 3, plot intersections with x-axis, y-axis, find out the coordinates of vertex, domain, range.
3. Solve the equation 3 - x = |- 2x| graphically and write a result.
4. Martin has 50,- Slovak crowns. He would like to save for long-wished-for encyclopedia worth 950,- Slovak crowns. Weekly he gets 150,- Slovak crowns pocket money from his parents. In how much time (in weeks) he can afford to buy it? Create graph and write the result.

I assumed that following mistakes will occur by solving exercises (E1, E2, E3 and E4):

E 1-1: incorrect marking of the point in the coordinate system,
E 1-2: correct graph but incorrect expression of the function,
E 2-1: numerical mistake and following incorrect defined vertex and incorrect range,
E 2-2: numerical mistake during calculating the intersection with x-axis,
E 2-3: numerical mistake during calculating the intersection with y-axis,
E 2-4: incorrect defined domain,
E 2-5: confusion of the domain and range notion,
E 2-6: incorrect oriented graph,
E 3-1: incorrect plotted function graph with absolute value,
E 3-2: incorrect plotted graph of the other function,
E 3-3: exploring only one intersection between the graph of functions,
E 3-4: incorrect numerical solution of the equation,
E 4-1: misinterpretation of the task context,
E 4-2: incorrect starting point of the graph,
E 4-3: numerical mistake during adding the weekly pocket money.

Analysis a-posteriori of didactic situation

In the following chapter I will bring up the evaluation of the experiment hence the mistakes done by students while solving the exercises. The mistakes I haven't assumed but they appeared, are following:

- N1 incorrect function graph (exercise no. 1),
- N2 incorrect range (exercise no. 2),
- N3 incorrect solution (exercise no. 3),
- N4 incorrect answer (exercise no. 4).

From the executed analysis of the results follows that four most frequently incident mistakes were:

• E1-2: students plotted correct function graph with absolute value but they assigned incorrect expression of function. This mistake occurred in 21 cases what represents 48% of all solution of the first exercise.

• E3-1: 16 students plotted incorrect function graph with absolute value. This mistake comprises 36% of all solutions of the third exercise.

• Following two mistakes appeared in 10 cases, hence in 23% of all solutions of the second and fourth task:

• E2-1: the students calculated the parabola vertex incorrectly and thus they wrote the range incorrectly.

• E4-2: the students plotted incorrect starting point of the graph hence they didn't take the money saved before saving into consideration.

I also have expected the mistake E4-1 of misunderstanding the exercise by students leading to not solving it. This mistake didn't happen. The mistake E2-6 (confusion of the domain and range notion) and N1 (incorrect function graph or more exactly the line $x = 0$ which is not a function) appeared once and represented approximately 2% of mistakes in the given exercise. From mistakes that occurred I expected 15 and I didn't await 4 of them. The expected mistakes are 87%.

Following the total number of points obtained I can say that students were most successful by solution the verbal exercise No. 4 with 83% successfulness. Something less students attained by solution the exercise No. 1 with 66% successfulness. In exercise no. 2 students obtained 49% success in average. The solutions in the exercise no. 3 were with 36% successfulness the least successful. The successfulness of students by solution of this written exam was 55%. My expectations of successfulness (50%) were fulfilled in the average [1].

5. The survey

At the end of the last lessons I gave survey to the students. Respondents had approximately 10 minutes to accomplish this survey. From the analysis I executed on the solution basis came out interesting results:

To the question: "How difficult was to learn to work with Equation Grapher in your opinion?ąnswered 74% of students positively (very easy or easy). An interesting point is that none of the students marked the answer "very difficult". It is quite simple to learn to use this program and it doesn't take a lot of time in the lessons.

To the second question: "Did the Equation Grapher program help you to understand the repeated curriculum?ąnswered 58% of students (yes or rather yes). I believe this result to be success, because the program helped more than a half of all students.

60% of students answered propitious (yes or rather yes) to the question: Ćan you imagine also the lecture of new curriculum by means of Equation Grapher?". Only 17% of respondents answered negatively. I suppose that this was caused because the students who the program has helped to during the revision would welcome it also in the new curriculum lectures and those who were dissatisfied with this program would rather not meet it in the new curriculum again.

A few inputs from students:

- The program helped me to better understand how the graphs look like.
- At last I am able to imagine graphs of functions as well as what a, b, c parameters are causing.
- It's represented better graphically and more transparent than on the blackboard or in the exercise book.
- The solving procedure isn't evident.
- It's a variegation of the lesson [1].

6. Conclusion

When I compare the students' knowledge before and after the revision through the use of Equation Grapher, it clearly follows that working with this program has helped them. They became conscious of what role play parameters, how the graphs shift, how they change their position, slope. In so far my hypothesis was confirmed and on the ground of my lesson acumens, solution analysis and survey results I can say that "IT, namely the Equation Grapher program can be used effectively in the phase of review, strengthening or revision of the curriculum."

Students enjoy such lessons, they find them interesting. The computer using alone is motivating for them, the students demonstrate unusual interest for mathematics and they show more efforts to cooperate.

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THE INNOVATION ACCESS TO MATHEMATIC EDUCATION AT AN ELEMENTARY SCHOOL

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Abstract. The article is about one of the possible forms of mathematics education at elementary schools. We are getting closer to possible shifts from the traditional education process (desk, crayon, notebook, pen and drawing material) to an innovated one, using a computer and software equipment. This change requires a new approach from teacher as well as realising, what materials are to be prepared for students, or how to manage the lesson etc. The article deals with the term "work sheet for student", "work sheet for teacher", its creation and its exact implementation in educational process. The article is trying to solve the problem of what software can be used in math's education, esp. for its better presentation /imagination/ and explanation to students. Students get familiar with new terms and discover new information by solving a problem-like exercise. They do not consider the information to be a theorem or a sentence, told by the teacher. They get confident about their own truth, because they solve the problem alone, in line with a "work sheetąnd in inspiring environment.

Facts presented in this article are based on realized training in various areas of mathematic. The article is focused on geometry education at elementary schools.

1. Work sheet and its function in the class

Nowadays a student is surrounded by many mass-communication media, much non-fiction literature, many various functional toys, etc. We think that it is not correct to be orientated only to a school-book during education process, but it is necessary to use all available sources of knowledge. Only in this case a student is able to show his skills and opinions. The main tasks of educational process at school are focused on humanization and effective improvement of educational process. The priority is to focus on to student's creativity, responsibility and independence.

The priority must also be the following: building up a virtual educational environment, creating new methods of education, preparation of teachers to new competences for education with the support of information and communication technology. Information and communication technologies are the other necessary literacy in the society on present days. Teachers should also create their own work sheets for theirs classes with the use of these technologies [1].

We will show you a Work Sheet designed as a help for teacher's interpretation of a new lesson. The goal of this work sheet is to let students discover new observation with the didactical software Cabri Geometry II Plus. Guiding tasks will lead students on their way to new knowledge. According to these tasks students will get new information. Work sheet is finished by students' own conclusions written by them. Students should get their result after finishing the tasks of the Practice Files. Teacher will help students to formulate the correct result. Teacher will use open discussion to get students to formulate the right result. If students are not able to find new terms, teacher will illustrate them. The premise is that students will work out tasks by means of didactical software and then they deduce result according to a discussion. Smarter students could put down results alone, but the goal is, that teacher regulates students and advices them on what is important.

Students are developing creative thinking with these Work Sheets (Practice Files). Work with the didactical software Cabri Geometry II Plus could be motivational for them. Many students have a better feeling from lesson, if they know that they can sit at a computer and create something alone. It is much funnier and more effective, than listening to teacher's speech and learning thesis by heart. One of the working materials for mathematic esp. circle education can be an alternative Work Sheet (Working File / Practice Sheet) for education at elementary schools.

These electronic work sheets could be used by students who absented the lesson for some reason and they want to finish their training. For these students it could be interesting to download tasks and Cabri Geometry II Plus drawings from the internet. Of curse there would have to be a model solution added to every working list. Students would be able to correct their solution (result) afterwards. The results of individual work sheet problems could be problematic for students. The solution could be that students would work out work sheets first and then compare their result with the model ones (published in the internet). The premise is that students know how to work with the respective didactic software.

Students have to know basic rules, let us say principles of Cabri Geometry II Plus software, to master the work with work sheets. Manipulation with this software is very easy and students could master the work very quickly.

Teacher could explain basic planar shape at one or two first lessons. Students would master basic terms, changing of colours and thickness, calculation of circle line lengths or calculation of grade size...etc. As an explanation we choose a work sheet (which can be used on the class), that will be used for education "perimeter and length calculating". The Work Sheet consists of a Cabri drawing, where students realize their experiments, measurements and discoveries.

2. Cabri Geometry II and/or Cabri Geometry II Plus

Cabri Geomtery II is not designed to demonstrate PC possibilities. It is purely didactical software. The advantage of the didactical software Cabri Geometry II is mainly its interactivity and dynamics - it allows students to experience many individual situations, by means of which they can get to creation of different universal models faster.

We can actively use dynamic functions of construction in Cabri Geometry (move, follow the path etc.). This software could be used mainly in that part of geometry, where it is necessary to picture the right image of a term. Cabri Geometry is the best loved interactive software in the world. It is a tool for geometry education, excellent for teachers and for students as well.

There is a lot of other geometry software (Geometer's Sketchpad, Euklides, Geom, Cinderella, Felix, Geolog, Géoplan, Geometric Supposer, Geometry Inventor, Geonext, Thales).

3. Work Sheet: circle perimeter and the circle length

The Work Sheet was created according to elementary school education class books. Students are asked to find from four to six things, which have the shape of a circle or a shape where is a circle is simple to be marked - coin, jar of jam etc. Students draw a chart then where the data will be filled (length of the circle - o, diameter - d, ratio - o : d)

Students will get this data from the things that they will have found. Students will get the circle perimeter according to experiment - they spool a sewing thread around and then measure its length.

We tried to apply this task in the work sheet. We constructed a wheel (bicycle wheel). The wheel has an ability to move, rotate and roll on the street. Didactical software is able to give us the dimension, the wheel passes. This is one of the dynamic functions of Cabri Geometry II Plus.

We can easily change the radius of the wheel. Every single student has one work sheet, where it is possible to set up various radiuses. This experiment can show us that given formula is really correct for any radius. Students are filing the chart with the measured data continuously. The chart is in the student's work sheet. Students have the possibility to compare their data.

Cabri drawing to the working list
"perimeter of the circle and length of the circle"

Work Sheet - Length of the circle and perimeter of the circle - model solution

Task:

Figure out perimeter of the bicycle wheel with Cabri drawing.
Click here to open Cabri help drawing.
Fill the following chart:
(ratio o : d round to 3 decimal positions):
On the basis of this drawing and task solutions try to deduce the formula for perimeter calculation for a circle and/or a circle line according this drawing.

Wheel radius r	4 cm	2 cm	1.5 cm	3 cm	3.66 cm	0.9 cm
Wheel diameter d	8 cm	4 cm	3 cm	6 cm	7.32 cm	1.8 cm
Wheel perimeter o	25.11 cm	12.56 cm	9.40 cm	18.85 cm	22.98 cm	5.61 cm
Ratio $o : d$	3.139	3.140	3.133	3.142	3.140	3.117

If the radius of the circle is known, we can calculate the perimeter of the circle with the formula

$$o = 3.14 \cdot 2r$$

If the diameter of the circle is known, we can calculate the perimeter of the circle with the formula

$$o = 3.14 \cdot d$$

4. Students' response to work with Work Sheets and Cabri Geometry II Plus

The presented Work Sheet is only an example of the whole working lists package, which has been designed to the topic Ćircle and Circular line". Each work sheet had been tested by students from two different elementary schools in Bratislava. We found out from short questionnaires, that the task was easy, funny, fast and practical. This work was more precise and easier. Students suggest that it would be interesting to use didactical software for other lessons as well. Some students would prefer to learn the whole geometry lesson with Cabri Geometry II Plus. Other students mentioned exact lessons e.g. Pythagorean Theorem, construction tasks, triangle identity etc.

5. Conclusions

The goal of working lists is to change the form of education. Students should have a feeling that education is much funnier than the common one. Working lists have to be effective for suitable work. They have to deliver new knowledge.

Students should get their result after finishing the working list task. Teacher will help the student to formulate the correct result. Teacher will use open discussion to get students to formulate correct result.

Students have to have the feeling that the result was their own idea.

On the basis of our research we have figured out, that Work Sheets are a good start for informatics and implementation of communication technologies in educational process.

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WORD PROBLEMS DESCRIBING MOVEMENT

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Abstract. Several years we have already been oriented to creation of multimedia teaching aids for teaching mathematics in the elementary school. We have successfully solve out variety of grants in this topic. We are developing international relations and cooperation. The results of our work is a program, in which we have tried to create alternative point of view of motivation and work of pupils, who are solving out word problems describing movement. While fore thinking the didactic questions related with creation of software Word problems describing movements we have focused on the matter of motivation and creation of sufficient amount of separated models of visualization and feedback.

The main aim of school education is to provide pupils with a systematic and balanced structure of concepts and their mutual relationships which enable them to sort particular pieces of information and integrate them in a sensible context of knowledge and life experience.

From this point of view, the teaching of mathematics has a unique position. In its essence, mathematics as a science is a mirror of the real world and on the other hand, thanks to its theoretical background, it is at the same time a source of many applications to the real world. For this reason, the learning process should be conducted as a gradual creation of concepts and their integration into already created knowledge structures. However, this should not be performed in the way of transmitting of encyclopaedic pieces of knowledge from teachers to their pupils leading to algorithmisation without thinking. Research in this field (Hejný, Kuřina, 2001) shows that this transmission creates a certain knowledge apparatus which pupils can reproduce but are not able to use effectively when solving mathematical problems.

In the learning process at school, there is sometimes not enough time devoted to manipulation of objects, numbers, concepts and this can lead to formalisation of learning. The effectively conducted teaching of mathematics with the support of computers can bring involved pupils within a short time many pieces of important information. Various models and the change of objects simulated by a computer enable the pupils to create separated models and make them think about the essence of the presented concepts and also ask questions. This all flows into a better and correct understanding of the given concept.

At the Faculty of Education in České Budějovice, we have been dealing for several years with the question how to make suitable multimedia aids and programs for the teaching of mathematics at primary and lower secondary levels. This fundamental topic has been developed using in various grants, in diploma theses and also in international projects. One diploma thesis is focused on the program Image Logo, enabling its users to create internet multimedia applications, presentations and projects with effective animations. It brings a different view and also motivation to mathematical movement problems.

Word problems seem to be complicated for pupils mainly because of the impossibility of using a single and universal algorithm to solve them. When solving mathematical problems, pupils have difficulties mostly with the understanding of the problem, its analysis, the mathematisation of the problem and finally the process of finding the solution. A significant role is attributed to the subsequent check, and rectification of the solution - feedback which is conducted very often only formally. When thinking about didactical questions connected with the creation of the program called Word Problems about Movement, we focused on the question of motivation, creation of a sufficient amount of separated models of visualisation and feedback (from the point of view of individual needs).

The program can be found at:

`http://www.pf.jcu.cz/stru/katedry/m/diplprace.phtml`

After executing the program, an introductory menu presenting the possibilities of work with the program is displayed. After choosing VSTUPTE / ENTER another site is displayed. According to the text of the given word problem, the user can decide which situation best describes the given problem and the relevant possibility is chosen accordingly. Another site (Figure 1) is very important from the point of view of informal understanding and the correct creation of a universal model. It is essential to make an estimation of the result taking into consideration the presented data of the given problem. This estimation is displayed in the upper left corner of the screen and it is possible to change during the work with the program. At the end, after the display of the final result, this result is compared with the estimation made beforehand. This serves as feedback and makes the pupils think where they have made a mistake and which consideration and estimation were wrong. We think that this part of the program is the most beneficial considering the didactical point of view, as it enables the pupils to see the problem in several steps of understanding and it enables them toform correct imagery and concepts (Kutzler, 1998).

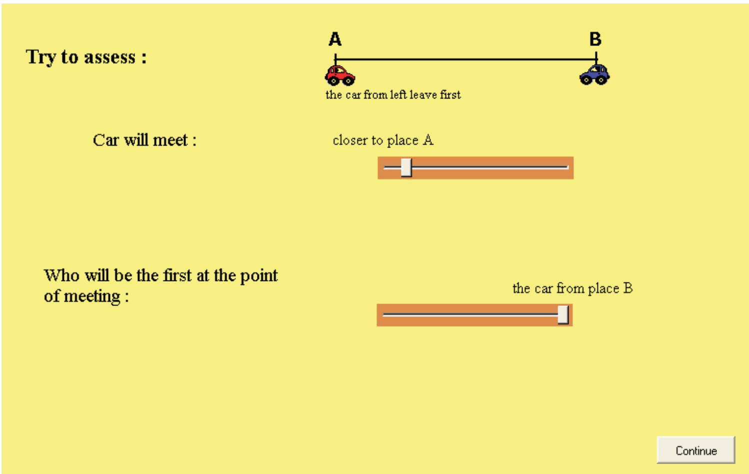

Fig. 1

After the estimation of the result, we enter another site which serves as a motivation part and which can be proceeded with as long as necessary according to the pupils' needs. After entering all the necessary information about the distance and velocity, pressing the button JEĎ/GO makes two cars go from their original places. The pupils can set the cars at any time to their original places and observe the given situation again and change their estimations. After pressing NEXT SITE / DALŠÍ STRÁNKA, the pupils can create even more profoundly their separated models. Buttons for setting the time of the cars' drive are displayed, offering to see the mutual position of the cars in 5, 15, 30 and x minutes.

At the same time there is a time indicator displayed above the road showing the time of the drive in minutes. Still, it is possible to watch the estimation made which enable the pupils to adjust their estimation and compare it with the reality as a check.

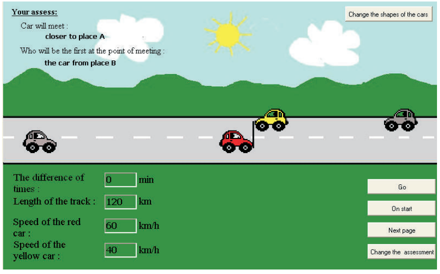

Fig. 2

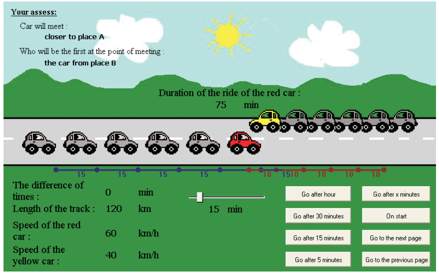

Fig. 3

Another very important fact is the possibility of solving the given problem by estimation of the time interval. By refining and time selection according to the time when the cars meet, the pupils can build and strengthen the presented concept. Another site contains the solution of the given problems and the comparison with the pupils' estimation displayed after pressing the button DISPLAY SOLUTION / ZOBRAZIT VÝSLEDEK.

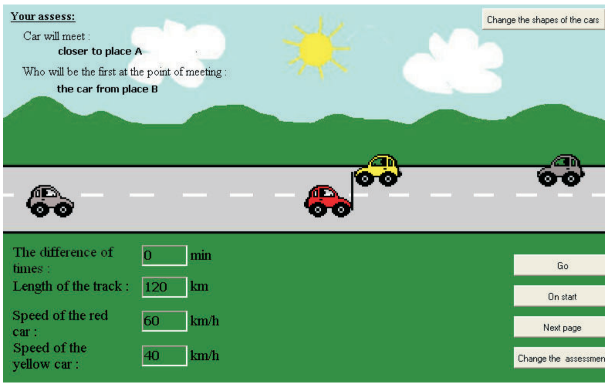

Fig. 4

We have tried the work with this program in several schools and the program has been received with enthusiasm by the pupils as well as by their teachers. The pupils stated that they could understand the setting of the presented word problems and they could analyse them. Other types of problems were not perceived as so much difficult and the work moved ahead more quickly.

One positive aspect of the work with the program was the possibility of trial and error. Scientists and mathematicians attempting to find a solution to a problem also experiment using the method of trial and error.

Why should the pupils abandoned from using these methods? Because of the lack of time? We are against ourselves. Pupils learn only for an effect on others, for their parents, school and very rarely for themselves and only some of them can use their knowledge effectively in real life as they do not see the mutual coherence.

Use of the program can save a lot of time and open the floor to experimentation and the making of separated models. The change could come with RVP as the teachers will not be binded by a uniform curriculum. However, stereotypes are still prevail in Czech education and for this reason the transformation will need some time.

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THE USE OF FINITE EXPANSION OF FUNCTIONS FOR EVALUATION OF LIMITS

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The theory of finite expansions of functions is very helpful in evaluation of complicated limits. One-variable functions are replaced by appropriate polynomials. Extensive chapters in French textbook are devoted to the theory of finite expansions and its applications. In Polish mathematical literature the problem of finite expansions is omitted. More complicated limits are evaluated using l'Hôpital's rule or Taylor and Maclaurin series. However, there exists close connection between those series and finite expansions. The goal of this paper is popularization of the theory of finite expansions on the Polish ground.

The symbol $(DL)_n^{x_o}(f)$ will be used for finite expansion of the nth degree of a function f in the neighborhood of point x_0, whereas the symbol $d^o(P)$ means the degree of one-variable polynomial P with real coefficients.*

Definition 1.

a)

$$(DL)_n^0(f) = P \Leftrightarrow \left\{ d^o(P) \leqslant n \wedge \lim_{x \to 0} \frac{1}{x^n} [f(x) - P(x)] = 0 \right\};$$

b)

$$g(x) = f(x + x_0) \Rightarrow (DL)_n^{x_0}(f) = (DL)_n^0(g);$$

*The symbol DL is an abbreviation of "developpement limite" (finite expansion).

c)

If a function f is defined in the interval $(x_0, +\infty)$ and $g(x) = f\left(\frac{1}{x}\right)$, then $(DL)_n^{+\infty}(f) = (DL)_n^0(g)$.

According to the above definition the finite expansion of a function f in the neighborhood of the point x_0 is a certain polynomial.

Example 1.
A polynomial $P(x) = 1+x+x^2+...+x^n$ is the finite expansion of the nth degree of the function $f(x) = \frac{1}{1-x}$ in the neighborhood of zero, i.e. $(DL)_n^0(f) = P$.

Actually, $d^o(P) = n$, $\lim\limits_{x\to 0} \dfrac{1}{x^n}[f(x)-P(x)] = \lim\limits_{x\to 0} \dfrac{1}{x^n}[\dfrac{1}{1-x}-(1+x+x^2+...+x^n)] =$

$$= \lim_{x\to 0} \frac{1}{x^n(1-x)}[1-(1-x)(1+x+x^2+...+x^n)] =$$

$$= \lim_{x\to 0} \frac{1}{x^n(1-x)} \cdot [1-(1-x^{n+1})] = \lim_{x\to 0} \frac{x^{n+1}}{x^n(1-x)} = \lim_{x\to 0} \frac{x}{1-x} = 0.$$

Theorem 1.
If a function f is defined on an open interval I and $0 \in I$, $n \geqslant 0$, then this function has at most one finite expansion of the degree n in the neighborhood of the point $x = 0$.

Proof.
Suppose that $(DL)_n^0(f) = P$ and $(DL)_n^0(f) = Q$. Then according to Definition (1a) we have:

$$d^o(P) \leqslant n \wedge d^o(Q) \leqslant n. \tag{1}$$

$$\lim_{x\to 0}\frac{1}{x^n}[f(x)-P(x)] = 0 \wedge \lim_{x\to 0}\frac{1}{x^n}[f(x)-Q(x)] = 0. \tag{2}$$

Consider a polynomial R of the form:

$$R(x) = P(x) - Q(x) = C_0 + C_1x + ... + C_kx^k, \qquad \text{where } x \in I,\ k \leqslant n.$$

Suppose also by contradiction that $R(x) \neq 0$. Then at least one of coefficients of this polynomial is non-zero, for example, $C_{j_1} \neq 0$ $(j_1 \leq k)$.
Let $l = \inf\{j : C_j \neq 0\}$.

Then we have:

$$\frac{R(x)}{x^n} = \frac{1}{x^n}(0+0\cdot x+...+0\cdot x^{l-1}+C_lx^l+...+C_kx^k) = \frac{1}{x^{n-l}}(C_l+C_{l+1}x+...+C_kx^{k-l}).$$

If $l = n$, then:

$$\lim_{x\to 0} \frac{R(x)}{x^n} = \lim_{x\to 0} \frac{1}{x^{n-l}} \left(C_l + C_{l+1}x + ... + C_k x^{k-l}\right) = C_l \neq 0. \tag{3}$$

If $l < n$, then:

$$\lim_{x\to 0} \left|\frac{R(x)}{x^n}\right| = +\infty. \tag{4}$$

However, we have:

$$\lim_{x\to 0} \left|\frac{R(x)}{x^n}\right| = \lim_{x\to 0} \left|\frac{P(x)-Q(x)}{x^n}\right| = \lim_{x\to 0} \left|\frac{f(x)-Q(x)}{x^n} - \frac{f(x)-P(x)}{x^n}\right| \leqslant$$

$$\leqslant \lim_{x\to 0} \left(\left|\frac{f(x)-Q(x)}{x^n}\right| + \left|\frac{f(x)-P(x)}{x^n}\right|\right),$$

from which on the basis of (2) we obtain:

$$\lim_{x\to 0} \left|\frac{R(x)}{x^n}\right| = 0 \qquad \text{and therefore} \qquad \lim_{x\to 0} \frac{R(x)}{x^n} = 0,$$

what contradicts (3) and (4). Hence, assumption that $P(x) - Q(x) \neq 0$ leads to contradiction, so $P(x) - Q(x) = 0$ and $P = Q$.

Definition 2.
If

$$P(x) = a_0 + a_1 x + ... + a_n x^n, 0 \leqslant m \leqslant n \quad \text{and} \quad \varphi_m : R^{n+1}[x] \to R^{n+1}[x]$$

$$\text{and} \quad \varphi_m(P(x)) = a_0 + a_1 x + ... + a_m x^m,$$

then the polynomial $\varphi_m(P(x))$ is called the mth degree restriction of a polynomial $P(x)$.*

Let us prove several properties of finite expansions which will be used for evaluation of limits.

Theorem 2.
If a function f is defined on an open interval I and $0 \in I$, $d^o(P) \leqslant n$, then:

$$DL_n^0(f) = P \Leftrightarrow \underset{\varepsilon: I \to R}{\exists} [\lim_{h\to 0} \varepsilon(h) = 0 \wedge \underset{x\in I}{\forall} f(x) = P(x) + x^n \varepsilon(x)].$$

*$R^{m+1}[x]$ is a set of all polynomials in one variable x of at most nth degree.

To prove it is enough to suppose that

$$\varepsilon(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{x^n}[f(x) - P(x)], & \text{if } x \neq 0. \end{cases}$$

At the end of this paper we cite Table 1 with finite expansions of important functions in the neighborhood of the point $x = 0$, i.e. having the form of $f(x) = P(x) + x^n \varepsilon(x)$, where $\varepsilon(x) \underset{x \to 0}{\longrightarrow} 0$.

Theorem 3.

$$(DL)_n^0(f) = P \wedge 0 \leqslant m \leqslant n \Rightarrow (DL)_m^0(f) = \varphi_m(P).$$

Proof.
From the assumption and Theorem 2 it follows that $f(x) = P(x) + x^n \varepsilon_1(x)$, where $\varepsilon_1(x) \underset{x \to 0}{\longrightarrow} 0$. Moreover, $d^o(\varphi_m(P)) = m$. A function f can be written as

$$f(x) = \varphi_m(P(x)) + x^{m+1} \cdot S(x) + x^n \varepsilon_1(x) = \varphi_m(P(x)) + x^m[xS(x) + x^{n-m}\varepsilon_1(x)],$$

with $\lim\limits_{x \to 0}[xS(x) + x^{n-m}\varepsilon_1(x)] = 0$.

On the basis of Theorem 2 we assert that $(DL)_m^0(f) = \varphi_m(P)$.

Theorem 4.

$$\text{If } (DL)_n^0(f) = P \;\text{ and }\; (DL)_n^0(g) = Q, \;\text{ then}$$

a) $(DL)_n^0(\lambda_1 f + \lambda_2 g) = \lambda_1 P + \lambda_2 Q$
(theorem about finite expansion of linear combination of two functions);

b) $(DL)_n^0(f \cdot g) = \varphi_n(P \cdot Q)$
(theorem about finite expansion of a product of two functions);

c) if $g(0) \neq 0$ and χ is a quotient of the nth degree from division of a polynomial P by a polynomial Q according to growing powers (i.e. $P(x) = Q(x) \cdot \chi(x) + x^{n+1}Q_n(x)$ i $d^o(\chi) = n$), then $(DL)_n^0(\frac{f}{g}) = \chi$
(theorem about finite expansion of the division of two functions).

Proof.
It follows from the assumption that

$$d^o(P) \leqslant n,\, d^o(Q) \leqslant n, \tag{5}$$

$$\lim_{x \to 0} \frac{1}{x^n}[f(x) - P(x)] = 0,\, \lim_{x \to 0} \frac{1}{x^n}[g(x) - Q(x)] = 0. \tag{6}$$

a) From (5) and (6) we have:

$$d^o(\lambda_1 P + \lambda_2 Q) \leqslant n,$$

$$\lim_{x\to 0}\frac{1}{x^n}[\lambda_1 f(x) + \lambda_2 g(x) - (\lambda_1 P(x) + \lambda_2 Q(x))] =$$

$$= \lim_{x\to 0}\frac{1}{x^n}[\lambda_1(f(x) - P(x)) + \lambda_2(g(x) - Q(x))] =$$

$$= \lambda_1 \lim_{x\to 0}\frac{1}{x^n}[(f(x) - P(x)] + \lambda_2 \lim_{x\to 0}\frac{1}{x^n}[(g(x) - Q(x))] = 0$$

Hence, on the basis of Definition (1a) we assert that

$$(DL)^0_n(\lambda_1 f + \lambda_2 g) = \lambda_1 P + \lambda_2 Q.$$

b) It follows from the assumptions of theorem that $d^o(\varphi_n(P\cdot Q)) \leqslant n$ Moreover, on the basis of Theorem 2 we have:

$$f(x) = P(x) + x^n\varepsilon_1(x),\ \ g(x) = Q(x) + x^n\varepsilon_2(x),$$
$$\text{where } \lim_{x\to 0}\varepsilon_1(x) = 0,\ \ \lim_{x\to 0}\varepsilon_2(x) = 0.$$

Hence,

$$f(x)\cdot g(x) = [P(x) + x^n\varepsilon_1(x)]\cdot[Q(x) + x^n\varepsilon_2(x)] =$$
$$= P(x)\cdot Q(x) + x^n[Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^m\varepsilon_1(x)\varepsilon_2(x)] =$$
$$= \varphi_n(P(x)Q(x)) + x^{n+1}S(x) + x^n[Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^n\varepsilon_1(x)\varepsilon_2(x)].$$

Therefore, we get:

$$f(x)\cdot g(x) = \varphi_n(P(x)\cdot Q(x)) +$$
$$+ x^n[xS(x) + Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^n\varepsilon_1(x)\varepsilon_2(x)],$$
$$\text{where:}\quad \lim_{x\to 0}[xS(x) + Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^n\varepsilon_1(x)\varepsilon_2(x)] = 0.$$

From Theorem 2 we assert that

$$(DL)^0_n(f\cdot g) = \varphi_n(P\cdot Q).$$

c) It follows from the assumption that $d^o(\chi) = n$. Moreover,

$$\lim_{x\to 0}\quad \frac{1}{x^n}[\frac{f(x)}{g(x)} - \chi(x)] = \lim_{x\to 0}\quad \frac{1}{x^n g(x)}[f(x) - \chi(x)g(x)] =$$

$$= \lim_{x\to 0}\quad \frac{1}{x^n g(x)}[P(x) + x^n\varepsilon_1(x) - \chi(x)(Q(x) + x^n\varepsilon_2(x))] =$$

$$= \lim_{x\to 0} \frac{1}{x^n g(x)}[P(x) + x^n\varepsilon_1(x) - \chi(x)(Q(x)) - \chi(x)x^n\varepsilon_2(x)] =$$

$$= \lim_{x\to 0} \frac{1}{x^n g(x)}[P(x) + x^n\varepsilon_1(x) - (P(x) - x^{n+1}Q_n(x)) - \chi(x)x^n\varepsilon_2(x)] =$$

$$= \lim_{x\to 0} \frac{1}{g(x)}[\varepsilon_1(x) + xQ_n(x) - \chi(x)\varepsilon_2(x)] = 0.$$

Hence, $\quad (DL)^0_n(\frac{f}{g}) = \chi.$

Theorem 5. (About finite expansion of composition of functions)

$$\big[(DL)^0_n(f) = P \wedge (DL)^0_m(g) = Q \wedge f(0) = 0 \wedge f \neq 0 \wedge m\cdot k \geqslant n\big] \Rightarrow$$
$$\Rightarrow (DL)^0_n(g\circ f) = \varphi_n(Q\circ P),$$

where k is the power of the lowest term of a polynomial P.

Proof.
It follows from the assumption and Theorem 2 that

$$f(x) = P(x) + x^n\varepsilon_1(x), \quad \text{where } \varepsilon_1(x) \underset{x\to 0}{\rightarrow} 0, \text{ hence } f(0) = P(0) = 0.$$

There exists

$$k > 0, \text{ such that } P(x) = x^k P_1(x).$$

Then we obtain:

$$f(x) = P(x) + x^n\varepsilon_1(x) = x^k P_1(x) + x^n\varepsilon_1(x) = x^k(P_1(x) + x^{n-k}\varepsilon_1(x)).$$

On the basis of Theorem 2, a function g has the following property:

$$g(y) = Q(y) + y^m\varepsilon_2(y), \quad \varepsilon_2(y) \underset{y\to 0}{\rightarrow} 0. \tag{7}$$

From (7) we get for the composition of functions $g \circ f$

$$(g \circ f)(x) = g(f(x)) = Q(f(x)) + [f(x)]^m \varepsilon_2(f(x)) =$$

$$= Q(f(x)) + [x^k(P_1(x) + x^{n-k}\varepsilon_1(x))]^m \cdot \varepsilon_2(f(x)).$$

In the interval $(f(x), P(x))$ we use Lagrange's mean value theorem for a polynomial Q

$$\frac{Q(f(x)) - Q(P(x))}{f(x) - P(x)} = Q'(C_x), \text{ where } C_x \in (f(x), P(x)),$$

therefore,

$$Q(f(x)) - Q(P(x)) = [f(x) - P(x)]Q'(C_x) = x^n \varepsilon_1(x) Q'(C_x).$$

A function $g \circ f$ gets the following value:

$$(g \circ f)(x) = Q(P(x)) + x^n \varepsilon_1(x) Q'(C_x) + x^{km}[P_1(x) + x^{n-k}\varepsilon_1(x)]^m \varepsilon_2(f(x)).$$

Transforming a polynomial $Q(P(x)$

$$Q(P(x)) = \varphi_n(Q(P(x)) + x^{n+1}S(x))$$

we obtain

$$(g \circ f)(x) = \varphi_n(Q(P(x)) + x^{n+1} \cdot S(x) + x^n \varepsilon_1(x) Q'(C_x) + x^{km}[P_1(x) + x^{n-k}\varepsilon_1]^m \varepsilon_2(f(x)).$$

Hence,

$$\lim_{x \to 0} \tfrac{1}{x^n}[(g \circ f)(x) - \varphi_n((Q \circ P)(x))] =$$

$$= \lim_{x \to 0}\{xS(x) + \varepsilon_1(x) Q'(x) + x^{km-n}[P_1(x) + x^{n-k}\varepsilon_1(x)]^m \varepsilon_2(f(x))\} = 0,$$

as $km - n \geqslant 0$.
On the basis of Definition 1a we assert that

$$(DL)_n^0(g \circ f) = \varphi_n(Q \circ P).$$

Corollary 1.

$$(DL)_n^0(f) = P \wedge (DL)_n^0(g) = Q \wedge f(0) = 0 \Rightarrow (DL)_n^0(g \circ f) = \varphi_n(Q \circ P).$$

This corollary follows immediately from Theorem 5 for $k = 1$ and $m = n$.

Theorem 6. (The Taylor–Maclaurin finite expansion.)
If a function f is differetiable n times on an open interval I containing zero, $f^{(n)}$ is continuous at zero and

$$P(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0), \quad \text{then} \quad (DL)_n^0(f) = P.$$

This Theorem follows from Theorem 2 and the Taylor–Maclaurin theorem. It should be noted that a finite expansion of a function f can exist, though the derivative $f^{(n)}(0)$ does not exist. For example, a function f defined as:

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ x^3 \sin(\frac{1}{x}), & \text{if } x \neq 0, \end{cases}$$

does not have the derivative $f''(0)$, since

$$f'(x) = \begin{cases} 0, & \text{if } x = 0, \\ 3x^2 \sin(\frac{1}{x}) - x\cos(\frac{1}{x}), & \text{if } x \neq 0, \end{cases}$$

but the limit of

$$\frac{f'(x) - f'(0)}{x} = 3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

does not exist when $x \to 0$.

On the other hand, the zero polynomial O is a finite expansion of the second degree for the function f in the neighborhood of the point $x = 0$ as

$$d^o(0) = -\infty \leq 2 \quad \text{and} \quad \lim_{x \to 0} \frac{1}{x^2}[f(x) - O(x)] = \lim_{x \to 0}(x \sin(x)) = 0.$$

Theorem 7 (about integration and differentiation of finite expansions):

a) If a function f is continuous on an open interval I, $0 \in I$, $(DL)_n^0(f) = P$, f is the derivative of a function $g : I \to R$ *and* $Q(x) = g(0) + \int_0^x P(t)dt$, then $D_{n+1}^0(g) = Q$.
b) If $f^{(n)}$ is continuous on an open interval I, $0 \in I$ and $(DL)_n^0(f) = P$, then $D_{n-1}^0(f') = P'$.

Proof.
a) A continuous function f can be written as

$$f(x) = P(x) + x^n \varepsilon(x),$$

where a function

$$\varepsilon(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{x^n}(f(x) - P(x)), & \text{if } x \neq 0 \end{cases}$$

is continuous on the interval I. From the assumption we have $g'(x) = f(x)$ for $x \in I$, therefore

$$\int_0^x g'(t)\,dt = \int_0^x f(x)\,dt = \int_0^x P(t)\,dt + \int_0^x t^n \varepsilon(t)\,dt,$$

hence

$$g(x) - g(0) = \int_0^x P(t)\,dt + \int_0^x t^n \varepsilon(t)\,dt$$

or

$$g(x) - Q(x) = \int_0^x t^n \varepsilon(t)\,dt.$$

Let us show that

$$\lim_{x\to 0} \frac{1}{x^{n+1}}[g(x) - Q(x)] = \lim_{x\to 0} \frac{1}{x^{n+1}} \int_0^x t^n \varepsilon(t)\,dt = 0.$$

In fact, substituting $u = \frac{t}{x}$ in the last integral we obtain:

$$\frac{1}{x^{n+1}} \int_0^x t^n \varepsilon(t)\,dt = \frac{1}{x^{n+1}} \int_0^1 (ux)^n \varepsilon(ux) x du = \int_0^1 u^n \varepsilon(ux)\,du.$$

Let $\varepsilon_1(x) = \int_0^1 u^n \varepsilon(ux)\,du$. As a function $\varepsilon(x)$ is continuous at the point $x = 0$, i.e. the Cauchy condition is filfilled:

$$\underset{\omega>0}{\forall} \quad \underset{\delta_\omega>0}{\exists} \quad \underset{x}{\forall} (|x| < \delta_\omega \Rightarrow |\varepsilon(x)| < \omega),$$

hence, for $|x| < \delta_\omega$ we have $|\varepsilon(x)| < \omega$. Then for a function $\varepsilon_1(x)$ we obtain:

$$|\varepsilon_1(x)| = \left| \int_0^1 u^n \varepsilon(ux)\,du \right| \leqslant \int_0^1 u^n |\varepsilon(ux)|\,du \leqslant \int_0^1 \omega u^n\,du = \omega \left[\frac{u^{n+1}}{n+1} \right]_0^1 = \frac{\omega}{n+1}$$

or $\lim\limits_{x\to 0} \varepsilon_1(x) = 0$.

As

$$\lim_{x\to 0} \frac{1}{x^{n+1}}[g(x) - Q(x)] = \lim_{x\to 0} \int_0^1 u^n \varepsilon(ux)\,du = \lim_{x\to 0} \varepsilon_1(x) = 0$$

$$\text{and} \quad d^0(g) \leqslant n+1, \quad \text{then} \quad (DL)^0_{n+1}(g) = Q.$$

b) Under continuity assumption for a function $f^{(n)}$ we can use the Maclaurin formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + x^n\varepsilon(x)$$

On the other hand, on the basis of assumption $(DL)^0_n(f) = P$ we have $f(x) = P + x^n\varepsilon(x)$. Therefore,

$$P(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0).$$

Then $f'(x) = P'(x) + x^{n-1}\varepsilon_1(x)$, where $\varepsilon_1(x) \underset{x\to 0}{\to} 0$, and $d^o(P') \leqslant n-1$.

Hence, we get $(DL)^0_{n-1}(f') = P'$.

Example 2.

Evaluate the limit $\lim\limits_{x\to 0}\dfrac{e^x - \cos(x) - x}{x - \ln(1+x)}$ using the finite expansion.

Individual functions can be represented by the following finite expansions of the second degree (see Table 1):

$e^x = 1 + x + \frac{x^2}{2} + x^2\varepsilon_1(x)$,

$\cos(x) = 1 - \frac{x^2}{2} + x^2\varepsilon_2(x)$,

$\ln(1+x) = x - \frac{x^2}{2} + x^2\varepsilon_3(x)$,

$e^x - \cos(x) - x = x^2 + x^2\varepsilon_4(x)$,

$x - \ln(1+x) = \frac{x^2}{2} + x^2\varepsilon_5(x)$.

The functions $e^x - \cos(x) - x$, $x - \ln(1+x)$ have the finite expansions x^2 and $\frac{x^2}{2}$, respectively. Dividing the polynomial x^2 by the polynomial $\frac{x^2}{2}$ we obtain the polynomial 2. Therefore, it follows from Theorem 4c that

$$\frac{e^x - \cos(x) - x}{x - \ln(1+x)} = 2 + x^2\varepsilon(x), \text{ where } \varepsilon(x) \underset{x\to 0}{\to} 0,$$

$$\text{hence, } \lim_{x\to 0}\frac{e^x - \cos(x) - x}{x - \ln(1+x)} = 2.$$

Example 3.

Find the finite expansion of the 5th degree for the function

$$h(x) = \tan(x - \sin(x))$$

in the neighborhood of the point $x = 0$.

Let $f(x) = x - \sin(x)$, $g(y) = \tan(y)$.
As $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^6\varepsilon_1(x)$ (see Table 1),
then $f(x) = \frac{x^3}{3!} - \frac{x^5}{5!} + x^6\varepsilon_2(x)$,
where $\varepsilon_2(x) \underset{x\to 0}{\longrightarrow} 0$. A function f has the following finite expansion of the 5th degree

$$P(x) = \frac{x^3}{3!} - \frac{x^5}{5!}.$$

The least power in the polynomial P is $k = 3$. Therefore, using the condition

$$m \cdot k \geqslant n = 5 \text{ (Theorem 5) we obtain } m \geqslant \frac{5}{3} \text{ or } m = 2.$$

For the function g we calculate the finite expansion of the second degree. As $\tan(y) = y + y^2\varepsilon(y)$ (Table 1), then this expansion is equal to $Q(y) = y$.

The composition of functions $h(x) = (g\circ f)(x) = g(f(x)) = g(x-\sin(x)) = = \tan(x - \sin(x))$ has the following finite expansion of the 5th degree (Theorem 5):

$$\varphi_5((Q \circ P)(x)) = \varphi_5\left(Q(P(x))\right) = \varphi_5\left(\frac{x^3}{3!} - \frac{x^5}{5!}\right) = \frac{x^3}{3!} - \frac{x^5}{5!}.$$

Example 4.
Find the finite expansion of the nth degree for the function $g(x) = \ln\frac{1}{1-x}$ using the finite expansion of the $(n-1)$th degree for the function $f(x) = \frac{1}{1-x}$. Using the finite expansion of the function g evaluate the limit

$$\lim_{x\to 0}(1-x)\ln\left(\frac{1}{1-x}\right).$$

Since

$$f(x) = g'(x) \quad \text{and} \quad f(x) = \frac{1}{1-x} = 1+x+x^2+\ldots+x^{n-1}+x^{n-1}\varepsilon(x) \text{ (Table 1)},$$

we can use Theorem 7a.
The finite expansion of the nth degree for the function g is as follows:

$$Q(x) = g(0) + \int_0^x P(t)\,dt, \quad \text{where} \quad P(t) = 1 + t + t^2 + \ldots + t^{n-1}.$$

Then

$$\begin{aligned} Q(x) &= \ln(1) + \textstyle\int_0^x (1 + t + t^2 + \ldots + t^{n-1})\,dt = \\ &= \left[t + \tfrac{t^2}{2} + \tfrac{t^3}{3} + \ldots + \tfrac{t^n}{n}\right]_0^x = x + \tfrac{x^2}{2} + \tfrac{x^3}{3} + \ldots + \tfrac{x^n}{n}. \end{aligned}$$

The function g has the form:

$$g(x) = \ln(\frac{1}{1-x}) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n} + x^n \varepsilon(x), \quad \text{where} \quad \lim_{x\to 0} \varepsilon_1(x) = 0.$$

The finite expansion of the second degree for the function $h(x) = (1-x)\ln\frac{1}{1-x}$ is:

$$\varphi_2\left[(1-x)\left(x + \frac{x^2}{2} + \ldots + \frac{x^n}{n}\right)\right] = x - \frac{1}{2}x^2.$$

Then

$$(1-x)\ln\frac{1}{1-x} = x + \frac{2}{3}x^2 + x^2\varepsilon_2(x), \quad \text{where} \quad \varepsilon_2(x) \underset{x\to 0}{\to} 0)$$

and $\lim_{x\to 0}(1-x)\ln\frac{1}{1-x} = 0$.

It is obvious that this limit can be easily evaluated, but we would like to show how to use Theorem 7a.

Table 1. Representation of several functions by finite expansions

No.	Function
1.	$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + x^n\varepsilon(x)$
2.	$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + x^n\varepsilon(x)$
3.	$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x)$
4.	$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x)$
5.	$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + x^6\varepsilon(x)$
6.	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^n}{n} + x^n\varepsilon(x)$

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RECURRENT EQUATIONS FOR THE ARITHMETICAL AND GEOMETRICAL SEQUENCES OF HIGHER DEGREE

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Abstract. Recurrent equations concern relationships between some (in general in a neighbourhood) elements of sequences. By these equations one can evaluate an arbitrary element of such sequences. In this paper we consider recurrent equations for the arithmetical and geometrical sequences of higher degree. We also give some properties of these sequences.

Let us start with a short introduction to definitions and properties of the sequences of higher degree. For the given sequence $\{a_n\}$ we define the sequence of the m-th differences $\{\Delta^m a_n\}$ and the sequence of the m-th quotients $\{q_{n,m}\}$ in the following way:

$$\begin{cases} \Delta^1 a_n = a_{n+1} - a_n, \\ \Delta^{m+1} a_n = \Delta^m a_{n+1} - \Delta^m a_n, \quad n, m \in N \setminus \{0\}. \end{cases} \tag{1}$$

$$\begin{cases} q_{n,1} = \frac{a_{n+1}}{a_n}, \\ q_{n,m+1} = \frac{q_{n+1,m}}{q_{n,m}}, \quad n, m \in N \setminus \{0\}. \end{cases} \tag{2}$$

The following two tables show elements of the considered sequences:

$$
\begin{array}{c|ccccc}
a_1 & & & & & \\
 & \Delta a_1 & & & & \\
a_2 & & \Delta^2 a_1 & & & \\
 & \Delta a_2 & & \Delta^3 a_3 & & \\
a_3 & & \Delta^2 a_2 & & & \\
 & \Delta a_3 & & \Delta^3 a_2 & & \ddots \\
a_4 & & \Delta^2 a_3 & & \vdots & \\
 & \Delta a_4 & & \vdots & & \\
a_5 & & \vdots & & & \\
 & \vdots & & & & \\
\vdots & & & & &
\end{array}
$$

$$
\begin{array}{c|ccccc}
a_1 & & & & & \\
 & q_{11} & & & & \\
a_2 & & q_{12} & & & \\
 & q_{21} & & q_{13} & & \\
a_3 & & q_{22} & & q_{14} & \\
 & q_{31} & & q_{23} & & \ddots \\
a_4 & & q_{32} & & \vdots & \\
 & q_{41} & & \vdots & & \\
 & \vdots & & & & \\
\vdots & & & & &
\end{array}
$$

The arithmetical and geometrical sequences of higher degree are defined as follows (see [1-3]):

Definition 1. The sequence $\{a_n\}$ is called the arithmetical sequence of the k-th degree ($k = 1, 2, 3, \dots$) if and only if the sequence $\{\Delta^k a_n\}$ is constant and $\{\Delta^k a_n\} \neq 0$. Any constant sequence is called arithmetical sequence of the 0-th degree.

Definition 2. The sequence $\{a_n\}$ is called the geometrical sequence of the k-th degree ($k = 1, 2, 3, \dots$) if and only if the sequence $q_{n,k}$ is constant and $q_{n,k} \neq 1$. Any constant sequence is called geometrical sequence of the 0-th degree.

Theorem 1. Any arithmetical sequence $\{a_n\}$ of the k-th degree has the following properties:

a) $a_n = \binom{n-1}{0}a_1 + \sum_{i=1}^{k} \binom{n-1}{i}\Delta^i a_1,$

b) $s_n = \sum_{i=1}^{n} a_i = \binom{n}{1}a_1 + \sum_{i=1}^{k} \binom{n}{i+1}\Delta^i a_1,$

c) $\Delta^1 a_n = a_{n+1} - a_n = \sum_{i=1}^{k} \binom{n-1}{i-1}\Delta^i a_1.$

Theorem 2. Any geometrical sequence $\{a_n\}$ of the k-th degree has the following properties:

a) $a_n = a_1 \cdot \prod_{i=1}^{k} q_{1,i}^{\binom{n-1}{i}},$

b) $\pi_n = \prod_{i=1}^{n} a_i = a_1^n \cdot \prod_{i=1}^{k} q_{1,i}^{\binom{n}{i+1}},$

c) $q_{n,1} = \frac{a_{n+1}}{a_n} = \prod_{i=1}^{k} q_{1,i}^{\binom{n-1}{i-1}}.$

Theorem 3. For any sequence $\{a_n\}$, the general formulas of the sequences $\{\Delta^m a_n\}$ and $\{q_{n,m}\}$ given by (1) and (2), respectively, have the following forms:

$$\Delta^m a_n = \sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{n+m-i}, \tag{3}$$

$$q_{n,m} = \prod_{i=0}^{m} a_{n+m-1}^{(-1)^i \binom{m}{i}}. \tag{4}$$

Proof of the formula (3). The proof is by induction on the parameter m (for every $n \in N_+$). Let $m = 1$. Then the formula (3) is true, since $\Delta^1 a_n = a_{n+1} - a_n = (-1)^0\binom{1}{0}a_{n+1} + (-1)^1\binom{1}{1}a_n$. Let us assume now that formula (3) holds for $m = k$. By inductive assumption we obtain $\Delta^{k+1}a_n = \Delta^k a_{n+1} - \Delta^k a_n = \sum_{i=0}^{k}(-1)^i\binom{k}{i}a_{n+1+k-i} - \sum_{i=0}^{k}(-1)^i\binom{k}{i}a_{n+k-i} = \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}a_{n+k+1-i}$. So, the formula (3) holds for $m = k + 1$. By the principle of mathematical induction we conclude that the formula holds for every natural number $m \in N_+$.

Proof of the formula (4). Let $m = 1$. Then the formula (4) holds, since $q_{n,1} = \frac{a_{n+1}}{a_n} = a_{n+1}^{(-1)^0\binom{1}{0}} \cdot a_n^{(-1)^1\binom{1}{1}}$. Assume now that (4) holds for $m = k$. Using the induction hypothesis the expression $q_{n,k+1}$ can be rewritten as:

$$q_{n,k+1} = \frac{q_{n+1,k}}{q_{n,k}} = \frac{\prod_{i=0}^{k} a_{n+1+k-i}^{(-1)^i\binom{k}{i}}}{\prod_{i=0}^{k} a_{n+k-i}^{(-1)^i\binom{k}{i}}} = \prod_{i=0}^{k+1} a_{n+k+1-i}^{(-1)^i\binom{k+1}{i}}.$$

Thus the formula holds for $m = k + 1$. So, by the principle of induction we can conclude that the formula (4) holds for every $m \in N \setminus \{0\}$.

Theorem 4.

a) If the sequence $\{a_n\}$ is the arithmetical sequence of the k-th degree, then the recurrent equation

$$\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}a_{n+k+1-i}=0 \tag{5}$$

is satisfied by this sequence.

b) If the sequence $\{a_n\}$ is the geometrical sequence of the k-th degree, then it satisfies the following recurrent equation:

$$\prod_{i=0}^{k+1}a_{n+k+1-i}^{(-1)^i\binom{k+1}{i}}=1. \tag{6}$$

Easy proofs will be omitted.

Theorem 5. The sequence $\{a_n\}$ is the arithmetical sequence of the k-th degree if and only if the sequence $\{u_n\}$ of the form $u_n = A\cdot r^{a_n}$ (where $r\in R_+\setminus\{1\}$ and $A\neq 0$) is the geometrical sequence of the k-th degree.

Proof. On the basis of Theorem 3 (formulas (3) and (4)) we have:

$$q_{n,k}=\frac{q_{n+1,k-1}}{q_{n,k-1}}=\frac{\prod_{i=0}^{k-1}u_{n+1+k-1-i}^{(-1)^i\binom{k-1}{i}}}{\prod_{i=0}^{k-1}u_{n+k-1-i}^{(-1)^i\binom{k-1}{i}}}=\frac{\prod_{i=0}^{k-1}(Ar^{a_{n+1+k-1-i}})^{(-1)^i\binom{k-1}{i}}}{\prod_{i=0}^{k-1}(Ar^{a_{n+k-1-i}})^{(-1)^i\binom{k-1}{i}}}=$$

$$=\frac{A^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}}\cdot r^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+1+k-1-i}}}{A^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}}\cdot r^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+k-1-i}}}=$$

$$=r^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+1+k-1-i}-\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+k-1-i}}=$$

$$=r^{\Delta^{k-1}a_{n+1}-\Delta^{k-1}a_n}=r^{\Delta^k a_n}.$$

Hence, it is shown, that $q_{n,k}=const\neq 1$ iff $\Delta^k a_n=const\neq 0$. Therefore, by Definitions 1 and 2 it is obvious, that Theorem 5 holds.

Corollary 1. If the sequence $\{a_n\}$ is the arithmetical sequence of the kth degree, then the sequence defined by the formula $u_n = Ar^{a_n}$ ($A\neq 0$, $r\in R_+\setminus\{1\}$) satisfies the equation (6).

It follows from Theorems 5 and 4b.

In order to solve the equation (5) we use the theory of homogeneous linear recurrent equations. Recall, that the recurrent equation of order k is of the form:

$$u_{n+k} = d_1 u_{n+k-1} + d_2 u_{n+k-2} + \ldots + d_{k-1} u_{n+1} + d_k u_n \ , \quad (d_k \neq 0) \tag{7}$$

has the following characteristic equation:

$$r^k = d_1 r^{k-1} + d_2 r^{k-2} + \ldots + d_{k-1} r + d_k. \tag{8}$$

Theorem 6 (see [4]).

a) If the characteristic equation (8) has k distinct roots $u_n^{(i)} = r_i^n$, then the general solution of (7) has the form:

$$u_n = \sum_{i=1}^{k} c_i r_i^n, \tag{9}$$

where the coefficients $c_1, c_2, \ldots, c_k$ are constants. One can see, that the general solution (9) is linear combination of particular solutions of the equation (7): $u_n^{(1)} = r_1^n$, $u_n^{(2)} = r_2^n$, $\ldots$, $u_n^{(k)} = r_k^n$.

b) If the characteristic equation (8) has distinct roots $r_1, r_2, \ldots, r_t$, $(t \leq k)$, of multiplicities $k_1, k_2, \ldots, k_t$, where $\sum_{i=1}^{t} k_i = k$, then the characteristic polynomial $\varphi(r) = r^k - (d_1 r^{k-1} + d_2 r^{k-2} + \ldots + d_{k-1} r + d_k)$ can be factored as: $\varphi(r) = (r - r_1)^{k_1} \cdot (r - r_2)^{k_2} \cdot \ldots \cdot (r - r_t)^{k_t}$. Then, the general solution of the equation (7) has the form:

$$u_n = \sum_{i=1}^{t} r_i^n \cdot \sum_{j=0}^{k_i - 1} c_{ij} n^j, \tag{10}$$

where c_{ij} $(i = 1, \ldots, t,\ j = 0, 1, \ldots, k_i - 1)$ are an arbitrary constants.

After expanding the formula (10) will be of the form:

$$\begin{gathered} u_n = r_1^n (c_{1,0} n^0 + c_{1,1} n^1 + \ldots + c_{1,k_1-1} n^{k_1-1}) + \\ + r_2^n (c_{2,0} n^0 + c_{2,1} n^1 + \ldots + c_{2,k_2-1} n^{k_2-1}) + \\ + \ldots + r_t^n (c_{t,0} n^0 + c_{t,1} n^1 + \ldots + c_{t,k_t-1} n^{k_t-1}). \end{gathered}$$

Any particular solution of (7) can be obtained from the general solution by suitable choice of constants. These constants are determined by the initial conditions.

Let the formula (5) has the expanded form:

$$(-1)^0\binom{k+1}{0}a_{n+k+1}+(-1)^1\binom{k+1}{1}a_{n+k}+\ldots+$$

$$+(-1)^k\binom{k+1}{k}a_{n+1}+(-1)^{k+1}\binom{k+1}{k+1}a_n=0. \tag{11}$$

Corollary 2. The general solution of (11) has the form:

$$a_n=c_0n^0+c_1n^1+\ldots+c_kn^k. \tag{12}$$

Proof. The equation (11) has the following characteristic equation:

$$(-1)^0\binom{k+1}{0}r^{k+1}+(-1)^1\binom{k+1}{1}r^k+\ldots+$$

$$+(-1)^k\binom{k+1}{k}r+(-1)^{k+1}\binom{k+1}{k+1}=0. \tag{13}$$

It is easy to see, that the left side of (13) is the expansion of the formula $(r-1)^{k+1}$ by the Newton's binomial. Thus $(r-1)^{k+1}=0$ if $r=1$. So, the characteristic equation (13) has the only one root $r_1=1$ with multiplicity $k+1$. Theorem (6) (the formula (10)) states, that the general solution of (11) is the sequence $\{a_n\}$ given by (12).

Example. Find a closed-form expression for the sequence

$$\{a_n\}=(0,0,2,6,12,20,30,42,\ldots).$$

The general formula of the sequence $\{a_n\}$ is determined by Theorem 4a and Corollary 1. The sequence $\{a_n\}$ is the arithmetical sequence of the 2-nd degree and its recurrent equation has the following characteristic equation (see (5) and (11) for $k=2$): $(r-1)^3$. The only root of that is the number 1 with multiplicity 3. Using the formula (12) we obtain $a_n=c_0n^0+c_1n^1+c_2n^2$, i.e. $a_n=c_0+c_1n+c_2n^2$. The constants c_0,c_1,c_2 are determined by the initial conditions: $a_1=0$, $a_2=0$, $a_3=2$. Then, we have:

$$\begin{cases} a_1=0=c_0+c_1+c_2\\ a_2=0=c_0+2c_1+4c_2\\ a_3=2=c_0+3c_1+9c_2\end{cases}$$

The solutions are: $c_0=2$, $c_1=-3$, $c_2=1$. Finally, the given sequence is of the form: $a_n=2-3n+n^2$. We can also use Theorem 1a to find the general formula of that sequence.

Theorem 7. The sequence $\{a_n\}$ is the arithmetical sequence of the kth degree if and only if its general formula is a polynomial of degree k.

Proof. Let $\{a_n\}$ be an arithmetical sequence of the kth degree. This sequence fulfils (11). By Corollary 2 we obtain $a_n = c_0 n^0 + c_1 n^1 + \ldots + c_k n^k$. By assumption and Definition 1 we have the following condition: $\Delta^k a_n = const \neq 0$, which implies $c_k \neq 0$, what means that the polynomial $a_n = c_0 n^0 + c_1 n^1 + \ldots + c_k n^k$ is of degree k.

Now, let us assume that $a_n = c_0 n^0 + c_1 n^1 + \ldots + c_k n^k$ and $c_k \neq 0$. Thus, by the equation (1), we obtain $\Delta^k a_n = const \neq 0$. According to Definition 1 we have that the sequence $\{a_n\}$ is the arithmetical sequence of the k-th degree, which completes the proof.

Lemma 1. If sequences $\{a_n^{(j)}\}$, $j = 1, 2, \ldots, l$ satisfy (5), then the sequence $\{u_n\}$ of the form $u_n = A \cdot r_1^{a_n^{(1)}} \cdot r_2^{a_n^{(2)}} \cdot \ldots \cdot r_l^{a_n^{(l)}}$, where $r_1, r_2, \ldots, r_l \in R_+ \setminus \{1\}$ and $A \neq 0$, satisfies (6).

Proof. Let sequences $\{a_n^{(j)}\}$, for $j = 1, 2, \ldots, l$, satisfy (5). Then, the following condition holds:

$$\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} a_{n+k+1-i}^{(j)} = 0, \quad j = 1, 2, \ldots, l. \tag{14}$$

So, we have:

$$\prod_{i=0}^{k+1} u_{n+k+1-i}^{(-1)^i \binom{k+1}{i}} = \prod_{i=0}^{k+1} \left(A \cdot r_1^{a_{n+k+1-i}^{(1)}} \cdot \ldots \cdot r_l^{a_{n+k+1-i}^{(l)}} \right)^{(-1)^i \binom{k+1}{i}} =$$

$$= A^{\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i}} \cdot r_1^{\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} a_{n+k+1-i}^{(1)}} \cdot \ldots \cdot r_l^{\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} a_{n+k+1-i}^{(l)}} =$$

$$= A^0 \cdot r_1^0 \cdot \ldots \cdot r_l^0 = 1.$$

It is easy to see that the sequence $\{u_n\}$ satisfies (6).

Corollary 3. If $W_1, W_2, \ldots, W_l$ are polynomials of variable n of degree at most k, then the sequence $\{u_n\}$ of the form: $u_n = A \cdot r_1^{W_1(n)} \cdot \ldots \cdot r_l^{W_l(n)}$, where $r_1, r_2, \ldots, r_l \in R_+ \setminus \{1\}$ and $A \neq 0$, satisfies (6).

Proof. The sequences $a_n^{(j)} = W_j(n)$ are the arithmetical sequences of at most the k-th degree (Theorem 7), therefore they satisfy the equation (5) (Theorem 4a). Now, by Lemma 1 it follows, that $\{u_n\}$ satisfies (6). Notice, that if $W_1, W_2, \ldots, W_l$ are polynomials of variable n and $d^0(W_j) \leq k$ for $j = 1, 2, \ldots, l$ and at least one of these polynomials is of degree k, then the sequence $u_n = A \cdot r_1^{W_1(n)} \cdot \ldots \cdot r_l^{W_l(n)}$ (where $r_1, r_2, \ldots, r_l \in R_+ \setminus \{1\}$ and $A \neq 0$) is the geometrical sequence of the k-th degree.

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ANALOGICAL PROBLEMS IN THE PLANE AND SPACE

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Abstract. It is possible to form analogical problems concerning 2D objects in the space for 3D objects. As for students profitable is to learn how to formulate and solve this 3D problems, especially those ones whose solution requests non-trivial modification of the solution of the original problem.

1. Formulation and solving analogical problems

When students learn to formulate to a given problem the analogical one in the space of another dimension, they create the spectrum of analogical conceptions in these spaces. They treat with not only the spectrum of basic objects *point – line – plane – space* but also another spectrum, for example *half line, half plane, semi-space* and subsequently then *segment* (intersection of two half lines), *triangle* (intersection of three half planes), *tetrahedron* (intersection of four semi-spaces), or *segment, square* and *cube* (like sets of points, whose coordinates form one-dimensional, two-dimensional or three-dimensional interval), or *pair of points, circle* and *sphere* (like set points that have given distance from a fixed point). Solving analogical problem in higher dimension can sometimes use the benefit from the same intellectual draft like the original problem, another time is solving of the analogical problem in new dimension non-trivial, however. The series of these problems is known, for instance some simple numerical problems. If we have for example in the plane the square with the side of the length n cm and if we both ćolorąll its sides and cut it to squares with sides of 1 cm, we can then calculate, how much of them has colored two sides or one or no side. Analogical task we can formulate in the space for cube, which has colored all walls.

2. Problems related to triangle and tetrahedron

Investigation of medians and median points can culminate by symbolic equations for median points of segment, triangle and tetrahedron in the form

$$T = \tfrac{A+B}{2}, \text{ respectively } T = \tfrac{A+B+C}{3}, \text{ respectively } T = \tfrac{A+B+C+D}{4}.$$

The construction of a triangle circumscribed circle by means of bisectors of sides is known. Then students analogously search sets of all points in the space, that are equidistant from two, three or four vertexes of tetrahedron. They find out also interesting piece of knowledge, that the join of centre of the circumsphere with centres of circles circumscribed to walls are perpendicular to this walls. Similarly: the construction of a triangle inscribed circle by means of bisectors of sides is known, then students analogously search sets of all points in space, that are equidistant from two, three or four planes, determined by sides of tetrahedron. Further, they can for example derive relation between radius of r triangle inscribed circle and its area S and circumference o

$$S = S_a + S_b + S_c = \frac{a \cdot r}{2} + \frac{b \cdot r}{2} + \frac{c \cdot r}{2} = \frac{o \cdot r}{2} \Rightarrow r = \frac{2 \cdot S}{o},$$

and then analogously derive analogical identity for tetrahedron: $r = \frac{3 \cdot V}{S}$. By the investigation of heights of a tetrahedron students will find out that in case general the heights can be skew. All of four heights cut in one point if and only if when pair of opposite edges of tetrahedron lie in respectively perpendicular lines.

3. Generalization of Pascal's triangle

The numeral configuration, which is called Pascal's triangle, can be defined in this way, that numbers represent the number of ways, which lead from the origin $[0;0]$ to a given point $[x;y]$. Students reveal through the use of extension ways by a unit the recurrent formulation

$$C(0,y) = C(x,0) = 1 \wedge C(x,y+1) + C(x+1,y) = C(x+1,y+1),$$

and subsequently derive other characteristics of this numbers, namely that they determine the number of subsets, appearing in the binomial theorem and have explicit formula in the form:

$$C(x,y) = \frac{(x+y)!}{x! \cdot y!}.$$

How can generalization of this conception look like, if we proceed to three-dimensional space? We will count the ways that lead from the origin $[0;0;0]$ to a given point $[x;y;z]$ again.

In co-ordinal planes three Pascal's triangles originate and every number inside the octant originates as sum of three earlier calculated numbers. Then recurrent formula for all the numbers has then the form:

$$C(x,y+1,z+1)+C(x+1,y,z+1)+C(x+1,y+1,z)=C(x+1,y+1,z+1),$$

whereas initial values are given as follows:

$$C(x,y,0)=\frac{(x+y)!}{x!\cdot y!}\wedge C(x,0,z)=\frac{(x+z)!}{x!\cdot z!}\wedge C(0,y,z)=\frac{(y+z)!}{y!\cdot z!}.$$

Another meanings of this numbers are numbers of ordered pairs of subsets, occuring in terms like $(p+h+d)^n$ and having explicit formulation in the form: $C(x,y,z)=\frac{(x+y+z)!}{x!\cdot y!\cdot z!}$. This "geometrical"method of generalization follows the line of permutations with repetition, so then combination without repetition are special case of permutations with repetition.

4. Filling plane with circles and space with spheres

The question "Which part of the plane can we fill with suitable organized consistent circles?ćan bring us to nice problems. It's evident, that quotient can be quantified as quotient of the area of one circle and appropriate ćell", which originates from it by "inflationóf all circles at once.

Understandably the shape of the cell depends on general circles layout, in the first case the cell is quadratic, in the second case it's hexagonal. Calculations aren't difficult:

$$p=\frac{\pi r^2}{S_B}=\frac{\pi r^2}{4r^2}=\frac{\pi}{4}\approx 0,7854$$

$$p=\frac{\pi r^2}{S_B}=\frac{\pi r^2}{2\sqrt{3}\cdot r^2}=\frac{\pi}{2\sqrt{3}}\approx 0,9069.$$

How to formulate and solve analogical problem in space? If spheres concern by their "polesąnd by four points on ȩquator", appropriate cubic cell and ratio will be:

$$p=\frac{\frac{4}{3}\pi r^3}{V_B}=\frac{\frac{4}{3}\pi r^3}{8r^3}=\frac{\pi}{6}\approx 0,5236.$$

More effective spheres' ordering can be done in two ways, when upper spheres lie either on four or on three spheres of the underlay.

It's very interesting, that for both of this ordering the appropriate cell is the same (so-called rhombic dodecahedron) and wanted quotient has value

$$p=\frac{\frac{4}{3}\pi r^3}{V_B}=\frac{\frac{4}{3}\pi r^3}{4\sqrt{2}r^3}=\frac{\pi}{3\sqrt{2}}\approx 0,7405.$$

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ABOUT A DISTRIBUTION OF POINTS ON A LINE SEGMENT*

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1. Introduction

The concept of infinity is a key term in mathematics and its teaching. The development of knowledge about infinity makes an essential milestone important for its further development. If we accept the theory of so called genetic parallel that the ontogenetic development is not independent of the phylogenetic development, it is possible to assume that the obstacles we can identify in the phylogenetic development of infinity can be found also in ontogenesis and that overcoming the obstacles is a necessary component of the cognitive process of individuals.

The research activities conducted under the framework of the three year project GAČR *Obstacles in phylogenetic and ontogenetic development of the concept of infinity* are focused on the study of phylogeny development of infinity and also on the study of ontogenetic development of infinity among today's population. The research should be crowned by formulations of suggestions how to overcome the identified obstacles.

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2. Attributes of infinity

Infinity has many different aspects. Many mathematical concepts, methods, particular thoughts and developed theories are based on infinity. For the purpose of research, it was necessary first to untangle of mutually connected exposures and their attributes. Today, there are the following six attributes in the centre of our research interest:

- set cardinality (distinguishing finite, numerous and infinite sets, acceptance of actual and potential infinity);
- set boundedness (distinguishing bounded, very large and unbounded sets);
- set measure (geometrical measures point sets, images of infinitely small and large sets);
- ordering (perception of density of ordering, polarity of 'atomic' discrete images and images of continuum of number and geometric sets);
- infinite process (phenomenon of infinite repetition, induction, recurring);
- convergence (perception of the phenomena such as approaching, overcoming potential argumentation).

3. Obstacles in understanding infinity

An obstacle is understood as a set of mistakes related to previous knowledge. The knowledge can be successfully applied in certain situations but in a new context, its application leads to wrong solutions. To move further it is necessary to overcome the obstacle, to distinguish which of the pieces of knowledge can be transferred into the new context and which not and which pieces of knowledge appear to be new [3]. The research we conducted in this field brought out the following set of obstacles in understanding of infinity. We consider them to be principally irremovable (epistemological):

- experience with 'finiteness' (finite sets, finite processes, bounded objects, etc.);
- experience with the ordering of natural numbers;
- potential perception of the infinite process;
- replacement of an object with its model, especially with a picture;
- 'location of the horizon', which determines generally in a certain context the border between distinguishable and undistinguishable objects.

Besides the epistemological obstacles, we can also come across obstacles of different features we could overcome without disturbing the concept-making process. These obstacles can be cultural and didactical, for example, experience with colloquial language presenting variations of the word infinity in unacceptable mathematical contexts; and also the expectation that every task has a solution.

4. Research problem sample

The solution of the following problem presented by our respondents is central to our interest. *Given a square $ABCD$, find a point X on its side BC so that the triangle ABX will have: (a) as large its area as possible, (b) as small its area as possible.*

The majority of the respondents answer the first part of the problem well that it is necessary to place the point X onto C. For this reason, we focus on the second part of the problem.

Our assumption is that in the way of solving this problem, it will be possible to identify the extent of understanding of attributes such as the measure and the set ordering. We expect that pupils can face remaining obstacles closely connected to these attributes, caused by the transfer of the characteristics of a model (picture) into the object itself (into a point, a line, a triangle), by the experience with the ordering of the set of natural numbers (good ordering, isolated points) and also obstacles connected with the movement of the horizon and its tilting behind.

4.1. Expected answers

In the following analysis we will assume that the respondent is a primary level pupil, a secondary level student, or a university student. We assume that the students will consider the presented problem in classical Euclidean space as they (with exceptions) have never come across different types of space. We can rank the expected answers (starting with a poor image of infinity up to a sophisticated image) as follows:

- <u>point X will be about 1mm</u> (resp. any concrete 'minute' distance) <u>from point B</u>: here, the significant element is the picture as mentioned above, and also the **creator principal**, by which we consider such an approach of pupils and students that they think they have to create and construct the required object, find the point, and their effort is limited by their human senses and abilities;

- point X is right next to point B: considering the previous answer, there is a certain shift of the horizon. The respondent realizes that any distance can be diminished. However, the approach is the same as in the previous case as he/she expects an answer based on the construction of a picture;
- point X will be infinitely close to point B: it is an analogy to the previous answers, there is again a shift of the horizon (possible analogy with infinitely small quantities);
- the location of point X cannot be determined (for the reason of limited human possibilities): point X is placed on the very horizon, again the creator principal;
- the location of point X cannot be determined (thanks to the possibility to move point X from any place to point B): argumentation based on dynamic imaginary and potential approach;
- such a point does not exist: the reason can be either the image of potential approach as well as conscious work with lower bounded sets with no minimum, resp. with finite open set;
- point X will be in point B: acceptance of a line segment as a triangle with zero area influenced probably by the knowledge of limits.

4.2. Interview samples

The research conducted since 2005 has been focused on the images of secondary level students aged 15 to 19. The experiment question was given in a written form with the picture of square $ABCD$. The responses were recorded using a camcorder or voice-recorder.

Interview 1: Martina, 18 years

M46: *I will place it somewhere just right above B. However, I cannot determine where...*

E47: *And why not?*

M47: *It is immeasurable... it is possible to measure it... I can say that side BC has 1 cm, but if I want the area to be as small as possible, so if I place it just above B it is always possible to get a smaller area.*

M50: *Oh, it is... the smallest possible distance above B.*

Martina argues for the potential decrease of the distance between points X and B. In the end, she accepts the hypothesis of the smallest distance between two pints. It can be caused by experience with limits (when infinite processes are assigned with solid concrete values), as she could come across this concept in year 4 at her grammar school. Answer M50 can also be influenced by the fact that Martina expects some (positive) answer. Also the considered distance has been changed from żight above"in M46 and "just above"in

M47 to śmallest distance possible"in M50. We can consider the "bit"to be in front of the horizon - we can imagine it even though it is smaller and smaller - however, we cannot imagine "the smallest possible distance- it has tilted behind out horizon. This tilt behind the horizon is identified by Martina with the searched position of point X. The fall behind the horizon concerning the differentiability can be indicated also by the response "it is immeasurable there"in M47.

Interview 2: Vašek and Martin, 15 and 16 years

M15: *If it was totally closed, there could be a distance of one single undefined point.*

E16: *(2s) Well, but a point is not a distance.*

M16: *It is not, but if it is not defined, there must be a certain minimal diameter. (2s) There must be a tiny little gap...*

V17: *I think there is nothing.*

...

V18: *But the point will have probably some length, otherwise a set of points do not make a straight line.*

E19: *Well, so we have come across a fact that a line is a set of points. OK... (3s) However, Martin thinks there is something between the X and B.*

M19: *There must be something minimal, otherwise if the points are on each other it would be the triangle.*

E20: *And do you think that XB is a segment?*

M20: *It is a line segment with the length of a single point.*

Martin and Vašek consider a point to be the smallest measure - they have a pure atomic image. Although Vašek does not accept directly Martin's justification (V17), in the end he argues that something having a zero size could not make a straight line (V18). Vašek is brought to the conclusion of a zero point by the argumentation about actual infinite set. Vašek is not experienced enough and does not have sufficient mathematical skills to overcome the seeming contradiction. However, he is aware of it. His knowledge obtained in his existing education contradicts itself which can be deducted from answer V18: "but the point ... ąnd "probably". On the other hand, Martin comes to the conclusion probably thanks to his image of finite objects and with his experience with models - i.e. pictures etc. Like in the previous conversation, Martin places the point beyond the horizon of differentiability when he speaks about "undefined point"in M15 and M16. Again, he expects a smallest non-zero value - łength of one point"in M20.

The conversation samples are accompanied with pieces of conversation between younger pupils. The problem they should solve was formulated as follows: Given line segment AB, divide it in the ratio 1:2. What objects are created? Even though the question sounds different, its aim is to explain again images about points and their distribution on a line.

Interview 3: Markéta, Lenka and Jirka, 13 years

L14: *No, I will cut to pieces also the point. (Laugh)*

E15: *And is it possible to do it?*

J16: *It is nonsense! Simply, the point will be one and not on the other. It will be a half -line, a half line-segment.*

J18: *Let's put it on this one. (Leaving line segment AC, crossing out the end point C by the other.)*

E19: *OK, now we have no point C on the other, where does it end?*

J20: *Nowhere!*

L21: *It ends here! (Pointing to point C.) (1s) The point is not there. (1 s) It is here. (Pointing to line segment AC) (1s) Well, we can define another point... Here (pointing to point C) we will name it with D.*

L35: *As there is point C, and there is the line, it will be halved. It is for instance one millimetre, so we will halve it and here will be half a millimetre and there as well.*

Lenka works from the very beginning with a point as if it was a very small object - not as if it was a unit, as she accepts further division (L14, L35). It is mainly because of the fact that the point blends with the model - the drawn line on the line segment (L21, L35). She even chooses a size "for instance one millimetre"(L35). The narrow focus on the model is supported by the above mention creator principle. Lenka wants to exercise physically or she imagines that she would do everything she mentions in the conversation. All responds are evaluated considering the presenter. On the other hand, Jirka can approach the problem in an original way (considering his age) when he speaks about an object he has never heard of - about a half segment-line (J16), which is an analogy to a half-open interval. He can even think about consequences of the óopeningóf the segment-line when he says that the half segment-line ends ńowhere"(J20). How far can he go in his thoughts, we can only assume. Dominant Lenka did not leave enough room to Jirka to develop his idea. Jirka could not be as sure as Lenka, as he was just grasping it. On the other hand, Lenka worked with her object images which have been developed since the first meeting with geometry and for this reason she worked more self-confidently. The dominance of Lenka is presented also by the fact that Markéta was given no space to express her ideas and so she became an observer of Lenka's argumentations.

We summarize the obtained results flowing out from the conversations in the final paragraph. Before we do it, let's outline relevant realized pieces of research.

5. Connection with other research

Similar problems are examined in two pieces of recent research. The first one is the research focused on the understanding of a line segment as a set of points and with related ideas and the second is research dealing with the density of number sets, which is an analogy with our problem in arithmetic context.

5.1. About cardinality of sets of points

Bikner-Ahsbahs formulated in her research of 10 year old children, images of geometrical objects [4]. One of the questions of her questionnaire as follows: Wie viele Punkte liegen auf dieser Strecke? (How many points are there on this segment line?) In the questionnaire, there was a drawn line segment with a length of about 2 cm. Bikner-Ahsbahs presents that the majority of pupils declared the number of 30. In her research exercise, the pupils sketched points on the image of the line segment and then count them. After the incentive question Are there more points?, they responded in a way that tried to squeeze more points onto the segment line. After filling the whole line with points, they come to the conclusion, there are no more points.

These results reflect the fact that the majority of primary level children understand a point its by its image on paper or a blackboard. We can legitimately assume that this is an obstacle as the transfer of characteristics from the model, resp. from the picture onto the object itself, prevalent with primary level pupils and very often also with older students [6].

5.2. About density of number set arrangement

The second piece research was conducted by Eisenmann in the years 1998–2000 in 30 primary and secondary schools in the region of North Bohemia [5]. Among questionnaire items, there were also the following questions:

- Which number is the smallest bigger than 0? (year 1 - 5)
- Which is the smallest decimal number bigger than 0? (year 7)
- Which is the smallest rational number bigger than 0? (year 9)
- Which is the smallest real number bigger than 0? (year 11 - 13)

The gradual changes in the formulations of the question obtained in the pilot tests, where every question was rectified concerning its suitability for the particular age group are shown above. From the mathematical point of view, there are two phenomena - set minimum and the density of ordering. If we detach our attention from year 1, 3 and 5; we deal with fact that the set of decimal, rational, and real numbers bigger than 0 have not the minimum but only the infimum. The reason is that these sets of numbers are densely arranged.

It is natural that the most common answer in years 1, 3, and 5 is the answer 1. The pupils take into consideration only the set of natural numbers. The rest of the answers in years 1 consists above all the answer I do not know (23%) and incorrect answers 3, 10 (10%) or 0 (8%). In years 3 and 5, there also answers such as 0.1, 0.01, $0.00\ldots01$ or $0.000\ldots1$. The answers such as $0.00\ldots01$ are the most common in each age group (starting with year 7).

In year 5, there are also sporadic answers such as 0.1, 0.0001 or $\frac{1}{2}$, $\frac{1}{3}$. The uniqueness of the last answers is natural. Pupils in year 4 are introduced to fractions.

However, they do not consider them to be numbers but as operators or a part of a unity. From the phylogenetic point of view, the concept of number corresponds with Euclid and his Fifth Book of The Foundations, as there is the ratio 2:3, a quantity relation between two concrete quantities (see [10]). One of the first situations, when pupils can start to perceive fractions as numbers, is their representation on the number line. And this representation is a suitable means for later understanding of the fact that the set of all rational and real numbers are densely arranged. Early in year 4, subtraction of bigger natural numbers is illustrated using the number line and the principle of a magnifying glass. If teachers use this principle also later on when talking about decimal numbers, pupils acquire better images about the density of the given set of numbers.

Let's talk about students' results of the last year (13) of their secondary education. In this group, there are 13% of respondents who state that the lowest real number bigger than 0 is number 1. Interviews with students showed that these individuals ignored in the given question the word real, under the word number they automatically see an integer or a natural number.

In this year group, the respondents used for the first time the answer $\frac{1}{\infty}$ (with a frequency of 20%), and they also used quite often (11%) the equivalent answers None and Does not exist. We interviewed the majority of the respondents answering the above mentioned way to find out the explanation and the reasons for their answers. With only one exception, all students explained meaningfully their correct answers. We can always find a smaller and smaller number; it will never be the smallest (Honza). We can make a half of the small number which is always smaller, (3s) it will never be zero (Jakub). So I can add another zero (in number 0.0000001 - author's note), and so on, it will never end with a smallest number (Hedvika).

An interesting parallel of the discussed problem is illustrated by the following chunk of conversation between two students of year 13, which took place an hour after the filling in of the questionnaire. Experimenter: *Jana, you have answered in this questionnaire* 0.000000000001 *to the question "What is the smallest real number bigger than 0?".*

Jana1: *Yes.*

E2: *Karel, do you think that it is right? (Karel wrote Does not exist.)*

Karel2: *No, for instance (writing on paper) 0.000000000000001 is smaller.*

E3: *(to Jana) Well, (2s) is he right?*

J3: *Hm, yes, he is...*

E4: *And what is then the smallest real number bigger than 0?*

J4: *(4s)*

K4: *None. I can always find smaller. (3s) Look. (Sketching the real number line on paper, marking 0 and point $\frac{1}{2}$.) I will halve this (marking $\frac{1}{4}$), and again this (marking $\frac{1}{8}$) and so on (sketching other five short lines towards zero). I can divide it forever and never reach the smallest number.*

J5: *Then, there must be some infinite small line segments. (3s)*

K5: *(Surprisingly) Well, (4s) well, but (2s) you understand, the right end point is the number, and I can always make it smaller, and so no smallest one exists.*

J6: *Well, but a line segment yes - we had in class mathematics a week ago when we talked about a circle (3s) its length (2s) and how to make its circumference by* *inscribing the polygons with bigger and bigger number of vertices (2s) up to the one with an infinite number of line segments (2s) or vertices (2s) which is the circle. Therefore, the infinitely short line segment exists.*

Jana aims, in her argumentation, at a different direction than Karel who keeps the idea of the smallest set element. Jana does not speak about the smallest but about an infinitely small line segment. It is not an analogy of the smallest real number bigger than zero, but about an analogy of an infinitely small quantity.

6. Conclusions and connections with the historical development of images of infinity

The chosen question enables us to distinguish sufficiently and in detail the level of understanding points of and their distribution on a line in connection with the attributes of infinity. The respondents have a natural possibility of justifying their answers. Thanks to this fact, we can confirm the presence of the obstacles.

We can see the importance of the horizon. By the movement of the horizon and its crossing, the respondents make their answers more precise. At the same time when they think beyond the horizon - as far as they can see they cannot do without a kind of indeterminateness - "undefined pointór "immeasurableąnd so on. The awareness of the horizon and the world beyond the horizon stands at the beginning of our European science. The need of crossing the horizon accompanied by fear and uncertainty of its crossing is evident in the history of Euclid's Fifth Postulate. Why did so many people want to "prove"it? (Even after it had been proved that this postulate is independent of the other axioms.) It is because of the fact that the truth about the world in front of the horizon is grounded about something beyond the horizon [8].

All mentioned conversations have a close connection with the atomism of Democritus. However, each of the respondents is in a different phase of understanding a point as an atom. It is apparent in interview 2. While Martin has a naive image of an atom to be a sufficiently small object, Vašek comes to his conclusion after his consideration of a line as a set of points. It is an essential parallel with Democritus' theory formulated according to the paradoxes of Zenon based on the dispute between potential and actual infinity. In this case, it is based on the paradox that the sum of infinitely small quantities can be a non-zero value [1, 7]. Aristotle formulates the basic principle "what does not have a size that does not add a size"[1].

The idea of atomism is not only typical of Democritus. Also Descartes, who was a founder of analytical geometry, uses in his work *Principles of Philosophy* 'next-bordered' points, when he claims that two touching spheres have two points of contact, each of them one [9]. It refers to the fact that the match of a straight line and the number line is an enormously difficult thought construct.

There is an obvious impact of a model, especially a picture, on the process of image creation of geometrical objects supported by the creator principle. These phenomena can be traced in the history of mathematics from the beginning of mathematics development to recent history. The criterion of existence has been changing. Publisher of Bolzano still respects the proof of the mean value theorem, but he adds: "It is obvious that such a function does not exist"[2], as it is not even possible to imagine it [9]. The analogy can be found in the late acceptance of the existential quantifier.

We assume that the further phase of our research, focused on extended quantitative and qualitative investigation, will bring more solid results. We plan to conduct more detailed analyses of interviews and to extend the problems' scope to cover all attributes of infinity and to open the possibility of identification of other obstacles.

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COMPENSATION OF STUDENTS' HANDICAP IN MATHEMATICAL DISCIPLINES

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Abstract. *Repetitorium of mathematics*, which has been suggested as a part of the *Teacher training for elementary schools* programme in all forms of study, is aimed at levelling basic mathematical knowledge and skills of students entering university from various types of secondary schools with various positions of mathematics in the curricula.

1. Introduction

Even though we do not realise so, we get quite frequently into situations that need to be solved mathematically. The result of our solution then depends on how we are able to use our knowledge of mathematics from school. Therefore, schools pays attention to pupils' ability to solve difficult real life problems. Faculties of Education are then responsible for preparing their students to become good teachers, who are able to develop their pupils' skills in mathematics. Good teachers have to teach their pupils elementary skills and competences, but they also need to be aware of the necessity to give them enough space and time for reasoning, understanding and solving problems, together with learning how to argue. The emphasis is put on practical usage of mathematical knowledge in many different situations and contexts and in many different ways.

2. Current state

University preparation of elementary school teachers focuses on general preparation and mastering many diverse study subjects. Majority of subjects stresses memorising (as it is the nature of learning pedagogy and psychology, for instance) or they require artistic and physical talent (such as musical or arts and crafts lessons, or physical training). Mathematical disciplines are the only subjects in which memorising is not the only way of learning. They require a different way of thinking: logical, critical and mathematical reasoning. Another important fact is that students of mathematics are usually graduates from many different types of secondary schools, including those where mathematics is just a marginal subject. As it results from Kurečková's research, during their studies many pupils and students explain their negative attitude towards mathematics mostly by incomprehension of the curriculum. And thus, incomprehension is one of the most frequent causes of the lack of interest in mathematics (cf. [2]).

These facts can be the reason of many students' difficulties. As it can be clearly deduced from students' comments and partial evaluation activities of the *Department of Mathematics*, mathematical disciplines are constantly viewed as the most difficult ones. This is, of course, negatively reflected in students' motivation and their relationship to mathematics at the beginning of their studies.

These ideas are not new, in our contribution we would like to inform about handling this situation at our department. We proposed a FRVŠ project called *Preparation of a new optional course.* The proposal was accepted – the new course was named *Repetitorium of mathematics.* Its managers are A. Stopenová, B. Novák and J. Eberová. The project is based on the *Long-Term Plan of Faculty of Education, Palacký University in Olomouc.* Its realisation is a part of completing individual recommendations of the *Czech Government Accreditation Board* in respect to accreditation of new study programmes and subjects. The new subject is a part of a new study plan at the *Department of Primary Psychology.* We expect the new subject to optimize primary teachers' preparation in respect to the outline education programme.

3. Aim

The aim of the project is to prepare a new optional subject called *Repetitorium of mathematics.* As it can be deduced from its name, the subject tries to deepen and broaden mathematical knowledge from secondary school education. We expect that the subject will help us to create supportive conditions for students who need to raise their mathematical abilities in order to deal more easily with mathematical disciplines in this field of study. We would like to balance the level of students' fundamental mathematical competences and thus to make mathematical studies more effective, which will also lead to reducing students' failures at the beginning of their studies.

The subject is expected to be included in the first semester of the first year and it is scheduled for 14 weeks of lessons.

4. Target groups of the project

The target groups are formed by 150 up to 200 students of fundamental specialisations of the faculty:

- *Teacher training for elementary school teachers* in the attended form of study,
- *Teacher training for elementary school teachers and English* or *German*,
- *Teacher training for elementary school teachers* in the combined form of study,
- *Teacher training for elementary school teachers and special education.*

5. Methods of solving the project

The following methods have been / will be / are intended to be used during the project:

- **Questionnaire**

A questionnaire was worked out to find out if students are able to name the obstacles in their studies and if the obstacles influenced their relationship to mathematics.

57 randomly chosen students of the 4th year of the *Teacher training* programme completed the questionnaire. They formed two groups. The first group were 16 students of the attended form of study (later referred to as AS) and the second group were 41 students of the combined form of study (later referred to as CS). They could choose more than one answer from multiple options and they were encouraged to think about obstacles making their mathematical studies more difficult at various levels of school education.

In table 1 and 2 we can see their answers to the following questionnaire items: "Think of your own mathematical studies and try to find factors and obstacles you have encountered. Some of them are suggested and you need to add others.Śtudents of the combined form of study did not differentiate study obstacles according to the levels of school education, even though they were encouraged to. The reason may be seen in the time gap from their previous education. Thus they suggested the obstacles they found most troublesome and frequent.

An interesting note is that AS students with higher level of education have also higher percentage of responses fearing failure – 69%. Similar situation is with the CS students – 71%. The second most common obstacle in learning is lack of knowledge. The percentage of AS students' answers in this respect ranges from 13% to 25%, and except for the first grade, the percentage of such responses grows with higher level of school education. On the other hand, 39% of CS students consider lack of knowledge a major obstacle, which is much higher percentage than with AS students.

	AS			
	number in %			
obstacle	*low elem.*	*upper elem.*	*secondary*	*univ.*
boredom	0	31	19	0
tiredness	13	13	6	6
illness	13	6	0	0
lack of interest	19	19	25	19
lack of knowledge	25	13	19	25
fearing failure	19	25	25	69
lack of usefulness	0	0	6	0
no response	0	0	0	
other	0	0	0	0

Tab. 1: List of obstacles and percentage of AS students who mentioned them at various levels of their school education (cf. [1]).

	CS
obstacle	*number in %*
boredom	20
tiredness	32
illness	24
lack of interest	20
lack of knowledge	39
fearing failure	71
lack of usefulness	0
small motivation	2
no response	1
other	0

Tab. 2: List of obstacles and percentage of CS students who mentioned these obstacles regardless the level of school education (cf. [2]).

AS and CS students' answers for these two obstacles were one of the reasons of establishing the FRVŠ project.

- **Pilot test**

By a pilot test we wanted to check if our new subject is suitable even for students in combined form of studies. We included 3 word problems. Two of them are formulated in such a way to be able to show the usefulness of mathematics in solving real-life situations and problems.

[1] Two lemonades and two cakes are 44 CZK. One lemonade and three cakes are 30 CZK.
- How much is the lemonade?
- How much is the cake?
- Describe your reasoning process leading to the solution.

[2] A seal has to breathe even during sleeping. Martin watched a seal during one hour. At the beginning the seal dived and fell asleep. Eight minutes later it slowly emerged and took a breath. Three minutes later it went back under the surface and the process was repeated regularly. After one hour the seal was:
a) under the surface
b) on the way up
c) taking breath
d) on the way down (taken from [4]).

Describe your reasoning process leading to solution.

[3] A football match finished 5:4 for home team. The home team had a leading position from the beginning and they kept it until the end. How many different ways could have the score been changed? (taken from [3]).

The pilot test was given to six 5th year students of the *Teacher training for elementary school* programme in the combined form of study. The time for solution was not limited. Their results confirmed our opinion about the new subject's adequacy for teachers as well. Not a single student described his / her reasoning process. The results are shown in the following table.

	number of students			
task #	*no solution*	*analysis made*	*result found*	*answer written*
1	1	5	0	0
2	1	5	4	0
3	2	4	3	1

Tab. 3: Students' success in pilot test

- **Entrance test**

We are going to prepare an entrance test that will monitor the level of knowledge and skills of students in the *Teacher training for elementary school* programme in the attended form of study. It will include selected topics from elementary and secondary school mathematics. In the entrance test there are going to be topics included in the upcoming battery of tests and a new collection of tasks.

6. Project output

[1] Battery of tests to monitor and control continuously the study of selected topics from elementary and secondary school mathematics. These tests will monitor current state of students' knowledge and skills.

[2] Collection of selected mathematical tasks aimed at individual study, followed by interactive lessons in seminars.

7. Expected project benefits

- Helping to improve individual insufficient skills in mathematics.
- Raising students' interest in mathematical disciplines and their positive motivation.
- Building self-confidence of both prospective and working teachers in their mathematical skills.
- Making studies more effective and reducing students' failure in mathematical disciplines

8. Conclusion

The subject puts emphasis on teaching students how to master particular competences and skills and how to be able to solve textbook cases from elementary and secondary school. The usage of these skills in real life then depends on the individual's ability to apply any knowledge on practical situations and to organise and control their further education, or even to study independently and to overcome obstacles in learning.

Neither pupils nor students can be taught everything they will need in real life at school. Nevertheless, they must be able to learn how to learn.

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GAINING MATHEMATICAL SKILLS BY USING 'CONCRETE' MANIPULATIVES

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Abstract. In this issue we propose the reasons why it is important to use a variety of 'concrete' manipulatives in the classroom starting from the earliest ages. And for the teachers who can see the importance of using manipulatives in their classroom, we offer some new innovative ideas.

Prologue

The elementary school mathematics curriculum devotes to content and methods on teaching mathematics, which contributes to development of creative skills, mental operations, spatial images, and also social abilities (patience, diligence, allowance, etc.) [8]. It is up to the teachers what methods they choose for gaining this goal. We think the one of the possibilities is using manipulative facilitation (further *manipulatives*).

However, manipulatives do not guarantee meaningful learning, physical actions with certain manipulatives may suggest different mental actions than those we wish students to learn. To understand the role of 'concrete' manipulatives and any concrete-to-abstract pedagogical sequence, we must define what we mean by 'concrete' manipulative.

The meaning of 'concrete' manipulative

Dr. Frobisher [4], the author of many mathematics textbooks and mathematics methodology books in Great Britain, refers to the word "*manipulative*" as some kind of classroom equipment used by teachers and children to assist with teaching and learning about the mathematical concept that it represents.

Most practitioners and researchers agree that manipulatives are effective because they are concrete. By 'concrete' they usually mean objects that students can grasp with their hands. Douglas Clements [1], in his article, refers to the wider meaning of ‘concrete’. He distinguishes two types of concrete knowledge: *Sensory-Concrete* and *Integrated-Concrete* knowledge.

By *sensory-concrete* he means the knowledge when we need to use sensory material to make sense of idea. For example, at early stages, children cannot count, add, or subtract meaningfully unless they have actual things.

Integrated-concrete knowledge is build as we learn. The root of the word is ‘to grow together,’ it is the combination of separate ideas in an interconnected structure of knowledge. For students with this type of interconnected knowledge, physical objects, actions performed on them, and abstractions are all interrelated in a strong mental structure. For example, Jacob read a problem on a restaurant place mat asking for the answer to $3/4 + 3/4$. He solved the problem by thinking about the fraction in terms of money: 75¢+75¢ is \$1.50, so $3/4 + 3/4$ is $1\,1/2$ [2]. Ideas such as ’75,’ ‘$3/4$,’ and ‘rectangle’ become as real, tangible, and strong as a concrete sidewalk. Jacob’s knowledge of money was in the process of becoming such as tool for him.

Therefore, an idea is not simply concrete or not concrete. Depending on what kind of *relationship* you have with the knowledge [14]. *Good manipulatives are those that aid students in building, strengthening, and connecting various representations of mathematical ideas* [1].

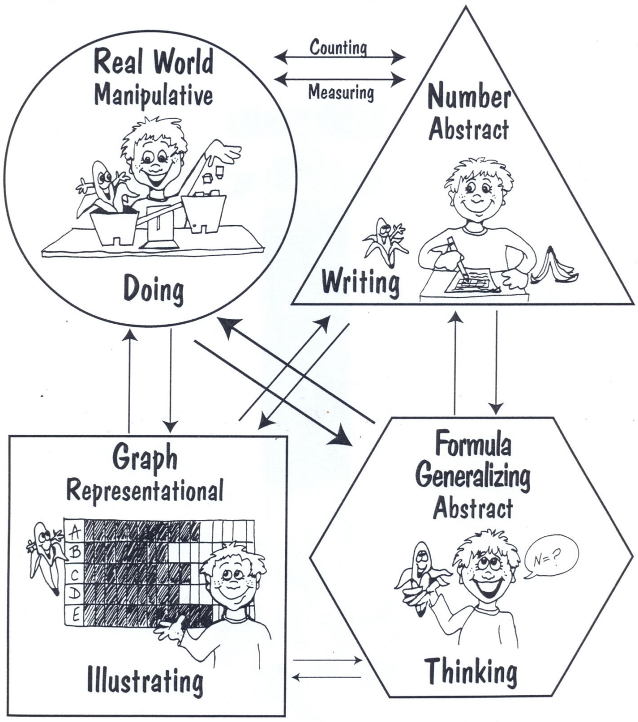
The use of ‘concrete’ manipulatives in mathematics education

‘Real-world’ mathematics has the greatest value to students. Students develop mathematical concepts and relationships best through activities in which they manipulate.

The following *Model of Mathematics* [10] serves as meaningful references for understanding the four environments in which mathematics is experienced:

The four distinct environments depicted in the models are:

- **Concrete:** experiences in the real world (circle),
- **Symbolized:** writing expressions using the symbols of mathematics and language (triangle),
- **Representational:** illustrating the real world by using drawings, graphs, and other types of picturing (square),
- **Intellectualized:** thoughtful analysis utilizing hypothesizing, drawing conclusions, making generalizations, etc. (hexagon).



The ability to translate from one environment to another is crucial. Greatest emphasis is placed on real-world mathematics, where students can see, touch, smell, feel, hear and manipulate with physical models (circle). Then are their representational counterparts, when students illustrate and draw charts, graphs, diagrams and maps (square). When students quantify and operate with numbers, they exemplify working in the triangle environment. As they problem solve, search for all solutions, analyze, formulate and discover principles, etc., they operate in the hexagon environment.

When to start with using 'concrete' manipulatives?

From an early age children gain experience with mathematics. Their exploring initially begins with tactile perception using fingers, sometimes toes and often with the mouth [7]. As parents automatically buy the toy blocks (lego, tangrams, etc.) to their children, they unconsciously help to develop their mathematical skills. By playing with constructive toys, other skills can be developed, too, e.g. manipulative skills, spatial sense, number sense, communication, etc. When children manipulate and build the objects, they question themselves: "Which brick will fit into this space? How many piece would I need? Do I need two-stud or three-stud piece of 'lego' to complete the wall?"

In our experience and after looking in several textbooks we can see that this pre-school experience is not fully continued in early years of schooling. According to this we agree with prof. Hanzel that developing mathematical thinking in pre-school and early years of schooling cannot lack the manipulation with 'real-world' objects, e.g. moving, classification and comparison. These activities are the basis for the developing number concepts meanwhile it represents initial stage for developing integer number concepts [6].

From above we can see the importance of implementing manipulatives into the mathematics education process starting at the earliest ages.

What manipulatives are efficient for mathematics education?

Piaget's point of view was that "giving no education is better than giving it at the wrong time" [9]. According to this we dare to say that not using manipulatives is better than using them in a wrong way. This fact supports Dr. Frobisher when he manifests that children become so absorbed by the manipulative that the concept it represents is so diffused that no mental image is formed [4]. Therefore, it is very essential to think of the effective and appropriate use of manipulatives that we would like to apply at mathematics education on different stage of children's development. We need to consider the age group as well as the goal we would like to claim, what mathematical skills we want to develop by chosen 'concrete' manipulative.

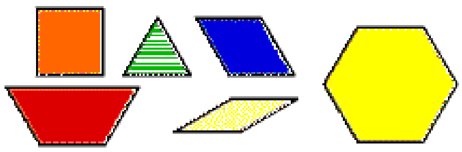
Regarding to character and consistence we can classify manipulatives into these groups:

- **toys** – building blocks, mosaics, lego, tangram, domino, etc.;
- **'real-world' objects** – fabrics, cereal boxes, clay, rods, rocks, beans, toothpicks, straws, etc.;
- **classroom equipment** – geoboard, geofigures, Attribute blocks, Base-ten blocks, Cuisenaire rods, Pattern blocks, Color tiles, etc.;
- **virtual manipulatives** – computer version of 'concrete' manipulatives, for representatives look at 'national library of virtual manipulatives' [12].

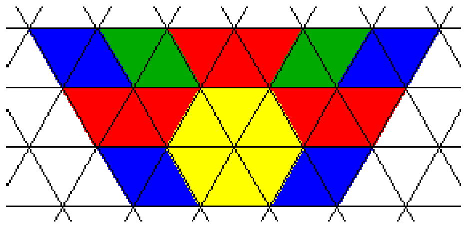
Examples of activities with 'concrete' innovative manipulative

We would like to introduce *Pattern Blocks* - the manipulative that we have first encountered when observing classes at Old Colony Montessori school (Hingham, MA) in 2006. We would like to show its importance and value at mathematics education in light of developing children's skills and their competences.

Pattern Blocks is a set of 250 blocks, it includes following geometric shapes with equal depth:

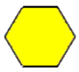


1) Activities that help to develop concept of space and symmetry:
Task: There is a mosaic on the picture. Can you locate the line of symmetry?



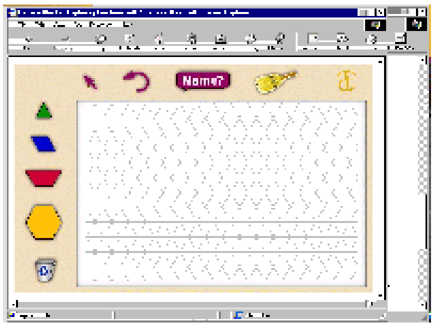
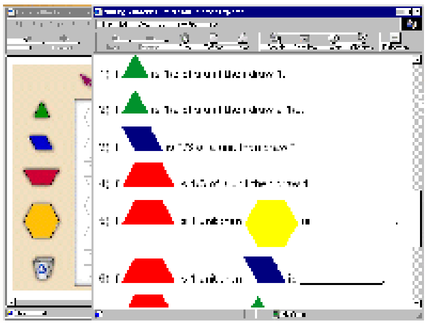
2) Activities that help to develop concept of fraction:

Task: If  is 1/2 of a unit then draw 1 unit.

If  is 3 units then  is _____.

3) Activities that develop mathematical vocabulary:
Architect and constructor
The architect builds the construction out of 10 blocks behind the screen. The constructor follows the architect's instruction on the other side of the screen. When they finish, the screen is taken away and the constructions are compared by both students. Then the roles are switched.

4) Activities on virtual bases [13]:
This program consists of a panel, as shown on the right, where you can drag in four different types of shapes of nicely fitting sizes - *pattern blocks* or *manipulatives*. You can place them on a triangular grid paper. Once inside the panel, you can rotate them and move them.

Conclusion

We introduced only a little fraction of the use of 'concrete' manipulatives in the classroom activities. There are lots of other manipulatives that help teachers to gain mathematics curriculum. Teachers can buy or make them through the internet or the catalogues. For other examples, as well as instruction how to create them, see the web pages

http://mason.gmu.edu/~mmankus/Handson/manipulatives.htm
http://nlvm.usu.edu/en/nav/vlibrary.html

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MATHEMATICAL STATISTICS AND MATHEMATICAL DIDACTICS

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In standings related with didactical experiments we have often need to use statistical check of our hypothesis. Though we consider that our method is correct or that some other method is wrong, we cannot make any claim whiteout mathematical background. Statistical proving of didactical hypothesis enables as to put our standings on mathematical standings. A most of didactical theories in some of their part use statistical proving.

Mathematical statistics enable as this goal to fulfill. Problem on which we could come upon when we start to chose statistical method which is most adequate for our needs, or our experiment needs, is fulfilling conditions which must be satisfied before we use some method. One of the most often conditions is condition than our sample is from normal distribution. This problem can be solved with central limit theorem of mathematical statistics.

In this work I would like to make more observation about cases where is not proper to use central limits theorem. This could be happening from several reasons: maybe sample that we have in not enough big, maybe we have interest, or our didactical experiment demand, knowing preciously to which distribution function belong our sample.

One of the very useful methods is Pearson's χ^2 test of congruence. Benefits of using this test are that this test can be applicable on every distribution function, which make our job much more easier. Other thing is that correspondent statistics are relatively easily countable, actually we have relatively easy calculation. But first take a look at method of maximal likelihood.

1. Method of maximal likelihood

We can estimate unknown parameter, with big size sample, of the numeric value distribution. Procedure of estimating unknown parameter depends on real or asymptotic (limiting) statistics. These statistics are functions of the sample.

Let's assume that theoretical distribution of numeric value is distribution function F, which belongs to set $P = \{F(x,v) : v \in \Theta\}$ acceptable of permitted distributional functions and let sample $(X_1, X_2, \dots, X_n)$ be from distribution F. We can get point estimator of the unknown parameter v from sample $(X_1, X_2, \dots, X_n)$ by following steps: we have to select statistics $T_n = T(X_1, X_2, \dots, X_n)$ and they name is estimator of the unknown parameter v. If realized value of the sample is $(x_1, x_2, \dots, x_n)$, then for approximation of the number v number $T(X_1, X_2, \dots, X_n)$ is taken. There are several methods how to get estimator, but basic is method of exchange. If unknown parameter v can be presented with functional $v = G(F)$, and we can mark with F_n empirical distributional function defined according to sample $(X_1, X_2, \dots, X_n)$. Estimator of the unknown parameter v is statistics $T_n = G(F_n)$.

If we want to have the most optimal properties we need to use the method of maximal veracity.

Let take a look at distribution $L(X)$ of numeric value X, which belongs to the set $P = \{f(x,v) | v \in \Theta\}$ of acceptable distribution. Veracity function is $L(x_1, x_2, \dots, x_n; v) = \prod_{k=1}^{n} f((x_k, v)$, when v is fixed this is density of random vector X. It can be observed that realized values of random vector X are mostly where veracity function take big values. Therefore we take statistics $\hat{v}_n = \hat{v}_n(X)$, for unknown parameter estimator. $\hat{v}_n = \hat{v}_n(X)$ is defined with condition

$$L(X : \hat{v}_n) = \max L(X; v).$$

Because function $\ln : R^+ \to R$ is increasing function $L(x,v)$ and $\ln\, L(x,v)$ have the same maximal value, we can make maximal veracity estimator from equation

$$\frac{\partial \ln\, L(X; v)}{\partial v} = 0.$$

Theorem. Let T_n is efficient* estimator of parameter v. Then T_n is the only solution of veracity equation.

Example 1. $(X_1, X_2, \dots, X_n)$ is a sample from $B(1, v)$ distribution. Let's find maximal veracity estimator of unknown parameter v. Distribution of numeric value is defined by $f(x,v) = v^x (1-v)^{1-x}$, where $x \in \{0, 1\}$. Function of veracity is

$$L(x_1, x_2, \dots, x_n; v) = v^{x_1 + \dots x_n} (1-v)^{n - x_1 - \dots - x_n}.$$

*Estimator is effecient if stands: $DT_n = G$, where $G = \frac{1}{nE\left(\frac{\partial}{\partial v} \ln(f(x,v))\right)^2}$.

Equation of veracity is

$$\frac{\partial \ln L}{\partial v} = \frac{\partial}{\partial v}(x_n \ln v + (n - s_n) \ln(1 - v)) = \frac{s_n}{v} - \frac{n - s_n}{1 - v} = 0,$$

where $s_n = x_1 + \ldots + x_n$. Solution of this equation is $v_n = \frac{s_n}{n} = \overline{x}_n$. Maximal veracity estimator of the unknown parameter v is $\hat{v}_n = \overline{X}_n$.

Example 2. $(X_1, X_2, \ldots, X_n)$ is a sample from $N(m, \sigma^2)$ distribution, where m and σ^2 are unknown parameters. Let's find maximal veracity estimators of this parameters. Function of veracity is:

$$L(x_1, x_2, \ldots, x_n; m, \sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(x_k - m)^2\right].$$

Equations of veracity are $\frac{\partial \ln L}{\partial m} = \frac{1}{\sigma^2}\sum\limits_{k=1}^{n}(x_k - m) = 0$, and

$$\frac{\partial \ln\ L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{k=1}^{n}(x_k - m)^2 = 0,$$

and their solutions are $m_k = \frac{1}{n}\sum\limits_{k=1}^{n} x_k = \overline{x}_n$ and $\sigma_n^2 = \frac{1}{n}\sum\limits_{k=1}^{n}(x_k - \overline{x}_n)^2$. From this we derive that maximal veracity estimators of parameters m and σ^2 are

$$\hat{m}_k = \overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{k=1}^{n}(x_k - \overline{x}_n)^2.$$

Note: If in distribution $N(m, \sigma^2)$ parameter σ^2 is known, and parameter m is unknown, then $\hat{m}_k = \overline{X}_n$ is the maximal veracity estimator of unknown parameter m. And if m is known and σ^2 is unknown, then the maximal veracity estimator of unknown parameter σ^2 is given by $\hat{\sigma}_n^2 = \frac{1}{n}\sum\limits_{k=1}^{n}(X_k - m)^2$.

2. Pearson χ^2-test

Hypothesis H_0 is to be tested on the sample $(X_1, X_2, \ldots, X_n)$ from the distribution $L(X)$. Distributional function of distribution X is equal to distributional function F. This can be written as: $H_0 : F_x = F$. Alternative is hypothesis H_1 that distributional function of statistic numeric value X is not equal to F. Significant value is α. Benefits of Pearson χ^2- test are that it can be used on every distributional function, there is no extra condition, and in comparison with other test calculations it is relatively simple.

Based on the sample $(X_1, X_2, \ldots, X_n)$ from distribution $L(X)$ we have to verify hypothesis H_0 that distributional function of distribution X is equal to distributional function F. We can write that as $H_0 : F_x = F$. Alternative hypothesis H_1 stands that distribution function of our population is not equal to F. Level significant value is α. Let $R = S_1 \cup S_2 \cup \ldots \cup S_m$ be fragmenting of set of real numbers. For $k \in \{1, 2, \ldots, m\}$, let M_k be number of elements from sample that have values in set M_k and let $p_k = P\{X \in S_k | H_0\}$. Then $M_k \in B(n, p_n)$. We can define random variable $X_n^2 = \sum\limits_{k=1}^{m} \frac{(M_k - np_k)^2}{np_k}$. This random variable can obtain good view on variation between random variables $M_1, M_2, \ldots, M_m$ and expected values $np_1, np_2, \ldots, np_m$. In this way we can find asymptotic distribution of random variable X_n^2 when hypothesis H_0 is valid.

If hypothesis $H_0 : F_x = F$ is true and if $p_k \in (0, 1)$ for $k \in \{1, 2, \ldots, m\}$, then is $X_n^2 \xrightarrow{D} \chi_{m-1}^2$ for $n \to \infty$.

When we have to solve a specific problem, like testing the match of two distributions, with Pearson χ^2 test, we have to follow next reasoning: from condition $P\{\chi_{n-1}^2 \geq \chi_{\alpha,m-1}^2 | H_0\} \approx \alpha$ we obtain constant $\chi_{1-\alpha,m-1}$. We have

$$P\left\{\sum_{k=1}^{m} \frac{(M_k - np_k)^2}{np_k} \geq \chi_{\alpha,m-1}^2\right\} \approx \alpha.$$

If for given value of statistics the inequality $\sum\limits_{k=1}^{m} \frac{m_k - np_k}{np_k} \geq \chi_{\alpha,m-1}$ holds true, then we dismiss hypothesis H_0. In practice we can use the following approximations, when $n \geq 50$ and $n \cdot p_k \geq 5$ for $k \in \{1, \ldots, m\}$.

With sample $(X_1, X_2, \ldots, X_n)$ from distribution $L(X)$ we want to test hypothesis H_0 that distribution function of statistical numeric value belongs to set $\{F(x, \theta) | \theta \in \Theta\}$. In this case we can make the same conclusion as we had with testing hypothesis that distribution function of numeric value belongs to given distribution function. But here we have that probabilities $p_k(\theta) = P\{X \in S_k | H_0\}$, $k \in \{1, 2, \ldots, m\}$ depend on parameter θ. For all $\theta \in \Theta$ is valid $p_1(\theta) + \ldots + p_m(\theta) = 1$. Similarly, we can define statistic X_n^2, it depends also on parameter θ:

$$X_n^2(\theta) = P\left\{\sum_{k=1}^{m} \frac{(M_k - np_k(\theta))^2}{mp_k(\theta)}\right\}.$$

Let the parameter θ be r-dimensional: $\theta = (\theta_1, \ldots, \theta_r)$, and $r \leq m - 1$.

Then it can be proved: if $\hat{\theta}_n$ is maximal veracity estimator of the unknown parameter θ, defined with sample dimension n, and if hypothesis H_0 is true, then $X_n^2(\hat{\theta}_n) \xrightarrow{D} \chi_{n-r-1}^2$, $n \to \infty^*$ is valid. By this we can, similar to the case

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*Def: Series of the random variables X_n is *converging in distribution* to the random variable X if is valid $\lim\limits_{n\to\infty} F_n(X) = F(X)$ *for* $\forall x$ (functions are continous).

without parameter, to test hypothesis H_0. From the condition $P\{\chi^2_{m-r-1} \geq \chi^2_{\alpha,m-r-1}\} = \alpha$ we can find the constant $\chi^2_{\alpha,m-r-1}$. If test statistics, found out the value $X^2_n(\hat{\theta}_n)$, is bigger than the constant $\chi^2_{\alpha,m-r-1}$, then we dismiss hypothesis H_0. In the opposite case we accept hypothesis H_0.

Example 3. At one faculty had been made test from mathematics. At the test student can achieve at most 500 points. On the sample of 750 student the following results are found:

Table 1

Grade	5	4	3	2	1
Number of points	[0,100)	[100,200)	[200,300)	[300,400)	[400,500]
Number of students, which have appropriate number of points	15	140	370	190	35

On given sample, with significant value $\alpha = 0.01$ test hypothesis that number of students, which have appropriate number of points, on this test, has normal $N(\hat{m}, 85^2)$ distribution (or is given sample from population with $N(\hat{m}, 85^2)$ distribution).

Solution:

In the case when we have to test hypothesis that some sample has or doesn't have given distribution, we can use Pearson χ^2-test. But before that, we must stop at unknown parameter $\hat{m}$.

If there was no parameter $\hat{m}$, if $\hat{m}$ is constant, we could apply Pearson χ^2–test immediately. $\hat{m}$ is unknown, so we have, first, to estimate them. We have to do that with maximal veracity method. Veracity function $L(x_1, x_2, \ldots, x_n; \hat{m}) = \prod\limits_{k=1}^{n} f(x_k \hat{m})$, in this case is

$$\begin{aligned} L(x_1, x_2, \ldots, x_n; \hat{m}) &= \prod_{i=1}^{n} \left(\tfrac{1}{\delta\sqrt{2\pi}}\right) \cdot \exp\left[-\tfrac{(X_i-\hat{m})^2}{2\delta^2}\right] = \\ &= \left(\tfrac{1}{\delta\sqrt{2\pi}}\right)^n \cdot \exp\left[\frac{-\sum\limits_{i=1}^{n}(X_i-\hat{m})^2}{2\delta^2}\right], \end{aligned}$$

further

$$\ln L = n \cdot \ln \frac{1}{\sigma\sqrt{2\pi}} - \frac{\sum\limits_{i=1}^{n}(X_i - \hat{m})^2}{2\delta^2} \quad \text{and} \quad \frac{\partial \ln\, L}{\partial \hat{m}} = -\frac{\left(-2\sum\limits_{i=1}^{n} X_i + n\hat{m}\right)}{2\delta^2} = 0,$$

$$\sum_{i=1}^{n} X_i - \hat{m}n = 0, \quad \hat{m} = \frac{\sum\limits_{i=1}^{n} X_i}{n} = \overline{X}_n.$$

We can calculate $\overline{X}_n$:

$\overline{X}_n = \dfrac{1}{750}(15 \cdot 50 + 140 \cdot 150 + 370 \cdot 250 + 190 \cdot 350 + 35 \cdot 350) = 262.$

Maximal veracity estimator of parameter m is 262. Now we can apply Pearson χ^2-test. In calculation of the value $\sum_{k=1}^{m} \frac{m_k - np_k}{np_k}$ it is often practical to put values in table. In our case the table would be as follows:

Table 2

	$[0, 100)$	$[100, 200)$	$[200, 300)$	$[300, 400)$	$[400, 500]$
m_k	15	140	370	190	35
np_k	21.75	153.41	329.78	206.38	39.18
$\frac{(m_k-np_k)^2}{np_k}$	1.84	1.17	4.90	1.30	0.48

Probabilities p_k $k = 1, \dots, 5$ are probabilities defined on adequate intervals of the function which have $N(\hat{m}, 85^2)$ distribution, e.g. $p_1 = P\{-\infty < x \leq 100\}$, because x has distribution $N(\hat{m}, 85^2)$, $p_1 = P\left\{\frac{-\infty - 262}{85} < x \leq \frac{100 - 262}{85}\right\}$ has $N(0, 1)$ distribution, because $\hat{m} = 262$, further $p_1 = \Phi\left(\frac{100-262}{85}\right) - \Phi\left(\frac{-\infty-262}{85}\right)$ and at the end $p_1 = \Phi(-1.9058) - \Phi(-\infty) = 1 - \Phi(1.90) - 0 = 1 - 0.9713 = 0.029$. Number of students n, in sample is 750, so $n \cdot p_k = 21.75$. Similarly,

$$\begin{aligned} p_2 &= P\left\{\tfrac{100-262}{85} < x \leq \tfrac{200-262}{85}\right\} \\ p_2 &= \Phi\left(\tfrac{200-262}{85}\right) - \Phi\left(\tfrac{100-262}{85}\right) = \Phi(-0.7294) - \Phi(-1.9058) = \\ &= 1 - \Phi(0.7294) - (1 - \Phi(1.9058)) = -\Phi(0.7294) + \Phi(1.9058)) = \\ &= -0.7666 + 0.9713 = 0.2045 \end{aligned}$$

and $n \cdot p_k = 153.41$ and so on.

When we summarize the last line we get $\sum_{k=1}^{m} \frac{m_k - np_k}{np_k} = 9.66$.

Now there is only to calculate $\chi^2_{\alpha,m-r-1}$, $m = 5$ (number of columns), $r = 1$ (number of unknown parameters, which are estimated), $\alpha = 0.01$, from table for χ^2 we find value $\chi^2_{0.01,3} = 11.345$. We see that $\sum_{k=1}^{m} \frac{m_k - np_k}{np_k} = 9.66 < 11.345 = \chi^2_{0.01,3}$, it means, that we can accept hypothesis H_0. Given sample belongs to population which has $N(\hat{m}, 85^2)$ distribution. We can say that students' point numbers have normal $N(\hat{m}, 85^2)$ distribution.

Note 1. Note that all values in the third line in table (values for np_k) are bigger then 5, so condition $n \cdot p_k \geq 5$ is fulfilled. If this wouldn't be the case, then we would have to use another test or we could modify table in the following way: column where value $n \cdot p_k$ is smaller then 5 will be joined with column near by, values m_k and m_{k+1} would be summarized, and probability p_k would be calculated from the beginning of the interval in k column to the end interval in column $k + 1$. Situation is similar for $k - 1$. This should be repeated in all columns until condition $n \cdot p_k \geq 5$ is not fulfilled.

Note 2. Interval $[0, 500]$ defined additionally on R, we need function f on whole R so we can calculate probabilities p_i $i = 1, \dots, m$.

Example 4. We have given results from entering exam in 2006 at the Faculty of Mathematics, Informatics and Physics, Comenius University in Bratislava. Entering exam attended 862 students. Maximal number of points was 20. Let's divide interval $(0, 20)$ into 10 parts:

$$(-\infty, 2), [2, 4), [4, 6), [6, 8), [8, 10), [10, 12), [12, 14), [14, 16), [16, 18), [18, +\infty).$$

With given sample on size 71 with significant value $\alpha = 0.01$ test hypothesis that numbers of students that have adequate number of points are selected from population whit normal distribution.

Solution:

In this case we have two unknown parameters m and σ. We will estimate these two parameters with maximal veracity method. In the previous example we have, that maximal veracity estimator of unknown parameter $\hat{m}$ is $\hat{m} = \frac{\sum_{i=1}^{n} X_i}{n} = \overline{X}_n$. Now we have to find maximal veracity estimator for σ.

Veracity function $L(x_1, x_2, \dots, x_n; \hat{m}, \hat{\sigma}) = \prod_{k=1}^{n} f(x_k, \hat{m}, \hat{\sigma})$ is

$$\begin{aligned} L(x_1, x_2, \dots, x_n; \hat{m}, \hat{\sigma}) &= \prod_{i=1}^{n} \left(\tfrac{1}{\hat{\delta}\sqrt{2\pi}}\right) \cdot \exp\left[-\tfrac{(X_i - \hat{m})^2}{2\hat{\delta}^2}\right] = \\ &= \left(\tfrac{1}{\hat{\delta}\sqrt{2\pi}}\right)^n \cdot \exp\left[\frac{-\sum_{i=1}^{n}(X_i - \hat{m})^2}{2\hat{\delta}^2}\right], \end{aligned}$$

further

$$\ln L = n \cdot \ln \frac{1}{\hat{\sigma}\sqrt{2\pi}} - \frac{\sum_{i=1}^{n}(X_1 - \hat{m})^2}{2\hat{\delta}^2} \quad \text{and} \quad \frac{\partial \ln L}{\partial \hat{\sigma}^2} = -\frac{n}{2\hat{\sigma}^2} | \frac{1}{2\hat{\sigma}^4} \sum_{k=1}^{n}(x_k - \hat{m})^2 = 0$$

and at the end we have parameter estimator $\hat{\sigma}_n^2 = \overline{S}_n^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k - \overline{X}_n)^2$. Then we get $\overline{X}_n = 10.464$ and $\overline{S}_2 = 32.69652$. We will test that sample is from $N(10.464; 32.69652^2)$ distribution. Similar to the previous example, we have to calculate numbers m_k (numbers of students that have points number in adequate interval), probabilities p_k and finish table filing

$p_{(2,4)} = P\{2 < x < 4\} = F\left(\frac{4 - 10.465}{32.6965}\right) - F\left(\frac{2 - 10.465}{32.6965}\right) = F(-0.1977) - F(-0.2589) = 1 - F(0.1977) - (1 - F(0.2589)) = 1 - 0.578 - 1 + 0.601 = 0.023$, etc.

Then we obtain the table:

Table 3

	$(-\infty, 2)$	$[2, 4)$	$[4, 6)$	$[6, 8)$	$[8, 10)$	$[10, 12)$
m_k	0	4	4	11	13	13
p_k	0.339	0.023	0.025	0.025	0.028	0.0195
np_k	24.069	1.633	1.775	1.775	1.988	1.3845
$\frac{(m_k-np_k)^2}{np_k}$						

$[12, 14)$	$[14, 16)$	$[16, 18)$	$[18, 20)$	$[20, +\infty)$
13	9	3	1	0
0.023	0.025	0.026	0.024	0.448
1.633	1.775	1.846	1.704	31.808

Condition $n > 50$ is fulfilled, but condition $n \cdot p_k \geq 5$ doesn't hold true in columns 2,3,4,5,6,7,8,9,10. Table has to be changed in the way that condition $n \cdot p_k \geq 5$, $k \in \{1, \ldots, m\}$ is true. We will join columns 2,3,4,5,6,7,8,9,10, values m_k and np_k will be summarized and other date must be calculated again. Then we obtain the next table:

Table 4

	$(-\infty, 2)$	$(2, 8]$	$(8, 14]$	$(14, 20]$	$(20, +\infty)$
m_k	0	19	39	13	0
p_k	0.339	0.073	0.0705	0.075	0.443
np_k	24.069	5.183	5.005	5.325	31.808
$\frac{(m_k-np_k)^2}{np_k}$	24.07	36.83	230.87	11.06	31.81

Both conditions are fulfilled, so we can continue. We had two unknown parameters during calculation, so when we look for $\chi^2_{\alpha,m-r-1}$, $r = 2$. $\chi^2_{0.01,5-2-1} = \chi^2_{0.01,2} = 9.21$. In the other side $\sum_{k=1}^{m} \frac{m_k - np_k}{np_k} = 334.6$.

We have $\sum_{k=1}^{m} \frac{m_k-np_k}{np_k} = 334.6 > 9.21 > \chi^2_{0.01;5-2-1} = \chi^2_{0.01;2}$.
It means that we dismiss hypothesis H_0. Given sample is not from population which has normal $N(10.464; 32.69652^2)$ distribution, with significance level $\alpha = 0.01$. With given sample we can not claim that number of points at the exam has normal distribution.

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ATTENTION OF TEACHERS TO PUPILS INTERESTED IN MATHEMATICS IN THE 4TH CLASS AT ELEMENTARY SCHOOL

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Abstract. This paper presents some results achieved with carefulness for pupils with larger interest in mathematics in the 4^{th} class at elementary school. We gave attention to two subjects – teacher and his (her) pupils. We mention the first one in this paper. We have found out certain reserves at work with these pupils including gifted pupils, mainly at out of educational time. Research shows needs to improve the quality of preparation for pupils interested more in mathematics. It is also connected with preparation of next teachers at universities.

1. Introduction

Educational process of pupils more interested in mathematics should be in attention of their teachers. Special attention to this fact is expressed in Conception of educational process development and in National programme of education in the Slovak Republic [1]. There are more reasons for it, like a determination gifted pupils, organisation of several mathematical competitions or mathematical hobby groups, and so on. A part of our research was focused on contemporary state of carefulness for pupils more interested in mathematics. We have found out ways that teachers used in their work with these pupils by questionnaire.

2. Basic categories of research

Aims of research were focused mainly on finding out

- how a teacher can determine his (her) pupils more interested in mathematics and gifted pupils in mathematics,

- what is a teacher carefulness for his (her) pupils during lessons,
- what is a teacher carefulness for his (her) pupils at out of educational time,
- what materials were used by a teacher at work with pupils.

We used empiric method questionnaire for finding out information. It contained 30 items. Answer part was combined, it had closed, half-closed and open questions. The questionnaire was anonymous.

Research sample: We asked 110 schools on Central Slovak territory in 2001. Schools were chosen by chance in order to be substituted schools in district towns and their surrounding, town and village schools.

3. Some achieved results

Basic information:
There were 71.82% of returned questionnaires. 135 teachers from 79 schools answered. 82.22% teachers were competent to teach, others studied at universities. Their average practice was 17 years. So we can say, that majority of these teachers were experienced and so they should be able to work with pupils with interest in mathematics.

There were few-classed and selective-classed schools among schools. So we focused our attention to them, too. It could show larger solicitude of a teacher about his (her) pupils there. We expected more pupils with larger interest in mathematics, because the pupils were chosen according mathematical exam, too. We expected that a teacher had more opportunities to work individually with pupils more interested in mathematics in few-classed schools. But finally there were not larger differences between these and classical classes.

Determination of pupils more interested in mathematics and gifted pupils in mathematics:
The teachers taught 3 219 pupils and they marked 34.61% of pupils as pupils more interested in mathematics and 12.33% as gifted pupils. It has showed that approximately one third of pupils were interested in mathematics. Information about gifted pupils is three times higher than psychological researches inform us.

This finding out shows us that many teachers have not sufficiency knowledge and experience with determination of gifted pupils. Literature [3, p. 17] gives information that there are 2–4% gifted pupils in population (not only in mathematics). Only 18.52% of the respondents are in area determined by experts. Almost one half of respondents asserted that they had had 11–20% gifted pupils in the class.

Teacher carefulness of his (her) pupils during lessons:

We asked the respondents if mathematical subject matter in curriculum has developed abilities of pupils more interested in mathematics. 54.82% of them said proportionately, only 5.93% thought that it has developed very much. Almost one third of the teachers (31.11%) thought that it has developed abilities little or not. 8.14% of the teachers were not able to judge this situation.

This finding out corresponds with results of another question, because the teachers very often used basic textbooks literature while they set homework to pupils more interested in mathematics (57.78%). The teachers, who were not satisfied how basic subject matter develops mathematical abilities of pupils with larger interest in mathematics, gave priority to other literature sources and own creation of problems. So to use only basic textbook literature is not sufficient.

51.11% of the respondents gave attention to pupils more interested in mathematics at least two times a week on mathematical lessons. Only exceptionally, mainly before competitions 40.74% of the teachers did that. Others (8.15%) did not give any attention.

We are glad, that approximately one half of the teachers do not forget about pupils, with larger interest in mathematics and work with them more or less regularly during the school year. Many teachers gave attention to these pupils irregularly and we can say that happening but once. We mean preparation before competitions during perhaps two weeks, because mathematical competitions Mathematical kangaroo, Pytagoriáda, Mathematical Olympiad are not running through all school year. Only Mathematical correspondent seminar and Minimix are whole year competitions, but the teachers mediate them only to pupils and do not secure special preparation for them, because organisation of these competitions is not connected with school organs. We would welcome if exceptional attention will be more regularly.

Individual problems were always given to pupils more interested in mathematics by 6.67% of the teachers on lessons and very often by 50.37% of them. It is more than answers in previous paragraph. Occasionally it is 39.26% of the teachers and 3.70% of them did not give any problems.

We think that regular attention of the teachers should be larger. Giving individual problems very often does not correspond with giving often attention to pupils more interested in mathematics.

Carefulness lessens at giving individual homework, only 14.82% of the teachers set it (30.00% per week, 45.00% twice per week, 25.00% three times per week). This form was occasionally used by 51.11% of the teachers and was not used by 34.07% of the ones.

This way was regularly used by small number of the teachers. Almost one third of them did not give any attention to it at all. But gifted pupils should be given individual homework and majority of the teachers said that they had have 11–20% of them at class. Choice and number of problems is very important, so that a teacher does not lower an interest in subject. According interval of giving homework we can judge that given problems were mostly algorithmic or half-algorithmic. Probably there were less non-standard problems, which texts are more pretentious to understanding a need to apply various solving strategies.

Individual homework connected with determined subject-matter only was given by 39.33% of the teachers. Problems not only connected with this subject matter were given by 60.67% of the teachers.

We can say that though there were not many teachers who set individual homework (14.82%), majority of them (according us) endeavoured to look up also other types of problems, to asset other solving ways, other topics in boundaries of mathematical knowledge and skills of pupil in the 4^{th} *class at elementary school.*

We have known that 13.34% of the teachers had to help always or very often at individual problems, 63.70% of them only sometimes and seldom or not 22.96%.

Here we could think of pretension of problems which were chosen by the teachers. So we can expect that they probably assessed inaccurately individual mathematical abilities of the pupils if they always had to help the pupils. Not only gifted pupils in mathematics were among pupils more interested in mathematics. It is question, if pupil, who does not experience success during independent solving of problem, does not loose motivation and interest. Spare if it is concerned homework a pupil should be able to solve it independently or with small advice of teacher. On the other hand gifted pupils need such problems in order to be able as more as to develop their own creative abilities, logical and critical thinking.

Used materials:
All teachers used textbooks written by Bero and Pytlová on lessons. 34.07% of them answered that there were sufficiency of suitable problems in this literature, other 52.59% of them thought that not and 13.33% did not know to judge this situation.

The textbooks copy subject-matter according curriculum. According view of the teachers (54.82%) subject-matter has developed abilities of pupils more interested in mathematics proportionately in one of the question, so then it would be sufficiency of suitable problems in textbooks for these pupils. But only approximately one third of the teachers agreed with it. Here answers of *respondents did not corespondent entirely. More then one tenth of the teachers who were not able to judge aptness of problems for pupils with interest in subject had probably smaller skills with classification of problems.*

The teachers, who were short of suitable problems in textbooks, answered what they had missed the most at work with pupils more interested in mathematics. They marked puzzles, games, crosswords (54.90%), untraditional tasks (52.10%), problems (47.90%), divergent problems (33.80%), creative problems (25.40%), combination problems (19.70%), experimental problems (7.04%), problems from practice (1.41%).

We can agree with the teachers who want more non-standard problems.

Teacher carefulness of his (her) pupils at out of educational time: Mathematical hobby groups were organised by only 8.15% of the teachers.

So we see that mathematical hobby group has not own stability place at elementary school and the teachers did not add a big importance to this form of preparation of pupils. But it is sufficient space for purposeful and continuos preparation of pupils at out of educational time.

Other respondents who did not organise mathematical hobby group asserted causes, mainly shortage of time (27.40%), financial valuation (23.40%), unconcern of the teachers (17.70%), unconcern of the pupils (6.06%), other reasons (35.50%) like organising other hobby group, organising mathematical hobby group by another teacher at school, shortage of materials, shortage of free time of the pupils, short-dated work at school and so on.

We see that there are various causes of non-existence of mathematical hobby groups at elementary school, some are objective, some less. In case of contemporaneous economical situation it can be comprehensible of unconcern of the teachers about work at hobby group and effort to use time another way than to prepare, organise and realise hobby group. Work is focused on to secure higher economical incomes. Shortage of time may be connected with fulfilment of organisation and administrative tasks. It concerns of one half asked respondents. But we see shortage of experience with organising of hobby group at elementary school, willingness to try and begin something new and perhaps non-precise ideas about contents, too. Although only small group of the teachers saw causes in shortage of suitable literature. We think, that this number is really higher. Evidently the teachers had no reason to find and judge necessary literature for work at schools where hobby groups were not organised.

All teachers who organised mathematical hobby groups did it regularly once per week (63.63%), once per two weeks (27.27%) or once a month (9.10%).

Although there were not many teachers working at hobby groups we are glad that they worked regularly and majority of them every week. We regard it as the best interval time. It is assumption to preserve pupils' interest.

The teacher working at hobby group was the most lacking of methodical materials (63.63%), set of funny tasks (54.55%), set of problems for mathematical hobby groups (45.45%), set of problems from thematic spheres (27.27%), competition problems (9.09%). Any help was not needed by 18.18% teachers.

We think that the teacher who works in hobby group can better evaluate necessary form of help and judge abundance of suitable suggestions and materials. The teachers who did not work at hobby groups did not see bigger obstacles in shortage of literature. Methodical materials and directions could help to improve work of teachers who have already certain experience with organising of mathematical hobby groups. Almost two thirds of the teachers were interested in it. The teachers demanded specific set of problems, which can help them more purposefully.

The teachers used as a source of differentiated problems for pupils on lessons or at mathematical hobby groups textbooks (57.78%), own creation of problems (34.07%), journal Komenský (8.88%), and other sources (49.63%). Just among them there were the most of problems from mathematical competitions (28.36%) and various tests for the first form in a grammar school (25.37%). Children journals were used less, likewise others for example Teachers' newspapers, older textbooks and so on. Approximately one half of the teachers focused on textbook literature. Only textbooks were used by 47.44% of them.

We think that it is not sufficient for development of mathematical abilities of pupils more interested in mathematics and gifted pupils. These pupils need to gain experience with larger variants of problems, their texts, solution strategies, degrees of pretension and creation. We are glad that the teachers aimed at creating own problems, too. 11.54% of them used only own problems. A creation of problems, mainly for more pretentious solver, is not simple affair for experienced author, too. It is necessary to think about many factors. So the teachers could not relay only on this source.

Both sources, textbooks and own creation combined 19.23% of the teachers.

We would recommend to combine as more sources as they can because there are stimulating thinks, motives and problems in every of them. There are various sources for gathering of suitable problems for pupils more interested in mathematics and gifted pupils at bookshops at present. But methodical materials (how to work with problems) are missing. Spare purchase of literature is mostly influenced by teachers' own financial possibilities.

Pupils took part mostly in traditional long-lasting mathematical competitions at elementary schools (Pytagoriáda 35.04% in 4^{th} class). Mostly each teacher joined pupils to one competition at least. The highest participation was in Pytagoriada. It rises with higher grade. We explain it by character of problems in this competition, too. They are simpler than in Mathematical Olympiad. 74.07% of the teachers joined their pupils (16.37%) to Mathematical Olympiad. 14.82% of the teachers did not joined their pupils to competitions at all. We were surprised that 11.11% of them had not known about this activity of pupils. Mathematical correspondence seminar was solved by 9.79% of the pupils, Mathematical kangaroo by 15.94% of them.

We think that if preparation of pupils is not indifferent to teacher in any area, he or she should know their interests and have a survey what form to satisfy interests. He (she) should be attentive to representation of class in various competitions. We are glad that the majority of the teachers gave chance to pupils to take part in Pytagoriada and Mathematical Olympiad. Teachers who did not know about competitions probably had no or only little knowledge about their organisation and course.

Finally we can say that there are some reserves at work of teacher with the pupils more interested in mathematics. It concerns above all suitable information and knowledge about gifted pupils in mathematics at elementary school, wider information about various mathematical competitions for 8-10 years old pupils, information about methodical work in mathematical hobby group, suitable sets of non-standard problems of various severity, teacher belief in important mathematical hobby group at elementary school.

Summary:

We recommend according achieved our findings out:

- to elaborate information materials for teachers in order to be able to improve a determination of pupils with larger interest in mathematics and gifted pupils in mathematics, Musil [2, p. 136] states, that accuracy of estimate was already raised about average 40% after short initiation of teachers to psychological principles and methods for determination of talent. It is important to give teachers an opportunity of training it by psychologists;
- to elaborate information materials about organisation and course of mathematical competitions at elementary school;
- to create methodical materials for work at mathematical hobby groups;
- to work out a suitable set of non-standard problems with various intentions and various degrees of creation for work in mathematical hobby group;
- to convince teachers about importance of mathematical hobby group also at elementary school and recommend establishment it.

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EQUATIONS $f(x) = f^{-1}(x)$ AS A GENERATOR OF MATHEMATICS TEACHING PROBLEMS

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Abstract. Within the secondary school mathematics, the notion of an inverse function and its relationships to the original function does not attract much attention. In this article we deal with equations of the type $f(x) = f^{-1}(x)$ as a source of problems the solution of which leads to a better understanding of the notion of an inverse function. We make use of the PC programs Derive and WinPlot.

1. Introduction

In this article we show, how to use the method of generating problems in teaching calculus at secondary level. The method has been described in [2]. A problem is given to a student and he is assisted when solving the when solving the problem as far as it is necessary. Afterwords, the student is motivated to ask himself further questions and to generate problems related to the original one. Exactly, this activity of generating related problems is the core of the method. The related problems can be obtained via analogy, variation, generation, etc.

Let us illustrate the method of generating problems on an example. Consider an equation of the form $f(x) = f^{-1}(x)$, where f is a function and f^{-1} is its inverse function. Solving such equation provides a student of a secondary school a broad variety of topics and activities leading to a much better understanding the notions and properties of functions, inverse functions, equations with parameters, and so on. Various computer programs supporting drawing graphs of functions will effectively help in such activities.

2. Linear function

Assume that f is a linear function $f(x) = ax + b$, $a \neq 0$. Then the inverse function is of the form f^{-1}: $x = ay + b$. Hence

$$x - b = ay, \qquad \frac{x-b}{a} = y.$$

The equation $f(x) = f^{-1}(x)$ can be restated as

$$ax + b = \frac{x-b}{a}. \tag{1}$$

Hence

$$\begin{aligned} x - b &= a^2 x + ab, \\ x(1-a^2) &= ab + b. \end{aligned}$$

If $a = 1$, we get $0 = 2b$. If $b = 0$, the solution are all real numbers. If $b \neq 0$, the equation (1) does not have any solution. If $a = -1$, we get $0 = 0$. For every real number b, the solution of the equation (1) are all real numbers.

If $a \neq 1$ and $a \neq -1$ we get

$$\begin{aligned} x &= \frac{b(a+1)}{1-a^2}, \\ x &= \frac{b(a+1)}{(1-a)(1+a)}, \\ x &= \frac{b}{1-a}. \end{aligned}$$

If $a \neq 1$ and $a \neq -1$ the equation (1) has one solution.

These solutions have also geometrical interpretation. We know that graphs of the function f and the inverse function f^{-1} are axial symmetric with axis $y = x$. If $a = 1$ and $b = 0$ we have a function $y = x$, the graph of which is axis of this symmetry. The inverse of this function is also $y = x$. Therefore the solution of (1) are all real numbers. If $a = 1$ and $b \neq 0$, the graphs of function $f(x) = x + b$ and $f^{-1}(x) = x - b$ are two parallel lines, which are parallel with $y = x$ and do not have common point. Therefore the equation (1) does not have any solution.

If $a = -1$, the graphs of functions f are lines perpendicular to the axis $y = x$ and these lines are in this axial symmetry the isometric sets of points. Therefore the solution of (1) are all real numbers. If $a \neq \pm 1$ the graphs of $f(x)$ and $f^{-1}(x)$ are non-parallel lines with common point on the axis $y = x$. Therefore in this case the equation (1) has one solution.

3. The rational functions of type $\dfrac{ux+v}{px+r}$

Every function $f(x) = \frac{ux+v}{px+r}$, $x \neq -\frac{r}{p}$ (p, r, u, v are real numbers), can be written in the form $f(x) = a + \frac{k}{x-b}$, $x \neq b$ (a, k, b are real numbers). For this function, the equation $f(x) = f^{-1}(x)$ has for $a \neq 0$ and $b \neq 0$ the form

$$a + \frac{k}{x-b} = b + \frac{k}{x-a}. \tag{2}$$

If we solve this equation, we get

$$\begin{aligned} a + \frac{k}{x-b} &= b + \frac{k}{x-a}, \\ a(x-a)(x-b) + k(x-a) &= b(x-a)(x-b) + k(x-b), \\ (a-b)(x-a)(x-b) + k(b-a) &= 0, \\ (a-b)\left((x-a)(x-b) - k\right) &= 0. \end{aligned}$$

If $a = b$, we get $0 = 0$ and the solution are all real numbers. If $a \neq b$, then

$$\begin{aligned} (x-a)(x-b) - k &= 0, \\ x^2 + (-a-b)x + (ab-k) &= 0, \\ x_{1,2} &= \frac{a+b \pm \sqrt{(a-b)^2 + 4k}}{2}. \end{aligned}$$

Now, we have three possibilities:

[1] If $k > -\frac{1}{4}(a-b)^2$, then the equation (2) has two solutions:

$$x_1 = \frac{a+b+\sqrt{(a-b)^2+4k}}{2}, \quad x_2 = \frac{a+b-\sqrt{(a-b)^2+4k}}{2}.$$

Notice that $(a-b)^2 > 0$ and for every positive number k the equation (2) has two solutions.

[2] If $k = -\frac{1}{4}(a-b)^2$, then one solution is $x = \frac{1}{2}(a+b)$.

[3] If $k < -\frac{1}{4}(a-b)^2$, then (2) does not have any solution.

The solutions have a geometrical interpretation. Graphs of the functions f and f^{-1} are hyperbolas. The question is, how many common points these hyperbolas have?

In case $a = b$, the hyperbola – graph of the function f is symmetrical with respect to the axis $y = x$. Therefore the solution of equation (2) are all real numbers.

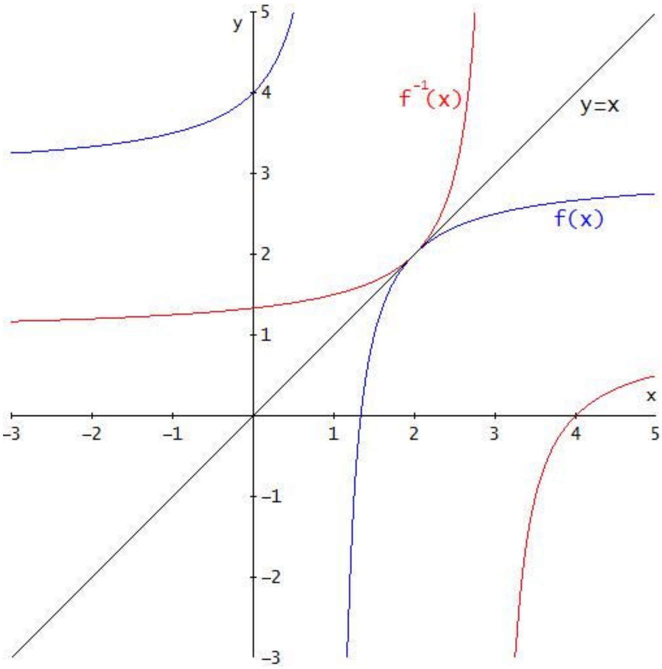

Fig. 1.

In case of $a \neq b$ we have three possibilities. First, the hyperbolas have two common points. For every positive number k the hyperbola – graph of the function f has two common points with the axis of axial symmetry $y = x$. These common points are the common points with the hyperbola – graph of the function $f^{-1}(x)$. Second, one arm of the hyperbola f touches the one arm of the hyperbola f^{-1}. They have a common tangent $y = x$ at the common point. This situation we explain by the function $f(x) = 3 - \frac{1}{x-1}$ (see Fig. 1).

Third, the hyperbolas do not have any common point.

4. Conclusion

These examples illustrate, how it is possible to use the method of generating problems with generator problem - solving equations $f(x) = f^{-1}(x)$ for different types of functions f. Applying this teaching method connect mathematical analysis, analytical and synthetical geometry in school mathematics. It is very important that the students' knowledge be not isolated.

Teacher has a possibility to explain the students the notions of inverse function, graphs of different types of functions. The students can see the graphs of the function f and its inverse f^{-1} are symmetrical with respect to the axis $y = x$. To draw these functions we can use some computer programs. We can use this programs during the teaching process also for finding the numerical solutions the equations $f(x) = f^{-1}(x)$ for some types of functions (trigonometrical, exponential functions, etc.)

The method of generating problems can be used in other parts of school mathematics, see [1], [3], [4], [5], [6].

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ABOUT SOME DIFFICULTIES IN DOING PROOFS ENCOUNTERED BY THE STUDENTS FROM THE FIRST YEAR OF MATHEMATICS

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Abstract. This paper contains some results of diagnostic research on the difficulties that students who begin studies at the tertiary level encounter when doing proofs from a section devoted to applying definitions in proofs. The considerations concern the issues connected with understanding of the role of definition and understanding of the texts of definitions by students. In these considerations the examples of solutions given by students to two diagnostic tasks applied in the research are used.

1. Introduction

The movement from elementary to advanced mathematical thinking, as Tall (1992) describes, involves a transition from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions (p.20). Students frequently struggle in making this transition what is especially seen as they tackle the first year of university. As a university teacher I often observe that my students experience many difficulties in their first encounters with definitions, theorems and proofs in undergraduate mathematics courses. The biggest problems seem to be connected with analysis and construction of mathematical proofs. It motivated me to undertake more detailed research in order to answer the question: *What didactic interventions and instructions could be introduced during classes to help students both to overcome their difficulties with proving and to develop their skills in that area?* There is also another justification for the necessity of conducting this kind of studies – there has been relatively little research on the teaching and learning mathematical proof having dealt with university students (Moore, 1994).

Before starting to plan activities, instructions aimed at eliminating students' difficulties with proofs I deemed as necessary to conduct more detailed studies on what are these difficulties in analysis and construction proofs. I would like to present some their results.

2. The study

My diagnostic research concerning the students' difficulties with proofs was conducted during the course "Introduction to Mathematics", which included topics such as logic and proof techniques, set theory, relations and functions, in the winter semester of the 2006/2007 academic year. The research group consisted of 57 students from the first year of mathematics. The research was based on the analysis of solutions of different tasks requiring the analysis of the texts of written proofs or independent construction of proofs, which were presented by students in their works. The required proofs were short, uncomplicated proofs in which inferences were based largely on definitions or previously accepted theorems.

Analyzing students' difficulties with proving I considered the issues connected with: (a) understanding the concept of a proof, its role and other methodological aspects, (b) concept understanding, (c) theorem understanding, (d) proof methods, (e) mathematical language and notation, (f) mathematical logic. In this study I will present some results of the diagnostic research from the section connected with the usage of definition of a concept in a proof. Discussing them I will refer to the students' solutions of the following tasks:

Task 1:
Read the definition and do the exercises:

Definition: Function f is increasing in its domain, if

$$\underset{x_1,x_2\in D_f}{\forall}(x_1 < x_2 \Rightarrow f(x_1) < f(x_2)).$$

1) Write this definition by words.
2) Using the definition justify that function $f(x)$=$2x+3$ is increasing in its domain.
3) Is function $f(x)$=tgx increasing in its domain? Justify your answer.

Task 2:
Read the definition and do the exercises:

Definition: Set $X \subset R$ is bounded below, if $\underset{m\in R}{\exists}\ \underset{x\in X}{\forall}\ x \geq m.$

1) Explain by words when the set is bounded below.
2) Give an example of bounded below set. Justify your choice.
3) Is set $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$ bounded below? Justify your answer.

The tasks required among others truth verification or justification of certain statements about the concept, whose definition was given. However, the definition from the Task 1 was familiar to the students from secondary school, the second one was met for the first time. Application of different in this sense definitions was done deliberately – I was curious if and how this difference would influence the way of solving both tasks by the students.

3. Theoretical background

Tall and Vinner (1981) have distinguished between the concept image and the concept definition. The former refers to the *"total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes"(p. 152) and it is built up by individual through different kinds of experiences with the concept. The latter refers to a formal definition which determines the meaning of the concept. Despite the fact, that in the process of solving tasks (also these which require analyzing and constructing proofs) both concept image and concept definition play the crucial role, the correct deduction demands orderly use of the definition. What is more, definitions provide the language – the words and symbols – for writing a proof, suggest the sequence of individual steps and provide the justification for each step in a proof (Moore, 1994). In this connection analyzing the issue if and how the students apply the definition in a proof I was investigating: 1) their understanding of the role of mathematical definition, 2) their understanding of the texts of definitions.*

4. Research results

a) Understanding of the role of mathematical definition

It can be said, that a student understands the role of mathematical definition if he/she consciously and orderly uses definition in reasoning. Considering the issue of understanding this formal discipline by my students I was analyzing their answers to points 2) and 3) in the Task 1.

In point 2) it was clearly said, that the justification should be formulated on the basis of the definition. Two thirds from 27 students, who were solving this task, conformed to this instruction, despite some of them made a mistake – they were checking the veracity of defining condition for the concrete numbers. A few students were applying other arguments using their intuition about the increasing function; they stated e.g. that given linear function is increasing in its domain because *"the raise of arguments is accompanied with the raise of function"*.

In point 3) only one student formulating justification referred to the definition – to show that the function $f(x) = \operatorname{tg} x$ is not raising in its domain he/she was constructing the counter-example. The rest of the students did not undertake solving the task or used different arguments in their answers. The latter ones most often referred to the graph of the function while formulating the justification, despite the fact if they considered the $f(x) = \operatorname{tg} x$ function to be increasing or not. They stated: *"It is not raising, it is visible in the graph"* or *"The function is not raising in its domain because while observing the graph of this function we can see how its particular parts are raising"*. The authors of these statements did not draw the graphs but visualized the function in their mind. The evoked image was a sufficient argument for them; they did not feel the need to verify their answer on the basis of the definition.

From these considerations it can be concluded that there was a lot of hesitation while applying the definition by the students. Many of them used the definition only in one point, the one in which they were asked to do so. In the second point they were referring rather to the concept image. Perhaps it was the result of recognizing such a solution as more simple and/or the lack of a remark about using the definition obligatory in the instruction. But if we want to speak about the methodological understanding of definition as about using it consciously, we would expect trials of referring to definition not only in the case of a clear instruction but also in the situation where it is not clearly stated. Thus it can be assumed that:

Conjecture: A lot of students do not understand the role of definition in proof.

b) Understanding of the texts of definitions

The facts that student is able to: (a) differentiate the name from the defining condition and knows that the defining condition determines the meaning of the name, (b) interpret the definition, express it in a different form, (c) construct examples of referents, (d) use the definition in solving of simple tasks, evidence about the understanding of the text of the definition (Krygowska, 1977). Having these kinds of competencies by my students I was studying with the usage of above quoted tasks.

Perfunctory analysis of solutions of these two tasks showed that more students were trying to answer the questions concerning the concept of increasing function than the concept of bounded below set. However even in the case of the first of these concepts, which was known and applied previously by students, there were answers showing difficulties in understanding the defining condition. These difficulties were revealed e.g. in point 1), where the students were asked to write the definition using words. Here are some examples of the answers:

"Function f is increasing in its domain, if for x_1 and x_2 which belong to the domain of function, x_1 is smaller than x_2 when and only when the function value at argument x_1 is smaller than the value at argument x_2",
"Function f is increasing in its domain, if for each x_1 and x_2 there is such an x_1 smaller than x_2, that $f(x_1)$ is smaller than $f(x_2)$".
These statements show the lack of knowledge and understanding of logical symbols present in the defining condition, what consequently led to incorrect interpretation of the logical structure of this condition. It can be concluded, that misunderstandings of mathematical language caused difficulties with understanding of the definition. However, even correct interpreting of logical symbols did not mean that students understood the concepts which were behind these symbols. It can be doubted, if they understood for example the concept of a general quantifier as 11 students made a verification for a few chosen numbers while justifying in point 2) that given linear function is increasing in its domain.

In the Task 2, where was an unknown definition of a bounded below set, as many as the half of 30 students, who were solving it, did not try to answer any question. A few students giving the reason stated that: *"I don't know this definition. It's unclear"*. These students behaved passively towards the text of the new definition. They did not take any action towards understanding it by themselves – they did not try to interpret the text of the definition, express it in a different way, look for referents of the concept.
Students who tried to analyze the definition individually not always were able to read correctly defining condition given in a symbolic way. In point 1) students wrote e.g. *"The set X is bounded below if for each x from the set X there is m from the set R that $x \geq m$"*. Interpreting definition condition the author of this comment changed the order of quantifiers revealing the lack of understanding that the order of the quantifiers in notation is important. It indicates some deficiencies in his/her logical education.
There was also a group of the students who in the process of solving the task used some associations connected with colloquial sense of the name of the concept without analyzing the text of the defining condition; for example the student in point 2) gave the set {1,2,3,4,...} as an example of a referent, and wrote in his justification: *"because it has a definite beginning"*. He/she acted as if he/she thought that set is bounded below if it starts from the concrete number. In this connection there are some doubts whether he/she understood that the defining condition determines the meaning of the name and that this meaning should be read from it.

From above examples it follows that:

Conjecture: Students experience a lot of difficulties connected with understanding a definition formulated in formal and symbolic language, regardless it is new for them or not.

5. Conclusions

The considerations presented in this paper show that constructing even simple proofs and justifications on the basis of definition can be difficult for students. It is partly because of some misunderstandings connected with methodological aspects, mainly lack of understanding of the role of mathematical definition. When doing proofs the students did not use definition of concept consciously but they referred to their intuition or mental pictures connected with the concept. What is more, intuitive visual arguments were so convincing that the students did not feel the need to verify their correctness on the grounds of the definition.

Another case are difficulties connected with understanding of the text of definition, especially when it is stated in formal, symbolic language. They can result from the deficiencies connected with the knowledge of the field of mathematical logic. Students encountering a new definition did not try to analyze and understand it. They were not able to create individually the mental picture of the concept given by the definition. The lack of intuitive understanding resulted further to the fact that some students did not try to answer the questions concerning the concept.

This paper contains only the fragment of my research on students' competencies in proving. The analysis of collected materials enabled me to indicate a lot of different difficulties in this area and define some relations between them, what has a great meaning during planning didactic activities aimed at eliminating these difficulties.

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MATHEMATICAL TASK STATEMENT

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Abstract. The text describes a task which must be solved on the analyze of propounded problem. Pupils and students solving these types of tasks are taught to think about the mathematical problem and to develop mathematical consideration. The following task also shows what way the mathematical problem statement can affect the difficulty of task solving.

Task 1. We have two sides of a triangle ABC: $a = 15$ cm, $b = 8$ cm. What size could gain the length of the third side c? What value c has the angle ABC for:

a) acute angle,
b) right angle,
c) obtuse angle?

Lead two cross cuts from one point, which divide the triangle into three parts. Use these parts for construction of a rectangle (without overlaying). What extension can have the rectangle? Find all possibilities.

a) Angle ABC is acute for $c \in (7; 17)$
b) Angle ABC is right for $c = 17$
c) Angle ABC is obtuse for $c \in (17; 23)$

Task solving deliberation 1:

- The cuts must be designed from one point. We must choose the point suitably. If we want to construct regular shapes from arisen parts the centre of the triangle side could be the most suitable point.

- The cuts can connect the centres of triangle sides. The cuts could be constructed along the middle transversals of triangle.
- If we construct the cuts in triangle with aid of centres and middle transversals we can make up the shape along congruent sides. The regular shape can come into existence.

The cuts construction in impedance triangle

We construct cuts along the middle transversals in an impedance triangle ABC (17 cm; 15 cm; 8 cm)

1. If we cut the triangle ABC along the middle transversals according Fig. 1, we can construct a rectangle by the help of arisen parts of rectangle (15 cm; 4 cm) see Fig. 2.

Fig. 1.

Fig. 2.

2. If we cut the triangle ABC along the middle transversals according Fig. 3, we construct a rectangle by the help of arisen parts of triangle (7,5 cm; 8 cm), Fig. 4.

Fig. 3.

Fig. 4.

Fig. 5.

3. If we cut the triangle ABC along the middle transversals according Fig. 5, we construct a rectangle by the help of arisen parts of triangle (15 cm; 4 cm) or rectangle (7,5 cm; 8 cm), Fig. 6.

Fig. 6.

Cuts construction in common triangle

If we cut a common triangle ABC along the middle transversals we can construct by the help of arisen parts after translocation a parallelogram with relevant size.

Fig. 7.

The rectangle arises, when the first cut is led along the middle transversal and afterwards we construct the second cut vertical to the middle transversal in one of its end points. If the arisen shape should be a rectangle, it must be constructed in the way that the right angle must be available in the shape. The rectangle extensions depend on the original size of triangle.

Fig. 8.

Summary of the Task 1 solution

1. If we construct the direct cut in the above mentioned way in the impedance triangle, we construct by the help of this way arisen parts a rectangle with relevant extensions.

2. If we construct the direct cuts in a common triangle with the above mentioned way, we can construct:

   a) A rectangle, if one cut is constructed along the middle transversal and we construct the second cut vertical to the middle transversal in one of its end point.

   b) A parallelogram, if both cuts are constructed along the middle transversals.

Task 2. We have lengths of two sides of triangle ABC: $a = 15$, $b = 8$. What size could gain the length of the third side c? What value c will have the angle ABC:

a) acute angle,

b) right angle,

c) obtuse angle?

From one point we lead two direct cuts, which divide the triangle into three parts. Construct a rectangle by the help of these parts (without overlaying). What extension could have the rectangle? Find all possibilities.

The second task is formulated in the way that the direct cuts should be constructed from one point and along the middle transversals. The solution instruction is given in task setting. The task setting is suitable for lower class of secondary schools or for slow students in upper classes of secondary schools.

Task 3. Two sides of a triangle ABC have: $a = 15$ cm, $b = 8$ cm. What size could gain the length of the third side c? What value c will be the angle ABC for:

a) acute angle,

b) right angle,

c) obtuse angle?

Divide the triangle into three parts by the help of two direct cuts. Use these parts for construction of a rectangle (without overlaying). What extension can the rectangle have? Find all possibilities.

The setting of the third task motivates students to discussion how to construct direct cuts in a triangle. Students can construct direct cuts from one point or from two different points. It is necessary to construct the centres of the sides and the middle transversals in triangle and the searched shape arises from composition along the parallel sides.

The task setting has an influence on the difficulty of the task. The task setting with help for students could be given to low students or in lower classes of secondary school. The generally setting leading to discussion about task solution we can give to talented students in lower classes of secondary school or students in upper classes of secondary school. The task leads students to search for different task solution, to develop the mathematical consideration and to teach student to discuss the task solution. It develops the logical thinking of students and possibility to lead skilful discussion.

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FROM MIRROR REFLECTION TO THE CONCEPT OF LINE SYMMETRY ON THE PLANE

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Rationale

In this paper I would like to discuss the growth of one of the geometrical concepts: mirror reflection and the way the child pass during the process of discovering certain properties leading to the concept of line symmetry.

Mathematical definition of this transformation concerns the entire plane and refers to the point. For children more natural is local symmetry being a transformation of a limited figure into the limited figure. At the beginning it is reduced to the situation in which the figure has not common points with the line of symmetry. Therefore it is considered as a mirror reflection. Perception is of a great importance in the process of learning and teaching on this stage.

The problem of the research work is: how the passage between visual perception of shapes on a plain or relation of the shape towards another on a plain and in contrast to noticing the dynamism of the relation look. In Poland this problem has significant didactic consequence e.g. in teaching line symmetry. Children start making ink - stains and paper cut - outs (10-year-old students) and they observe mirror reflections (they use a mirror). During the activities the terms are introduced: mirror reflection, symmetric figures and a line of symmetry.

The concept of mirror reflection and all names were introduced in the examples: *'During making ink stains there is the same shape on each half of the paper. Shapes are placed in the way that they are the mirror reflections of each other. A bend line of the paper specifies the placement of a mirror.'*

Figures having a line of symmetry are considered to be specific examples of symmetric figures (two halves have the same shape and they are mirror reflection). In the next stage (11 years old) students deal with figures having a line of symmetry, they learn about concept of line symmetry of a figure and congruent figure (as figures that overlap). As you can see, the intention of conception is internalisation of the activities done and leading to acceptation of definition of line symmetry.

The authors of this conception assumed that a figure and its reflection are of the same shape. Motivation of an ink stain leads to the fact that the copied figure is of the same size. The assumption that the figure and its reflection are of the same shape may cause some kind of cognitive conflict. In the sense of child's perception and by creating the environment meaningful it may be not the same shape. Potentially, the cognitive conflict which can be discovered by every child, but can not be identified or named, causes problem with the specialization of the concept and with leading to subtle understanding of reality.

I will analyse some specific characteristic situations observed in following stages of my scientific work, which are supposed to present the evolution of understanding the mirror reflection by children through acquiring experiences. I will also show how children discovered properties which preserve in mirror reflection (shape and size) and the ones which change themselves (orientation of the figure). Research lasted 4 years. Detailed description of whole research and research tools one can find in [1–3].

Methodology

My research has embraced four kinds of situations which could be named as:

[1] the diversity of objects,

[2] identical objects placed differently;

[3] two congruent figures placed differently and the process of overlapping one figure with another one;

[4] the process of overlapping one figure with another one.

Properly constructed game for 2 students was the instrument of research on the each stage. The first stage of research was devoted to verify in what way children will talk about the mirror reflection on the plane, to what extend they will differentiate and how they will describe this relation. A board game, which was based on choosing objects and pointing common features, was the diagnostic instrument. Diversity of objects on the board enabled to indicate different features which match the objects. One of them could be the specific placement on the plane such as the relation of the mirror reflection. On the second stage of research I limited the diversity of objects. I placed 2 families of congruent figures. They were placed differently but in a characteristic way (line symmetry, glide symmetry, rotary symmetry, translation). The placement would have been the common feature. I introduced dynamism in the third and fourth stage of research. Children overlapped one figure with the another. The movement was in the 2 dimension space. Transformations made by children can be examples of geometric transformations on the plane [5].

Situation: the diversity of objects

The pupils' goal was to find figures the same, in a certain respect and explain his/her choice by pointing on the feature combining one object with the other one. The students are required to look and describe common characteristics of the objects in the pictures.
It turned out that **in the case of specific geometric figures** (triangles, rectangles) there was no need to refer to the placement of objects and connecting it with the relation of the mirror reflection. Children at this age had enough geometrical knowledge and knew the properties of figures so that they did not need to join identical triangles in pairs because of the specific placement. Figures had a lot of properties - the same number of sides, angles, having an acute angle etc. It was enough to join them in pairs and point out the common feature. Whereas taking other objects into consideration (not figures) which a child can not give a specific geometrical name to, I noticed different argumentation.

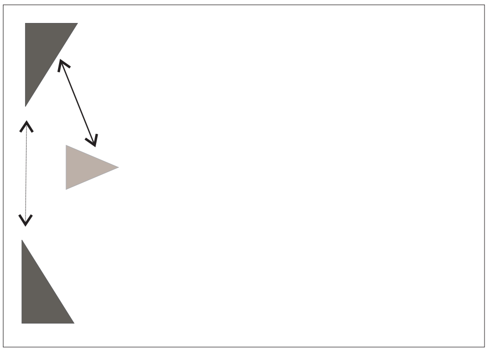


Fig. 1. Example of the choise of two figures which are elements of the set of triangles.

Example 1.

Sabrina looks at objects on fields 55 and 62 - musical note having "tummiesón different sides. She goes to the other side of the table, but finally before making a move she says:

S11: ... Not these because it is on a different side....
She wonders about a selection. She changes her mind and selects two loops (57, 30) saying:
S12: They are different but they both have this (shows a loop).

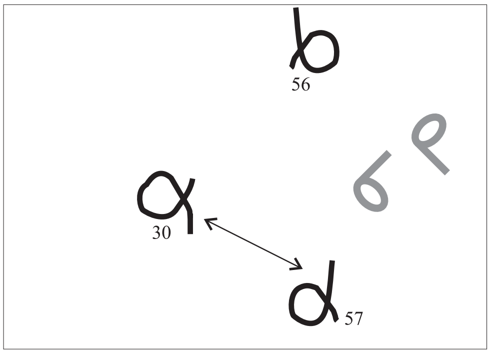


Fig. 2. The objects on a board.

Sabrina noticed a difference in the orientation of the objects-that makes them different physical objects, despite the same shape (Fig. 1). She decided to choose objects which we can name colloquially as loops, according to their shape identity. She considered other objects of the same shape but different orientation. She rejected these figures discovering that they differ too much. (S9: 'because it is on the other side'). Her choice indicates clearly that the difference in the orientation was the factor diversifying the objects strongly. The identical shape proved to be the weaker feature combining the objects.

Example 2.

K3: I'm choosing 19 and 49, because these are the same notes, but this (19) is in different side.
KK4: 62 and 55, because they are in different position, but they are empty inside.

Kuba referred to two properties: shape and position. However, he decided on the basis of one more fact which he could not precise. There were three more areas presenting the same objects as chosen, but he did not point them (Fig. 2). He also did not choose the figure in the privileged position the vertical - the horizontal level which is differentiated by the majority of pupils. He did not indicate the third figure which had an orientation consistent with the two chosen. The explanation of this fact is not clear. He could have decided to choose consciously the two objects which were relatively the closest to each other and demanded only a little rotation in the mind to discover that they are *'the same notes'* (K3). He differentiated them from among the rest which were not exactly *'the same'*. Consistent orientation of objects decided about his choice. The difference in the position was noticed and emphasised but it did not disqualify the objects identity.

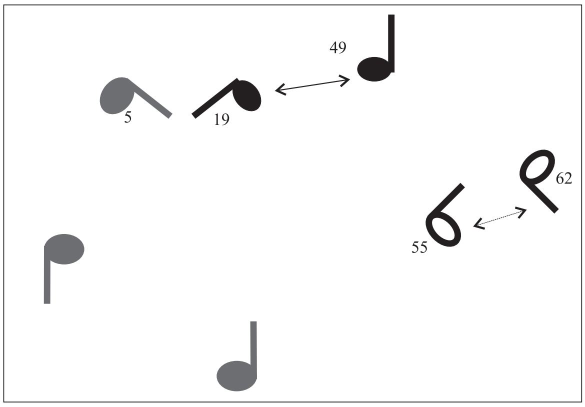


Fig. 3. Objects19 and 49 and objects 55, 62 on a board. The other objects with the same shape are marked gray color.

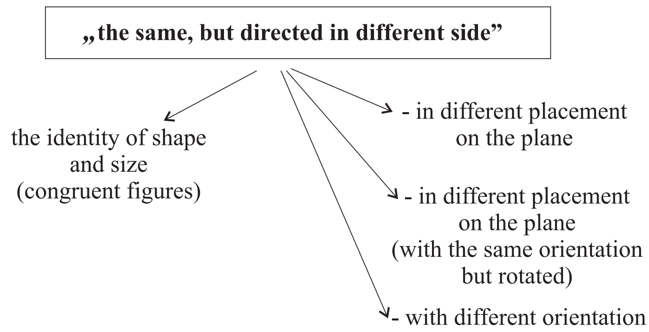
Kacper - the second student - clearly followed Kuba making his choices. He found the objects of the same shape on the board. He chose empty figures in order not to follow his opponent's choice. 'Empty inside' was the feature combining them. Kacper did not name it specifically. He emphasised different position in his argumentation. The same shape was a deciding factor. He pointed to differently situated figures but which were identical.

Example 3.

A66: I'm choosing 55, 62 because they are like notes empty inside. (Compare with Fig. 3)
B67: These are not notes. There is a stick placed incorrectly in one of them.
A68: Yes, they are. If you drew this (62) and then reversed the board, it would be that! This is similar to a note. These figures are similar to notes and similar to one another. They are **even the same***, but their sticks are placed differently.*
B69: But you did not say it like that at the beginning.[...]
A71: You only have to look from the other side at this board **and you can see the same** *- are you convinced now?*
B72: That is all right.

Adela explained how we should look at the one of the figures to notice that they are identical. The crux of the choice was the relation a figure to a figure. This relation was of a dynamic character. The figures were the same, because they were their own images in some transformation (here it was mental transformation), which did not change neither the shape nor the size, but only 'reverse on the other side'. Adela's justification enabled her to find the feature combining the figures of identical shape and size, but different orientations. Both girls recognised the identity of the figures, despite different orientations (stage of 'being (directed) in different side').

Relatively often students used the description 'the same but directed in different side'. Its meaning was different and embraced three kinds of situations:



Situation: congruent objects placed differently

Example 4.

Filip does not understand how it is possible to refer to the position. He asks if it is correct to say that they have the same angles. Krzysiek makes a choice:
K1: 5, 15 -they are in **the same identical position**.
F2: 19, 15 - because the position is the same (he points the direction of the position of each figure, he places his hand in a way that his fingers indicate the pointed parts of the figure) [. . .]
K5: 22, 26 - they are placed in the same way, horizontally, in this direction (he shows the direction of the position of each figure separately, but it is the same direction). There is (a higher pointed part) on the right.

Chosen figures were their own images in translation (Fig. 4). For the student they ware in ***the same position***. When there was no choice of figures in translation, explanations appeared:

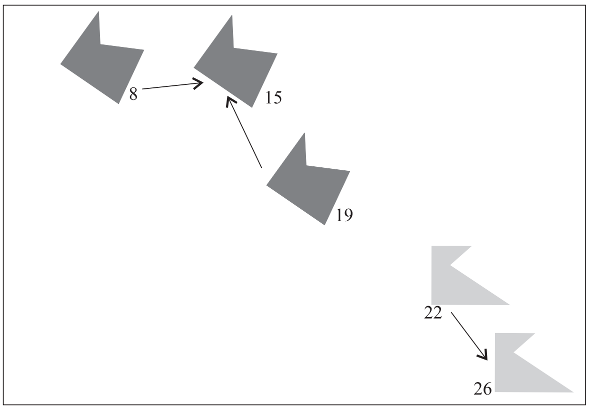


Fig. 4. The location of the objects from areas: 5, 15, 19, 22, 26 on the board.

K12: 29, 24 - here is the centre (he is pointing the line between figures) and they ... as if ... are diverging, they look like they have bigger obtuse.
F13: What? I do not agree!
K14: They are placed in a similar way, but on the other sides. This one is placed in the same way, but to the left (he is pointing to the object from area24), and this to the right (on the area 29).

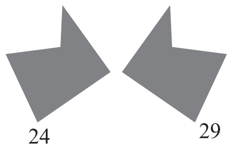


Fig.5.

Student's utterances were much elaborated. It indicates his difficulties in describing the specific position of the figure (Fig. 5). The distinction of the axis of symmetry was not difficult - Krzysiu used the term 'centre'. The position of two figures was referred to this imagined 'centre' - the axis of symmetry. We can combine diverging of the figures with their movement understood as the parallel shift of each figure. In the case of this movement one figure does not overlap the other one. They diverge in the same way but one to the left and the second to the right.

Situation: two congruent figures placed differently and the process of overlapping one figure with another one

The figure remained the same. The situation was changing. By using a computer I introduced some specific kind of movement on the plane and combined a static situation with overlapping of a figure with another one. Moves were performed by selecting an operation: *shift* (the figure is moved to the right, to the left, up or down), *rotate* (clock or counterclockwise), *reflect* (vertically or horizontally). I observed some characteristic students' behaviour.

Example 5.

In the problem on Fig. 6 it would be enough to reflect the figure. After thinking a while the student recognized different orientation of figures. He/she knew that only by *reflecting* it is possible to change the orientation of a figure and that it is enough to use the operation *reflect* only once. The student reflected the figure horizontally and then by *rotating* obtained a translation placement.

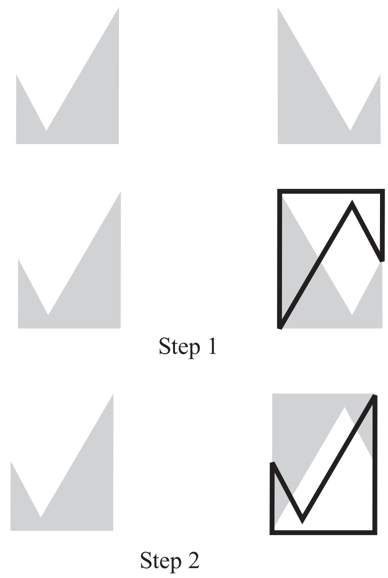


Fig. 6. An example of student's solution.

In the situations showed on Fig. 7 overlapping a figure with another one requires both: reflecting and rotating one of the figures. This arrangement of figures required from the student a while of thinking. It indicated moment of a deeper analysis of the figures' configuration, imaginative transforming of the figure, predicting the effects of the transformations, comparing certain characteristic of figures. After thinking a while the student reflected a figure vertically and rotated it to the translation position.

In those both situations the student had to change his/her perception of sameness of figures with operations of overlapping one figure with the other one. It required a certain mathematical strategy. Very soon, strategy of trans-

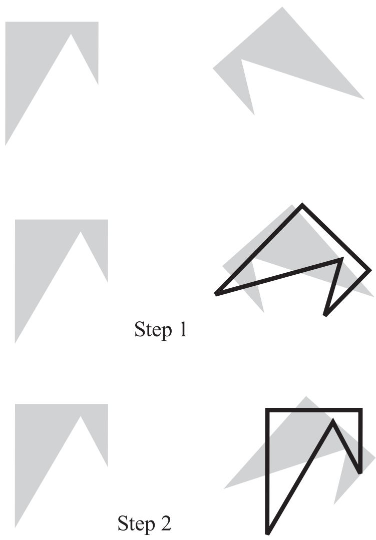


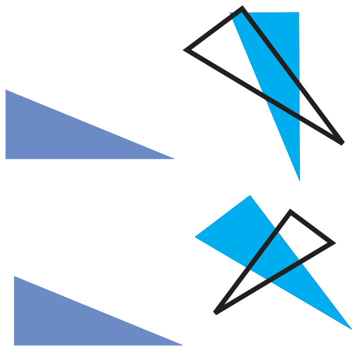
Fig. 7. An example of student's solution.

forming arose: through reflections, rotations attain the parallel arrangement of the figures, and late on it sufficed to translate them. Arranging the figures in a position of mirror reflection seemed too difficult. The operation *reflect* fulfil only role: it enabled to change the orientation of a figure. The placement of a figure on a plane after reflecting did no matter.

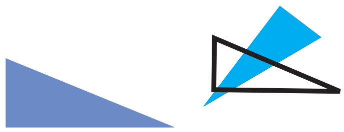
Situation: the process of overlapping one figure with another one

I had to perform the process of overlapping one figure with another one a little. It would not be available to use the operation *reflect* as a thirst transformation (it was blocked). In order to putt students' attention on the movement, the figures had different shape in each task.

It was a cognitive conflict between a visual analysis of a placement of figures and a possibility of movement. The students recognized a need of reflection but it was not possible to perform this kind of operation firstly (Fig. 8). They had to begin with rotating a figure. Finding a specific arrangement of a figure was the focus of their attention. They arranged the figure in parallel manner and then reflected it vertically or horizontally (they made a vertical reflection manly). After that transformation they were surprised because they would not be able to reach a good placement (Fig. 8, steps 1-2). They went on to rotating but it was a mistake rotating a figure twice. It turned out that they did not reach a good position for reflection. A lot of experiments enabled the students to find a new strategy of overlapping a figure with another one by rotating fist and then reflect the figure (Fig. 9).



Step 1.



Step 2.

Fig. 8. An example of student's solution.

It was a long way of investigation of arranging the figure in a position of mirror reflection in respect to horizontal or vertical line. It appeared that the students could recognise or intuitively guess some relation between figures; however these relations were not appropriate to the beforehand given line of symmetry.

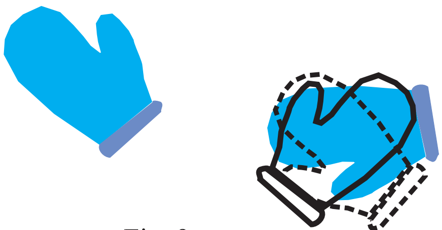


Fig. 9.

Conclusions

Analysis of my research results enabled me to determine the existence of some cycles in the development of the concept of symmetry on the plane by 10-12 years old children. The thirst cycle bases on perception. Students require them to build the basic concepts [4]. It is necessary to supply the specific situations concentrating their attention on the placement of figures and the relationships included in these placement to the students. The goal is not for a student to simply memorise the definition of line symmetry but to understand how to describe what he/she can see and understand the relation of a figure to another; to be able to replace movement with a relation between two objects as well as relation to movement. It is close to the actions become a process so the individual can describe or reflect upon all the steps in the transformation without necessarily performing them.

The *perceived object* [6] is the object based on perceptual information - seeing figures in a specific relations like mirror reflection, physically cutting figures and putting them appropriately, observing reflection in the mirror, in the water etc. The *conceived object* occurs when there is a reflection on perceptions and actions, so the focus is no longer on the specific physical manifestations but on the actions and processes performed upon them. In the process of learning and teaching from mirror reflection to line symmetry there are some specific efforts needed. The aim of them is to teach to discover definition conditions to use mathematics as a language to describe the real world.

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ON OBJECTIVE AND SUBJECTIVE DIFFICULRTIES IN UNDERSTANDING THE NOTIONS OF THE LEAST UPPER BOUND AND THE GREATEST LOWER BOUNDS

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Abstract. The notion of the least upper bound (the greatest lower bound) of a subset of real numbers is discussed from different points of view and some difficulties of this notion are presented.

One can observe misunderstanding or insufficient understanding of the notion of the least upper bound (the greatest lower bound) of a set in the practice of school teaching. This notion is in fact very difficult but has a great importance in the modern mathematics. Let us start our considerations from some theoretical background. We will consider only least upper bounds, the greatest lower bounds one can define and discuss in the same way.

1. Theoretical foundations

In textbooks on mathematical analysis one can find different kinds of definition of least upper bound (supremum). Let us remind them.

Definition 1. A number d is called an upper bound of a nonempty subset A of the set of real numbers if

$$x \leq d$$

for each element x from the set A.

Definition 2. A number d is called the greatest element of a nonempty subset A of the set of real numbers, if $d \in A$ and $x \leq d$ for each element x from the set A.

Definition 3. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean the greatest number of this set or the least one from the set of all upper bounds.

The least upper bound of the set A is denoted by $\sup A$.

Definition 4. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound and

$$\big(\forall d' < d\big)\, (\exists x \in A)\, (d' < x \leq d) \tag{8}$$

Definition 5. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound of the set A and

$$(\forall \varepsilon > 0)\, \big(\exists x \in A\big)\, (d - \varepsilon < x \leq d) \tag{9}$$

Definition 6. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound and

$$\big(\forall d'\big)\, (\forall x \in A)\, (x < d' \Longrightarrow d \leq d') \tag{10}$$

Definition 7. By the least upper bound of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound of the set A and

there exists a sequence $(a_n)_{n=1}^{\infty}$ *of points of the set A such that* $\lim\limits_{n\to\infty} a_n = d.$ (11)

One can see that all definitions are equivalent for subsets of the set of real numbers. In fact. In each definition the least upper bound is an upper bound of the set A. So, to prove equivalences of those definitions we have to prove that each of the conditions (8), (9), (10) and (11) is equivalent to the statement that d is the least of all upper bounds of the set A.

Suppose that d is the least upper bound (in the meaning of definition 3. Now, if d' is any number less than d, then it is not an upper bound of the set A, hence there exists element x of the set A such that $d' < x$ (and of course $x \leq d$); in this way we have proved that d fulfils condition (8).

It is easy to see that conditions (8) and (9) are equivalent since the role of d' can play $d - \varepsilon$ and conversely, in place of ε one can take by $d - d'$ in adequate conditions.

To prove that definitions 4 i 6 are equivalent, let us notice that d' in condition (10) is an upper bound of the set A too, so it must be greater than d.

Conversely, if condition (10) is fulfilled, then no less element than d is an upper bound of the set A.

2. Commentary

At school programmes one can find the item *least upper bound (supremum)* of a subset A of the set $\mathbb{R}$ of real numbers. Many teachers prefer to introduce definition 3. This definition gives no suggestion how to check, whether an upper bound d is the least one. From this point of view the definition, which is very popular, is not very useful to apply. So it is no wonder that the students do not like to apply this definition and have several difficulties in it.

Next definition which is popular among the teachers is definition 7. This definition is much more difficult for students than definition 3. The most difficult problem in this definition lies in application the idea of limit of a sequence. So the two of most popular definitions have many disadvantages in applications and understanding of them.

In view of this remarks, the best condition for defining least upper bound of a set appears to be condition (8). It says that an upper bound d of a set A is the least upper bound if any less than d element is not an upper bound of the set A. For school purposes this definition seems to be the best one. So we can come to the conclusions, that the definition 4 should be the main one and after this definition had been introduced one can show the equivalence of all other conditions formulated in conditions (8) – (11).

3. Conclusions (Propositions)

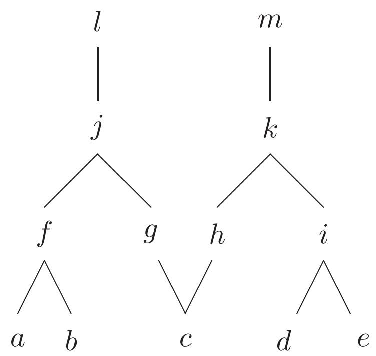
Let us consider now the problem of supremum of a set yet this time from the other point of view; I mean from the generalized kind of order. If we consider partially ordered set X, it means a set X equipped with relation of partial order. Let us remind; a relation $\prec$ is a partial order in a set X if it fulfils the following conditions:

- $x \prec x$,
- if $x \prec y$ and $y \prec x$ then $x = y$,
- if $x \prec y$ and $y \prec z$ then $x \prec z$,

for all elements x, y and z from the set X.

Nowadays we observe the enormous development of computer science. In many applications of this kind of science we can find the idea of (partial) order in small sets, namely finite sets. One can expect that such ideas will be found in school programmes quite soon. In partially ordered sets there is no possibility to define ideas like supremum in the way we discussed before. The problem lies in the fact that not every two elements of a set X can be comparable. Because of it we must define supremum in a way which had been pointed in definition 6. In consequence we have to define supremum by condition (10) since there is no other possibility to define it. Then the idea of partially ordered sets is the most general one. Later on, by specification, one can define supremum for subsets of a linearly ordered set; here one can use Definition 4. Definitions 5 and 7 can be useful only for subsets of the set of real numbers in which not only ordered structure but also algebraic (Definition 5) and topological structure (Definition 7).

There are some advantages for this sequence of conditions defining supremum. If X is a finite set with partial order in it, then one can illustrate it in a diagram. For example, if the set X is partially ordered in the way presented in the diagram:



then we can observe that:

$$\sup\{a,b\}=f, \qquad \sup\{a,b,c\}=j, \qquad \sup\{a,c\}=f$$

but there is no supremum of the set $\{a,d\}$.

Diagrams are very acceptable by students and improve the understanding of the very difficult idea of supremum. From this it is possible to come a little further and introduce the idea of Boolean algebra, which have many applications in computer sciences. I suppose that this kind of thinking is worth to discuss.

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DEVELOPMENT OF CHILD CONSTRUCTIONS– INTERCONNECTION: RESEARCH AND STUDENTS' TRAINING AT SCHOOL

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Abstract. Course of pre-mathematics includes theory as well as practice. The development of different competences is uneven. Sometime students have difficulties to imagine how to take advantage of research results (part of research VZ MSM002160862). The goals of student practice training: to observe children and to enroll the development of special abilities, (pre)-concepts, to use in practice one of diagnostic activities with children. The development of construction with build-set is used as a one of the themes which can link all these goals.

1. Research and university courses

Necessity of research transmission to the school practice is imperative. What is the reality? Are we able to create conditions for it?

Students enter into contact with university research in several different ways. "The classical type"which has been traditional at the universities for centuries takes the form of recommended literature and lectures or simply of lectures. In that case the student is relatively passive in his/her contact with the given research. Another type is the unfinished research where the student can participate in the realization of some if its parts. The measure of students' participation and their number depend not only of the character of the research, of the technical and financial conditions but also of the level of students' capacities and organizational possibilities. Both of these mainly used types of students' contacts with research are functional in the case of future teachers' education if the students' participation is linked with their pedagogical practice in school (as shown by experiments in the sphere of transfer of research results into school practice, see Kaslová) and if the students' contact with research is a long-term one. Let us consider the possibilities among which the future teacher can choose (see table 1).

Table 1

Lessons, courses, homework	Research is not finished 1) students take part in partial tasks and not in the complete realization of the experiment: (a, b)	Research is finished 2) presented to the students indirectly;
In cooperation with school practice training	3) students participate in the pilot research during their school practice training	4) students realize/copy the part of research; compare their results with research results

Remarks:

1) Unfinished research - the students take part in partial tasks and not in the complete realization of the experiment: a) The said experiment is at its beginning - the student participates in the research concerning its theoretical part. b) The experimental part is finished - the student takes part in the data treatment: classification, tabulation, use of statistical methods, creation of graphs and diagrams, transcription of audio-registration. The aims of such participation lie mainly in the following spheres - both approaches serve to acquire the techniques that the future teacher will be able to handle, especially when participating in the future researches that will help him/her to understand the data used in professional articles, he/her will get used to follow the professional literature. In certain cases the student is made to analyze data in the framework of discussion. In that case the future teacher lacks in general direct contact with school practice.

2) The experiment is finished - the research has been presented to the students indirectly. The aim is to teach the student to follow the actual research and to discuss it, especially its conclusions in a given context and the possibilities of its application. In general the knowledge, the information received in such a manner is not accepted if they are in the contradiction with his/her school experience as pupil. In another case the student tends to isolate the new / pleasing elements from their context or without taking into consideration the normal composition of the pupil population, to expect that in their following practice the school results are to be a 100% identical with the successful ones presented.

3) In case the student has the possibility to participate in a non-finished research, in the sphere of the realization of its experimental part - link the

mathematical subject taught in lectures of seminars with pedagogical practice then the problem is in general participation in a pilot research. His remarks in the framework of reflection can serve to correct the experiments' scenario. It is up to the student if he chooses to study the necessary literature concerning the given theme. The student will not be informed about the result of the experiment, about his part in this phase of experiment which can lead to hasty generalization, to a lack of preparedness to follow different phenomena at the same time and to evaluate them in the general framework. Less gifted students tend therefore to simplify and underestimate the experiences or their results. We have here the Pygmalion effect, and in the case of "distance studentsąlso the transfer phenomenon when interpreting or presenting data. On the other hand this type can serve as starting point for future work, it can be part of the university teachers' strategy following up the said situation.

4) In the case that the research is finished, the students can be informed about it (eventually with partial presentation of school experiment) and invited to repeat the experiment within small groups during their pedagogical training or practice. The aim is to teach students to repeat the experiments or part of them on small samples, to compare the results obtained and to discuss them. The discrepancies among the data obtained within student groups enrich the discussion and frequently tend to encourage the student to consult further professional literature and teach him to reflect on the transfer of experiment results into practice. Moreover they create the habit of communicating with colleagues and comparing not only work results but in particular pedagogical strategies.

The type of contact chosen depends on further factors (see scheme).

2. Research as a part of curriculum

In the lectures on pre-mathematical education we teach mathematics, pre-mathematics and their didactics. The theme of the researches (eventually experiments) to be chosen in order to embrace several themes at the same time and enable the students to conceive the chosen problematic under different angles so as to be capable to go back to the given problematic. One of successful researches was the long-term research "The development of child construction - children aged from 2 to 8 years". Type number 4) was chosen to be included in the program of studies of pre-school pedagogy. At present we are following up the results of that research and considering of linking it to the themes under study. We finished longitudinal research - development of constructions using the build sets – in the moment when many kindergartens liquidate the build sets, cubes etc Different kind of build sets (TOFA, KAPLA, MAXIBlock, LEGO, etc.) have been presented to children from 2 to 8 years old. We have been following among others the phenomena appearing in the finished constructions, therefore how to characterize the different constructions. Among the characteristics we distinguish: the direction of the development of construction, the existence of spaces between the different parts, their location, dimensions, rhythm, compactness, bridging of the space, symmetries, orientation of the parts – wall/angle vis-a-vis the observer. The position of the parts vis-a-vis the predominant dimension – horizontal, vertical, slant, the existence and eventual number of dominants, stability, non-standard solution.

With Tofa set we had determined 5 phases of construction development, as for other construction sets different phenomena follow the same model but due to the characters of the sets certain phases or elements of a phase are missing (for example in KAPLA, towers are excluded).

3. Phases of the child development of constructions (build set TOPFA)

The characteristic of phases is shortcut and illustrated by choice of photo-documentation.

• **Phase zero:** Scrutinizing, Studying the different forms by touching them, Studying of the sound made by the piece when hitting the surroundings, This phase is common to all people whatever their age. But the older the person is that phase gets shorter.

- **Phase one:** composition and decomposition of tower or train, or serpent - 1 dominating dimension.

- **Phase two:** compactness, 2 dominating dimensions, symmetric, one or two dominants; smooth façade.

- **Phase three** starts step by step; the order of phenomena is variable: Existence of small pauses ($p<1/2a$), regular rhythm of pauses' position; a symmetry of construction, 3 dominating dimensions, plastic facade;

- **Phase four:** different pauses, different rhythms of pauses, Technical difficulty of the composition - stability of construction as a principle.

Their advantage is the evelopment of child construction - it is a complex theme: spatial orientation and imagination, spatial memory, composition, manipulation, typology of grasp, verbal and non verbal communication, notion of number, some geometrical notions, identification and classification of object forms, transformation from 3d to 2d and vice versa, development of manipulation, development of algorithmic work, using of selection, choice, exclusion, evaluation, correction, development of hand cooperation, ... etc. main phenomena which can be observed, context of child behavior, laterality, spatial memory, preferences of direction during the construction process, etc. This theme offers us the possibility to focus children as well as well pre-mathematics in the relation with pedagogy, special pedagogy and psychology (collaborate with other department of faculty):

4. Students

Students - future teachers of kindergartens are prepared during baccalaureate studies – 3 years of studies at the university. The pre-mathematics (pre-mathematics literacy) is included in the second year of studies; it means that we teach mathematics, pre-mathematics and didactics of pre-mathematics together. The results of research are part of studies, but students have not yet enough of experience as to be able to imagine the real situation and accept results by including them into their work. Part of the chosen research serves as a methodological example and as an introduction of a new didactical theme. We decided to use a combination of strategies 3) and 4). We presented the results The demonstration – part of experiment presented in a group of 6 or 8 children 5 years old including the description of child behavior and their work results and the classification of discovered phenomena and the analysis of data. Students had to reproduce it during the practice training at kindergarten and compare obtained results with the results in the text and the results of pilot demonstration. This work has to be presented in a written form documented by photos. At the end of practice, students present their work and discuss it together.

Examples:

Student 1: I was surprised that the children created the same thing as the one during the demonstration ... but the table is similar to your article ... how is it possible?
Student 2: I started to observe children's activities differently from that of my first practice at school. It was very interesting for me ... the child repeated the same type of construction but the form of the construction changed.
Student 3: I observed children in detail, I wanted to discover new phenomena in their construction and enrich your results.

Student 4: The teacher told me that there was no build set in the kindergarten. I had to bring my own, but it was not sufficient because children were very interested in this activity.
Student 5: I do not like making construction. It was nice... children worked a long time with cubes, they were well concentrated, they asked me to play with them, to modify constructions, ... and discovered a lot of different activities.

5. Conclusion

This form of work and this theme present a good example of effective research result transmission to the practice.

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EXAMPLES OF INVESTIGATIONS FOR BEGINNERS

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Abstract. There are examples of several types of investigations available for beginners: 1. search for patterns, 2. iterating a certain procedure and analysing the results, 3. looking for exceptions, or special cases in a pattern, 4. generalizing given problem.

Investigating a pattern of numbers or shapes can lead to discoveries and may raise a number of challenging questions. Investigative work is a suitable introduction to the art of problem solving. There are examples of several types of investigations available for beginners:

1. search for patterns;
2. iterating a certain procedure and analysing the results;
3. looking for exceptions, or special cases in a pattern;
4. generalizing given problem.

We want to demonstrate these types of investigations with the help of examples.

Example 1: Investigate sums of the first odd natural numbers.

Solution:

Systematic experimentation:

$1,\ 1+3=4,\ 1+3+5=9,\ 1+3+5+7=16,\ 1+\ldots+9=25$

Our sums are square numbers.

Conjecture: $(\forall n \in \mathbb{N})\ 1+3+5+\ldots+e_n = k^2$, where $k \in \mathbb{N}$ and e_n is the nth odd number.

Other systematic experimentation:

$1+3=2^2,\ 1+3+5=3^2,\ 1+3+5+7=4^2$, and also $1=1^2$

New conjecture: $(\forall n \in \mathbb{N})\ 1+3+5+\ldots+e_n = n^2$

Proof can be done by mathematical induction. After that we have

Mathematical Theorem: $(\forall n \in \mathbb{N})\ 1+3+5+\ldots+e_n = n^2$

If we use geometrical way of investigation, we get the following figure (see Fig. 1)

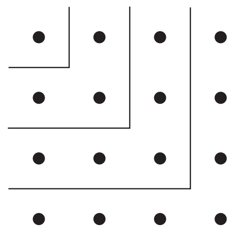

Fig. 1:

Example 2: Let T_0 be an equilateral triangle of unit area. Divide T_0 into 4 equilateral triangles T_1 by joining the midpoints of the sides of T_0. Now remove the central triangle. Treat the remaining 3 triangles in the same way and repeat the process. It means iterace – n-times. Find out:

a) What is the sum S_n of the removed triangles after n-step?

b) What happens to S_n as n tends to infinity?

Solution: Iteration of our procedure leads to

1. step ..... 1 removed triangle T_1 of area $1/4$
2. step ..... 3 removed triangles T_2 of area $(1/4)^2$
3. step ..... 3^2 removed triangles T_3 of area $(1/4)^3$
4. step ..... 3^3 removed triangles T_4 of area $(1/4)^4$

Generalization:

At the nth step

$$3^{n-1} \text{ removed triangles } T_n \text{ of area } (1/4)^n$$

a) $S_n = 1 \cdot \frac{1}{4} + 3 \cdot \left(\frac{1}{4}\right)^2 + 3^2 \cdot \left(\frac{1}{4}\right)^3 + \ldots + 3^{n-1} \cdot \left(\frac{1}{4}\right)^n = \frac{1}{4} \cdot \frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}} = 1 - \left(\frac{3}{4}\right)^n$

b) If n tends to infinity, $\left(\frac{3}{4}\right)^n$ tends to 0 and S_n tends to 1.

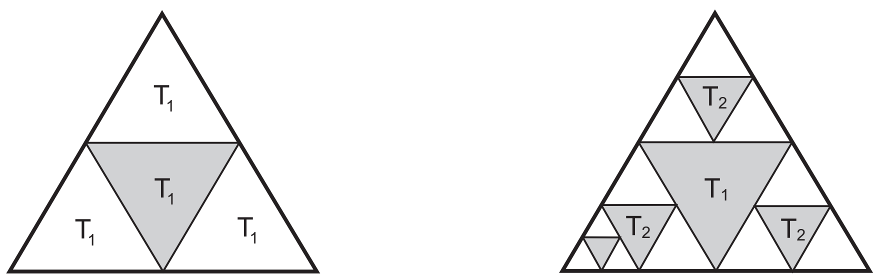

Fig. 2:

Remark: Consideration in b) is very good preparation for limits of sequences and functions.

Problem 3: Figure 3 shows three mirrors of length l forming a triangle ABC. A light source is placed at a point S of AB, at a distance d from A. A light ray, emerging from S at an angle of 60° with SB, gets reflected from the sides of triangle ABC until it returns to S. Find the length of the light ray's path in terms of l.

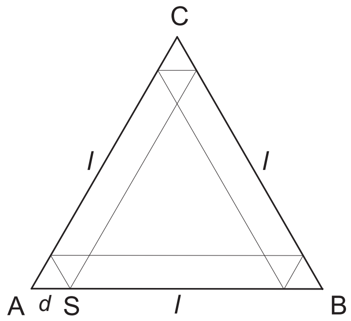

Fig. 3:

Solution: It is not difficult to prove that the length of the light ray's is $3\,l$. This answer does not involve the distance d of S from A. So it seems that the length of the light ray's path is the same for all positions of S on AB. But there is one exception. If S is at the midpoint of AB, the length of the path is only $3/2\,l$.

Problem 4 (Pappus problem)**:** ABC is an arbitrary triangle and ACDE and CBFG are arbitrary parallelograms constructed on two of the sides. Lines ED and FG meet in H.

Construct a parallelogram ALKB on the third side AB such that AL and KB are equal to CH and parallel to it (see Fig. 4).
Prove that the area of ALKB is the sum of the areas of ACDE and BFGC.

Solution: On EH construct the point O such that AO is parallel to CH, and on FH construct P such that BP is parallel to CH. Line HC meets AB in N and LK in M (see Fig. 5).
ACHO is a parallelogam having the same base AC and the same corresponding height as the parallelogram ACDE. ACHO and ALMN are also parallelograms with equal bases (CH = NM) and equal corresponding heights. It means

$$\text{Area ACDE} = \text{Area ACHO} = \text{Area ALMN}$$

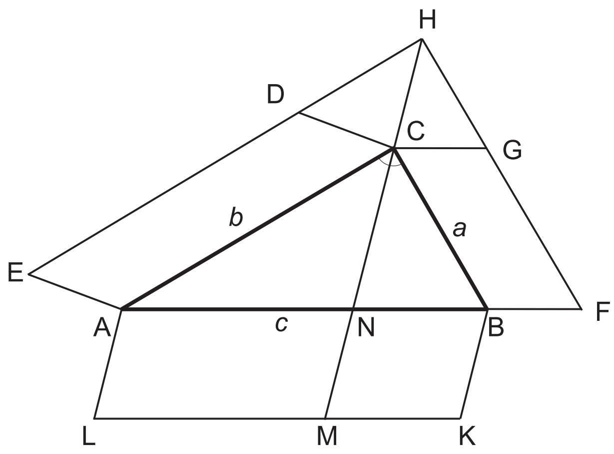

Fig. 4:

Similarly: Area BFGC = Area BPHC = Area NMKB
This implies that: Area ALKB = Area ACDE + Area BFGC

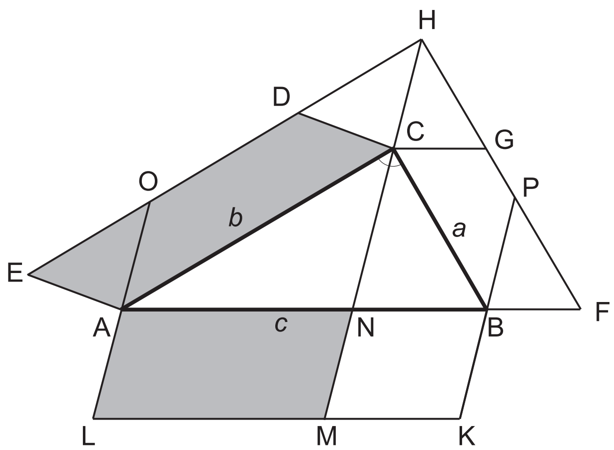

Fig. 5:

Remark: Problem 4 is one possible generalization of Pythagoras Theorem.

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COEFFICIENTS OF LEARNING IN MATHEMATICAL AND NONMATHEMATICAL SUBJECT MATTER

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Abstract. Ćoefficient to remember something"was introduced in cybernetic pedagogy. This coefficient expresses what part of information (from group of letters arranged without meaning) a learner is able to remember after one repetition. He can remember about 1/23 (4,34%) received information. We have derived ćoefficient of learning". Its values are greater, because understandable learning (we mean it) is more effectively than memory learning. We used this coefficient as expression of improvement of soft motive hand. Its value was about 6%. We found out it in ten pictures arranged chronologically during 4 months. We valued subject matter pretension by ćoefficient of understanding". We found out what children could understand subject matter with one repetition. Similarly ćoefficient of disclosing"was introduced for revealing of coherence reading of picture. It was 38% after the first experiment. It means that this number of children revealed coherence in the picture. It is possible to value subject matter pretension effectively and briefly according to introduced coefficients in standard class. On the other hand it is possible to value knowledge level of pupils by using standard subject matter.

1. Introduction

Many of appreciated sciences at present have become sciences due to mathematics and its tools, which enable us to express values of observed phenomena by numbers and according it to formulate law. A law is a valid statement in generally according exactly determined conditions. Even measurement and quantity of conditions is important, too. It enables us to express a law in form of function. All input conditions occur on its admission and result is represented by number as an output. It is consequence of admissible conditions.

These laws are taught incorrectly sometimes. Maybe the reason is that they have been already expressed incorrectly in textbooks or they have been taught incorrectly from time immemorial. I remember one school experiment. We warmed water in a vessel and measured time. We found out temperature of water equably grew above constant flame in dependence on time. It was a law, so our different measured values were żounded offąnd it was in harmony.

Really it was not true. Water had certain temperature in the beginning of measurement, for example 12^0 C and value was near 100^0 C later. If we take into consideration law about lead of warm, we find out that amount of warm led away from one body into other body depends on difference between temperatures of warmer and cooler. We neglected this difference of temperatures. Amount of warm delivered to water (for unit of time) in the beginning of warm was higher than at the end of warm, because exchange was made during higher difference of temperatures. Theoretical line of warm becomes a curve. It comes near to temperature of warm very slowly at small difference of temperatures. Then line and curve becomes tangents. Getting cool is similar process.

2. Comparison curve of forgetfulness and getting cool, warm and learning

A curve of getting cool passes very similarly as known curve of forgetfulness. A curve of warm resembles to curve of memory learning [1, 2005] very much. It seems that reason of the shapes of these curves causes diminution of differences between "temperature of mediumąnd "finish temperature". On the other hand difference between "mastering of subject matterąnd śubject matter that should be learnt"(aimed subject matter) makes smaller.

Value of learning coefficient $k = 0,043$ ($K = 4,3\%$) was measured during memory learning. It means a pupil could remember about 1/23 of subject matter after one repetition.

If amount of unlearnt subject matter became smaller, the respondents remembered about 1/23 from smaller amount of subject matter. Research was carried out the way that the respondents learnt about groups of letters without their meaning by heart [2, 1996]. If we draw "ignorance"in graph, the curve declines gentle to axis x - there is zero ignorance on it. When the respondents learnt the groups of letters with meaning, the coefficient of learning (memory) was much higher. Analogy thinking and episodic memory had there own part.

3. Coefficient and graph of understanding subject matter – ideal graph

If pupil understands subject matter, it is a manifestation of intelligibility and subject matter convenience to age. All pupils do not understand the whole subject matter for the first time. The process of understanding - ideal graph marks that every other repeating (explaining) of subject matter was understood approximately by equal proportionate part of pupils who did not understand to subject matter yet. We introduce 128 pupils and coefficient of understanding $k = 0,5$ as an example of ideal graph. Numbers of pupils who did not understand subject matter after every other explaining make a sequence {128; 64; 32; 16; 8; 4; 2; 1}. It is a geometrical sequence $a_{i+1} = k \cdot a_i$ with coefficient $k = 0,5$. Its graph declines to axis x and becomes tangent of graph for large ones. Steepness of curve depends on greatness of coefficient k.

4. Understanding subject matter – research

Research sample was created by 126 pre-schoolers (from kindergartens and zero classes at elementary school). We chose subject matter: explaining of notions "bigąnd śmall". Numbers of pupils and coefficients of understanding created next sequence {126; 75; 32; 13; 3; 2} after every repeating during learning. We counted coefficient of understanding k after every repeating. The values were {0,60; 0,63; 0,68; 0,50; 0,33}.

Investigated sample of pupils - pre-schoolers is characteristic by higher value about understanding of subject matter than "idealśtate with coefficient $k = 0,5$. We regard subject matter easy understandable according shape of curve and weight average of coefficient about understanding. Curve shape about understanding of notions approaches "idealśhape of learning curve. (Average deviation is 15,7%, weighted average deviation is 4,1%.)

5. Improvement of soft motive hand

We chose research sample of 30 pupils - pre-schoolers and 10 pictures which were painted by pupils chronologically during 4 months and we measured the third longest óverstep"**P** in millimetres. The children painted outlined pictures in workbook for pre-schoolers "Bude škola 1"(watering-can, frog, high boot, ..., hare). We counted values of the third longest óverstepóf all children. Then we found out sequence which uncovered improvement of soft motive hand focused on its punctuality. The sequence was {120; 108; 104; 93; 89; 84; 79; 80; 78; 70}.

Average punctuality P was improved about (120–70). 100/120 = **41,7%** during experiment in 4 months. Improvement between two next pictures had average value $k = 0,046$, it is only a little better than coefficient of memory learning. Anomaly in graph was created during painting of the 8^{th} picture and its surroundings. We awaited equable improvement here, but results made worse. The coefficient of learning had negative value there. Value of coefficient "k- improvement between two neighbouring pictures had following values {0,10; 0,037; 0,106; 0,044; 0,056; 0,059; -0,01; 0,025; 0,1025}. Shape of graph is not linear so we can expect smaller absolute value of improvement of soft motive hand at equal average coefficient of improvement in next four months.

6. Disclosure of situation in the picture

219 children took part in experiment. They had two pictures with educational situation. They were to uncover "Why is a figure (coloured pencil) sad?ón the 1^{st} picture. They disclosed that the coloured pencil hurt its finger. The car stopped rapidly in front of the pedestrian crossing on the 2^{nd} picture. The children were to uncover what happened and why. They disclosed snake's right of way. The teachers and parents who investigated these picture situations with children were instructed about to must not betray true answer. The number of pupils changed during 3 possibilities of disclosure of notions as follows {all pupils 219; rest after 1^{st} disclosure 135; rest after 2^{nd} disclosure 87; rest after 3^{rd} disclosure 60}. Coefficient values of disclosure were $k \in$ {0,38; 0,36; 0,31}. Average coefficient value of disclosure was approximately $k = 0,35$. It means that wanted notion was disclosed by approximately 35% of children from the group of children who did not uncover this notion yet during one observation of picture. (Average deviation is 7,6%, weighted average deviation is 6,9%).

7. Summary

According observation of shape about physical processes the hypothesis was created about similar shape of learning process, too. It was attested reliably at memory learning. The expectation of similar shape of other mental processes was (approximately) attested (in small sample) during measurement of improvement of soft motive hand in week intervals. Shape of curve with other coefficient was attested during investigation of understanding subject matter, too.

According achieved coefficient $k = 0,046$ we can regard improvement of soft motive hand as a very pretentious educational activity. It can be compared with memory subject matter, when coefficient was $k = 0,043$.

Coefficient of understanding of notions "biga̧nd śmall" was $k = 0,61$. So we can say, that subject matter is not pretentious. Only two children did not understand subject matter in the group of 126 children during lesson.

Ideal state is coefficient $k = 1$. It is a desire of teachers. In this case all pupils will master subject matter after the 1^{st} explaining.

The coefficient of disclosure of situation on the picture (hurt finger and snake's right of way) was $k = 0,35$. We can say that pretension of subject matter determined by pictures is appropriate to age and abilities of pre-schoolers.

The students at Fakulty of Education in Banská Bystrica solved more pretentious problems for pupils of the 4^{th} class at elementary school. Gerová - Klenovčan [3, 2004] indicate how students and pupils understood the text of problems. Problem 1: pupils 41,3%, students 41,3 - 88%; problem 4: pupils 65,2%, students 69–94%; problem 5: pupils 84,8%, students 91,3–100%. If we do not regard investigated problems for pupils as test but learning subject matter, we can say that it is adequately pretentious. We can regard it this way in weaker groups of students, too.

The results can be used in practice by two ways.

- If it is standard sample of pupils, we can state pretension of subject matter from gained coefficient values of learning.
- If subject matter has standard pretension, we can find out knowledge and abilities of pupils from gained coefficients.

It is evident that teaching style, skills of teacher and didactical abilities influence "pretension of subject matter"without regard to text of problem of subject matter.

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Scientific Issues, Mathematics XII, Częstochowa 2007

THE CONTEXT OF MEANING AND UNDERSTANDING OF MATHEMATICS WHICH HELPS STUDENTS TO UNDERSTAND ITS SENSE AND TO SEE THE AREA OF ITS APPLICATION

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Abstract. The purpose of the presentation is to show that the key to the process of the developing acquisition of mathematics is the emphasis that students should put on the right choice of the appropriate realistic contexts. There are many ways of considering this problem, for instance, the constructivist approach, the socio-cultural approach. Moreover, there is an idea of the epistemological triangle, which is considered a necessary tool for the analysis of the form and the degree of the development of mathematical meanings.

1. The concept and meaning of the context in teaching mathematics

This paper aims to focus on two issues: the importance of the concept of *context* as to the role appointed to it in development of meaning of mathematical concepts among students, and the second issue is the context as a didactic obstacle.

Context is an ambiguous concept and can cause much misunderstanding. The concept of *context* as a linguistic category (pragmatic, semantic, grammatical), ontological, epistemological (the problem of relativism), cultural (esthetic, anthropological) has been described and systematized since the very beginning of reflection on different aspects of the human life. In the study of context, which has so far comprised systematization and classification, no framework of any theory of context has been worked out (IV Cognitive Seminar [13]). However, it is an extremely useful concept in broadening human interpretation on a basis of nearly all sciences. In its narrow meaning, the concept of *context* means as much as "*a text, in which a given word of phrase* *is used (especially the one quoted somewhere else)*" [6]. In its broadest meaning, a word context comprises all concurrent circumstances of an activity, event, situation [4]. Such understanding of context appears mainly in word phrases preceding another name e.g. *situational context, realistic context, task context and context of reference*. The word *context* means in such premises as much as environment, background of a given object, situation, an object, which is a referent of the name appearing after the word ćontext". For example, a context for the word *difference* can be the following sentence: a student understands the difference in meanings of the concept of number. There is one more meaning of this word, where the context for a jocular sentence: *I exist, so I have to think* may be the ongoing lesson of mathematics or a free interpretation of a famous phrase of Descartes. The latter meaning of the context can result in different inconsistencies. Let me quote the following example of a situation, in which the following task is given: *Imagine a situation in which you have to tell someone over the phone what absolute value is. What arguments will you use?*

The context, which the author of the sentence can have in mind may be, for example, a fact that the student may use any arguments, including so called: ąrguments of force". It can also be e.g. the fact that one can fantasize - who talks on mathematics by telephone - giving this concept an arbitrary or a pleasant meaning. However, we do not mean this when we want to use context in the process of teaching mathematics. This example of situational context is artificial and ambiguous, thus - unwelcome. It is clear that not all circumstances accompanying a phenomenon, an activity, an event or a situation are the contexts in which such a situation, an event or an activity or a phenomenon took place or will take place. What these circumstances are, is eventually decided by a person who creates or applies the context and this person should be required to avoid any ambiguity or interpretation errors. Otherwise, the context will not determine what is to be a true part of information available and anticipated by participants of a communicative event.

The expression *context* in didactics of mathematics makes its way up. Z. Krygowska [9] in her paper devoted to main problems and directions of research into contemporary didactics of mathematics used the expression context as many as six times. She paid particular attention to the fact that nearly any subject of teaching mathematics refers to several different theoretical approaches in mathematics as a science, e.g. real number, function, continuity etc. It is necessary to present these concepts in a spiral way, in indifferent contexts with consideration of their formal and synoptic differences. In this place, I postulate that any transition connected with a 'leap" at the higher level of comprehension of a mathematical concept were conditioned by analysis and synthesis of different situational contexts and contexts of meaning in which a given concept is entangled. However, contexts seem to have special position in the substance of problems, warning against apparent and untrue contexts, especially in problems concerning application of mathematics. Krygowska puts emphasis on importance of didactic studies in which individual attitudes of students towards mathematical problems with context influence will be analyzed, in which the problem has been formulated at the level of difficulty perceived by the student as well as the way of its solving [3]. This question will constitute, among others, the topic of the second part of this paper.

A. Sierpińska [14] in her study of the subject of utility and limitations of the epistemological obstacle in didactics of mathematics presented the ways of analyzing the meanings of a mathematical concept from synchronic and diachronic perspective. Synchronic perspective means to see the concept in the context of its logical associations with other concepts with consideration of their application and the place they occupy in contemporary mathematical theories. Diachronic perspective means analyzing meanings of the concept against its historical development, its transformations in the past and problems, which constituted an impulse to its creation. She emphasizes*that diachronic perspective, especially in mathematical education, answers two important problems: selection of both basic concepts in teaching at school and contexts within the framework of which they acquire meaning and accurate selection or elaboration of situational contexts which encourage the student to learn the concept* [14, p. 73].

Semadeni's solutions [12] constitute development of postulates of Krygowska [9] and Sierpińska [14], [15] concerning the subject of the role of meanings in mathematical argumentation. The author clearly indicates that each mathematical concept - except for definition - has several different meanings, which should be approached from different points of view. Such approach in depiction of meaning, can contribute to acknowledge that:

...in school and academic teaching, the basic concepts should be introduced less formally, and more emphasis should be made on their presentation in a rich context [12, p. 146].

The student is acquainted with more concepts when he can see how they are applied in concrete situations (during a class, a lecture, in a course book), and then he proceeds analogically, (...) It can be assumed, that comprehension of a concept means understanding of its meaning. Since the word "meaning"has many aspects, 'Comprehension"has many aspects in this context too [12,p.164].

In various studies on diagnosing images of mathematical concepts (e.g. understanding of the concept of a number, fraction as ratio and proportion, absolute value of the real number, bound of a bounded set, limit of a sequence, function limit in a point, derivative in a point etc. [3], [5], [11], [17], [18]) constructed by pupils or students of different mathematical experience at different levels of education - situational contexts became one of six most important constituents of these images (Bugajska-Jaszczołt, Treliński, [3]). Situational contexts of images of concepts are treated there as situations influencing relations between studied concept and other concepts, examples, studies in which we have to deal with a given concept. In this place, let me introduce a personal remark. Namely, the problem is, that in literature of the subject the concept of context is generally applied by various researchers playing key role in organization of many research processes or simplification of numerous didactic projects of teaching mathematic content (emphasized by Krygowska [9]) - with particular focus on nearly each lesson of mathematics, but it has not lived to fundamental, scientific analysis of its role and meaning in didactics of mathematics.

In other studies of didactics of mathematics, e.g. on development of mathematical knowledge of students, a constructivist approach is applied, according to which the students should construct meaning of mathematical concepts and operations by themselves. This compromise in didactic research with socio-cultural approach stressing a social character of mathematical knowledge postulates, that the intellectual processes of students connected with learning experience and creation of their generalizations comprising individual meanings of learned concepts - were stimulated by the teacher during consecutive discussions and exchange of often contradictory points of view. In the study by Slezakova-Kratochvilova and Swoboda [16, p. 185] concerning obstacles in communication between teacher and students, it was emphasized that ignorance of different levels of understanding of meanings of concepts, relations and processed can distort their interactive discourse on these subjects and lead to intellectual conflicts and much misunderstanding. Admitting that school mathematics is... *a science which requires negotiation of meaning of concepts and terms, accurate command of language, signs and symbols, perception of relation between a real a situation and mathematical abstraction*, it has to be taken into account that its image will always evolve with a learning student. Consequently, this fact should especially influence the teacher's attitude connected with interpretation of the student's statements [7], [16, p. 187]. The problem with communication between students and the teacher regarding different contexts of understanding of meanings of the same concept, was recognized as an important element having impact on construction of individual mathematical knowledge of students The following obstacles of cognitive character, which are especially interesting for me, which have impact on understanding of mathematics by teachers were recognized, among others: **different understanding by different students of the same context, situation/problem and different meanings given by students to the same word**. Analysis of these obstacles which are consequences of the teacher and each student's different scopes of their own, so-called "fields of experienceąnd ćoncept fields"(as in [16]), one can find in the paper quoted above.

Another example of research on the verge of constructivist and socio-cultural approach, taking advantage, among others, of the concept of *context of reference*, is the idea of epistemological triangle. Following the theory of learning by common argumentation (the author - Max Miller), using the idea of the triangle (concept - sign/symbol - context of reference, successful attempts of diagnosing individual development of understanding of mathematical concepts with children were made. The authors of adaptation of the described idea: Jagoda, Pytlak, Swoboda, Turnau, Urbańska [5] into the field of Polish didactics, show dependence of understanding of the meanings of concepts on anchoring them in specific situations, objects, contexts of reference in which the properties of concepts can be exposed. In order for the student to understand what a given concept is, he has to be able to use this concept in precisely selected and elaborated situations, in which this concept will disclose its gist. The concept acquired by means of contexts of occurrence (e.g. models, drawings, text problems) which are referred to as representations, cannot be identified with any of them. So, they are very important in accurately selected diversity of providing students with a big chance of discovery of mathematical relations and properties which they represent. Contexts of reference, first as real objects and empirical representations, become mental representations over time, to become carriers of certain mathematical concepts embodied into them in the end. Can one overestimate in this process the problem of accuracy in selection of contexts as important element of the epistemological triangle?

The following reflection comes into mind: Perhaps following the obstacles mentioned earlier and the idea of epistemological triangle, it is worth looking into authentic proceedings of students when they face a mathematical concept in a context, for example, of a realistic problem in order to learn about differences of students in comprehension of the background of this context and the ways to cope with them. Accordingly, an interesting question arises: in what way can realistic contexts, in which a mathematical concept is entangled, distort their image constructed by the student following the concept of Bugajska-Jaszczołt, Treliński [3]?

2. Realistic contexts of concepts and their acceptance by students

Krygowska [8] claimed that one of the most important objectives of teaching mathematics is: "*students' understanding of mathematical concepts and their ability to apply them while solving problems"*. Therefore, a concept of their reasonable application also became an object of study for teachers of mathematics.

In the process of school teaching, most mathematical concepts are developed by their application in various examples, problems, calculations, tasks etc.(e.g. Semadeni [12]) . The meaning of concepts is in the best way recognized by the student while solving various problems.. The student should owe it to his own mathematical and extra-mathematical activity during which he earns logical experience and then tries to create their generalizations consisting of various images of these concepts and procedures of mathematical operation. Solving a problem in a realistic context, a student must refer to his earlier experience, taking advantage of existent or nonexistent transfer of procedures of mathematical activity in order to apply in similar situations. If the scope of experience, his own images of concepts and mathematical procedures connected with them are rich enough, then it is easier for him to transfer this individual mathematical knowledge into different or similar problem contexts. Boaler [2] claims, that it is an exceptional situation which not only provides for better understanding of school mathematics by students, but it also reinforces didactic transfer of this knowledge onto extra school situations, to mathematics of ȩvery day life". One of the important problems of my research is a question: Why do not realistic contexts introduced to teaching or learning mathematics at school levlel meet the expectations of teachers as to the scope of improvement of students' understanding of meanings of mathematical concepts and procedures?

Many accurate and interesting comments connected with this problem were postulated by Pawlak [11]. Following the reflection on Professor's remarks and conclusions from individual findings connected with diagnostic research being held in that field, I feel enormous respect to the significance of the issue raised in this question, because, according to Pawlak [11, p. 141] - *enrichment of the mathematical context with accompanying contexts does not guarantee didactic success; on the contrary, it can appear not to be a successful move, which in effect makes the process of knowledge acquisition difficult. Consequently, the second conclusion, that (...) the basic problem is (...) introduction of the context in inconsiderable way, without basing the activities on reliable knowledge (concerning didactics psychology and issues introduced in concrete examples).*

Pawlak [11, p.145] postulated introduction of a very important and usable concept of: **obstacle of realistic context**, which was included into the category of didactic obstacles. The most expressive explanation of the idea of a didactic obstacle and what its criteria are for the purpose of didactics of mathematics was provided by Bessot [1, p. 49]: *We refer to an obstacle, when a problem was solved after restructuring of the idea of concepts or change of theoretical point of view.*

However, there are the following criteria of a didactic obstacle:

1. *An obstacle is the knowledge, which functions (...) in a given class of situations and for certain changeable values in these situations*;

2. *An obstacle is the knowledge, which trying to adapt itself to other situations or other changeable values, provokes certain specific errors which can be identified and analyzed*;

3. *An obstacle is stable knowledge*;

4. *An obstacle can be overcome only in specific situations of rejection and then it will be a constituent factor of knowledge (...) it is an obstacle which can be avoided without any consequences for the construction of information, and which can be deleted by affecting the teaching context.*

Perception of this obstacle was possible, as the author indicates, thanks to application of elements of the theory of symbolic reactionism in reference to observation of pupils and students' behavior (cf. [15]).

The model presented below represents the way of systematization of the results of diagnostic research on one hand, and on the other hand, it is an individual attempt of presentation of the reasons and consequences of two different obstacles of realistic context most vital and statistically most often disclosed by students and teachers. The foundation of the reflections comprised the study held among 113 primary school pupils, 91 pupils of junior secondary schools and 53 students of post junior secondary schools. Due to the imposed limitations (maximum 6 printed pages), I selected a draft presentation of relations which were found as to the scope of appearing obstacles of the realistic context.

Finally, I would like to present an idea - proposition, which arose during observation and analysis of the records of nearly 80 lessons of mathematics. It nearly became a research hypothesis as follows:

Context in mathematics is a mirror in which, the student can reflect only what he already carries in his mind.

Fig. 1. Causes and effects of appearance of an obstacle of realistic context.

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THE ROLE OF THE MILTISTAGE TASKS IN DEVELOPING THE CREATIVE ACTIVITY OF MATHEMATICS TEACHERS

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Abstract. This papier presents the results of the research carried among the mathematics teachers. These research deals with the skills of undertaking creative mathematical activity by the teachers. It also deals with the awareness of the need of developing different kinds of this activity among students. The main tool which is used and studied in the research is the multistage task.

1. Introduction to the research problem

My research deals with **the formation and developing of skills of undertaking creative mathematical activity** by mathematics teachers and the tools of provoking this activity. Developing the skills of undertaking different kinds of creative mathematical activity among teachers is the necessary condition:

- to awake their **awareness** of the necessity of formation of this activity among their students,

- to develop their **skills** of organizing the situations which favour undertaking different kinds of this activity.

Only then the teachers would form and develop effectively that activity in their work with the students.

The main tool of that formation which I use and study in my research is **the multistage task**.

However, the research connected with the observation of school reality stress a worrying aspect that developing the creative mathematical activity is neglecting. In school practice the attention is paid mainly to the students knowledge of basic concepts and skills of applying procedures. *In general,* *mathematics teaching pays little attention to the more advanced aspects of mathematical activity such as the formulation and resolution of problems, the formulation and testing of conjectures, the pursuit of investigations and mathematical proofs, and the argumentation and critique of results. While these are fundamental and current themes of mathematics education expressed in many curriculum documents across the world they still find very little emphasis in classroom practice* ([3]).

2. The conception of formation of creative mathematical activity

The conception of formation of creative mathematical activity was worked out by M. Klakla ([1]). It is based on two elements:

First of them constitutes distinction of particular kinds of creative mathematical activity, which are present in essential way in activity of mathematicians. They are: (a) putting and verification of hypotheses; (b) transfer of the method (of reasoning or solutions of the problem onto similar, analogous, general, received through elevation of dimension, special or border case issue); (c) creative receiving, processing and using the mathematical information; (d) discipline and criticism of thinking; (e) problems generation in the process of the method transfer; (f) problems prolonging; (g) placing the problems in open situations.

The second element of that conception are the multistage tasks:

- which are the specific structure of series of tasks, problems and didactic situations,
- based on the problematic situations,
- connecting different kinds of creative mathematical activity with each other in the complex and rich mathematical-didactic situations,
- provide specific laboratory of creative mathematical activity for the students.

3. Information about the carried out research

To date, I carried out the pilot research and main research. **The pilot research** has dealt with the skills of undertaking creative mathematical activity by the teachers of mathematics. It also has dealt with the awareness of the need of developing different kinds of this activity among students.

The results of that research presented in [2] show that:

- Among the mathematics teachers the awareness of that what is the creative mathematical activity, and the awareness of the necessity of formation of this activity, is insufficient.

- Among the mathematics teachers it is generally erroneous believed that the creative mathematical activity develops by itself during the mathematics lesson and it does not require any special didactic endeavours, methods or tools to develop it.

- The mathematics teachers do not have experience and skills of undertaking that activity.

The aims of **the main research**:

1) Recognition:

- the awareness of the necessity of formation of creative mathematical activity;

- the initial skills of undertaking creative mathematical activity.

2) Verification of the put hypothesis, i.e. that the multistage tasks can be used as a tool:

- to introduce the teachers of mathematic into given creative mathematical activity (introductive means);

- to develop the skills of undertaking the creative mathematical activity among teachers of mathematics (developing means);

- to form among the teachers mathematics the awareness of the need to develop creative mathematical activity and the skills of provoking this activity among students (awareness means);

- to diagnose the skills to undertake the given kind of creative mathematical activity by teachers of mathematics (diagnostic means).

4. The description of the process of the main research

The group of 7 teachers of mathematics (of gymnasium and high schools) has taken part in the series of workshops from March to September 2006. The workshops, concerning three multistage tasks, were organized as a part of the Professional Development of the Teachers-Researchers (PDTR) project, during the mathematics course.

Before the series of workshops, the diagnosis of the skills of undertaking the creative mathematical activity and the awareness of the need to develop this activity among students had been carried out (the research tool developed during pilot research and the set of two open tasks). After the series of workshops, the next diagnostic research was carried out (the research tool - the questionnaire and the same set of two open tasks).

5. The fragment of analysis of solving the set of open tasks

The teachers received the set of two open tasks both before (solution I) and after the workshops (solution II). The aim was to diagnose their skills of undertaking the creative mathematical activity. There is one of that tasks:

The Hippocrates' lunes (Fig. 1) *constructed on the right triangle ABC with the right angle C are planes contained by the arc ACB of a circumscribed circle and the arcs of semicircle on diameters equal the lengths of two adjacent sides BC and AC and the centers in the centers of these sides.*

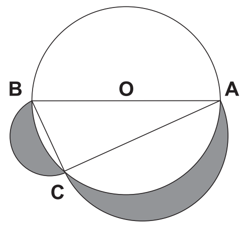

Fig. 1: The Hippocrates' lunes

Formulate some relevant questions connected with the situation presented in the picture and try to answer these questions.

I will present the solution I and II by one of the chosen teachers, who has taken part in the workshops.

The solution I (before the workshops):

The situation presented in the task was familiar for the teachers. The chosen teacher, like majority of the researched group, put only the question about relation between area of the Hippocrates' lunes and area of the right triangle ABC, referring to the known for her property.

The solution II (after the workshops):

She started the solution with the questions: *What is the area of the Hippocrates' lune? What is the sum of area of the Hippocrates' lunes?* And then she proved the relation: *The sum of area of the Hippocrates' lunes equals the area of triangle ABC* (alike in solution I). Afterwards she was prolonging the task modifying the initial situation in the ways presented in Fig. 2, Fig. 3, Fig. 4.

$P_I + P_{II} = ... = P_{\square} - P_{\triangle ABC}$,

where $P_{\square}$ means the area of the square with the side AB.

$P_I + P_{II} = ... = P_{\square} - \frac{1}{2}P_k + P_{\triangle ABC}$,

where $P_{\square}$ means the area of the square with the side AB and P_k means the area of the circumscribed circle.

$P_I + P_{II} = ... = 2P_{\triangle ABC} - \frac{1}{2}P_k$,

where P_k means the area of the circumscribed circle.

The first direction of prolonging is the replacement the circles with the squares (Fig. 2). The second direction of prolonging is building polygons (squares, right triangles) on two adjacent sides BC and AC of the triangle

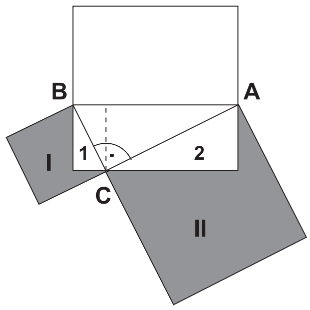

Fig. 2: The teacher's solution II, part 1

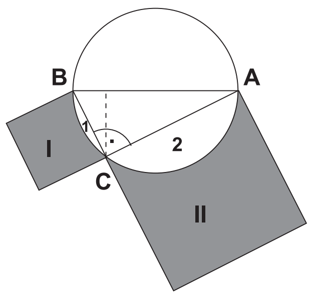

Fig. 3: The teacher's solution II, part 2

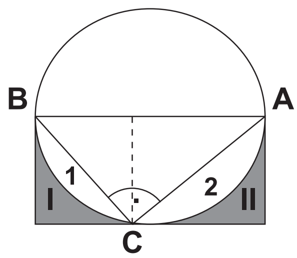

Fig. 4: The teacher's solution II, part 3

ABC inscribed in a circle (Fig. 3, Fig. 4). The work of the teacher consists in modification of the initial situation and verifying if it is analogous, to the initial situation, relation between areas of the figures I and II and the area of the triangle ABC, where P_I, P_{II} are the areas of the figures built on two adjacent sides BC and AC of the triangle ABC after substraction the areas 1 and 2 (rights triangles - Fig. 2 or parts of the circles - Fig. 3, Fig. 4).

During the solution II the teacher undertook several kinds of creative mathematical activity: she put hypotheses and made an attempts to verify them; she used the transfer of the method of solutions onto similar, analogous issue, she was prolonging the task. The researched person started to notice the potential of that situation, value of the open task and she used the possibility of free leading her solution, not to be confined only to the known by her relation. The teacher formulated untypical problems characterizing of creativity. It can be evidence of that she approaches more flexibly to the mathematical tasks and also it has been revealed visibly the skill of undertaking by her the creative mathematical activity.

The researched person declared in the questionnaire after the series of workshops in following words:
During the workshop *I have learned that the tasks of this type can be used in the school, they are interesting and educational for the pupils as well as for the teacher. Solving this type of the tasks gives lots of fun.*
Considered problems aroused my interest - the first task I used on the additional lessons of mathematics (both in the first and third class of gymnasium).
The tasks of this type you can use with the work with gifted students and some elements of those tasks also during typical lessons of mathematics.

As long as before the workshops, the researched person declared that she does not consider with the pupils the tasks which are prolongation of a given task and that she rarely solves the open tasks with the pupils, so during the series of workshops she started to use some fragments of multistage tasks in her didactic work. It can be evidence that her awareness of the need to develop creative mathematical activity among students has been rosen. It can be also evidence that a change of her attitude in the direction of taking an action oriented to developing creative mathematical activity among her students has been ensued.

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SOME REMARKS ON DEFINITION OF THE ABSOLUTE VALUE OF A REAL NUMBER

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Abstract. The article presents a didactic proposition of introducing the definition of the absolute value of a real number.

1. Introduction

This paper is a continuation of research into understanding of the absolute value of a real number. In the article (Major, Powazka, 2006) the necessary and sufficient conditions for the existence of the solutions of equations contained the absolute value functions were given. This paper contains the examples of didactic conceptions of the implementation of the definition of the absolute value of a real number at different levels of mathematical education. There were used functional equations of one or several variables. The didactic propositions described in this paper could be exploited by teachers of the secondary schools or the university teachers and mathematics students especially teaching oriented ones.

During their studies students learn different definitions of the absolute value of a real number. These definitions are based on the distance between two points on the number line, maximum of two real numbers or a square root of a nonnegative number. In the secondary school level one proves the following properties of the absolute value of a real number

$$|x \cdot y| = |x| \cdot |y|, \quad x, y \in R,$$

$$||x|| = |x|, \quad x \in R,$$

$$|ax| = a|x|, \quad x \in R, a \in R^+.$$

Let a function $\phi : R \to R$ satisfy the following equations

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y), \quad x, y \in R, \tag{1}$$

$$\phi(\phi(x)) = \phi(x), \quad x \in R, \tag{2}$$

$$\phi(a \cdot x) = a \cdot \phi(x), \quad x \in R, a \in R^{+}. \tag{3}$$

In this paper we study the conditions for which a solution of the system of equations (1), (2), (3) is in the form

$$\phi(x) = |x|, \quad x \in R. \tag{4}$$

2. Main results

In this part we prove four theorems. All of them could be use as a didactic proposition of introducing of the definition of the absolute value of a real number.

Proposition 1

We start with the following theorem of Cauchy [1].

Lemma 1. *If a continuous function $h : R \to R$ is for all real x, y a solution of the Cauchy functional equation*

$$h(x + y) = h(x) + h(y), \tag{5}$$

then there exists a real number λ such that

$$h(x) = \lambda \cdot x \tag{6}$$

for all real x.

Now we prove the following theorem.

Theorem 1. *If a nonconstant and continuous function $\phi : R \to R^{+} \cup \{0\}$ is a solution of equation (1) for all real x, y with $\phi(a) = a$, where a denote a constant number and $a \in R^{+} \setminus \{1\}$, then ϕ is the absolute value function (4).*

Proof. Let a nonconstant and continuous function $\phi : R \to R$ satisfy equation (1). Putting in (1), $x = y = 0$ we get $\phi(0) = \phi^2(0) \Leftrightarrow (\phi(0) = 0$ or $\phi(0) = 1)$. If $\phi(0) = 1$, then $\phi(x) = 1$ for all real number x, which is impossible, because ϕ is a nonconstant function. Hence we have

$$\phi(0) = 0. \tag{7}$$

Let x, y be positive real numbers. The substitution $x = e^u, y = e^v$, where $u, v \in R$ transform (1) into

$$\phi(e^{(u+v)}) = \phi(e^u) \cdot \phi(e^v). \tag{8}$$

Putting in (8) $g(u) := \phi(e^u)$ where $u \in R$. We get equation

$$g(u + v) = g(u) \cdot g(v). \tag{9}$$

Because g is a positive function, we have

$$\ln(g(u + v)) = \ln(g(u)) + \ln(g(v)). \tag{10}$$

Let $h : R \to R$ by the function given by $h(u) = \ln(g(u)), u \in R$. Then it follows from (10), that h is a solution of the Cauchy functional equation (5). By the definitions of functions g and h we get

$$h(u) = \ln(\phi(e^u)), \quad u \in R. \tag{11}$$

The continuity of the function (11) follows from the continuity the function ϕ and the logarithm function or the exponential function. It implies, that the function (11) is continuous solution of the Cauchy functional equation (5). In view of (6) and (11) there exists a number λ a such that

$$\phi(x) = e^{\lambda \cdot \ln x}, \quad x \in R^+,$$

thus

$$\phi(x) = x^\lambda m \quad x \in R^+. \tag{12}$$

Putting in (1) $x = y = t$ or $x = y = -t$, where $t \neq 0$ we have $\phi(t^2) = \phi^2(t)$ or $\phi(t^2) = \phi^2(-t)$, respectively. It follows that $\phi^2(t) = \phi^2(-t)$. By the assumption of Theorem 1 the function ϕ is a nonnegative solution of equation (1). Therefore

$$\phi(t) = \phi(-t), \quad t \in R \backslash \{0\}. \tag{13}$$

From (7), (12), (13) we get that continuous, nonnegative solution of equation (1) is given by

$$\phi(x) = |x|^\lambda, \quad x \in R. \tag{14}$$

Since a is a positive fixed point of the function ϕ, we get from (14), $a = \phi(a) = |a|^\lambda = a^\lambda$. From this we get

$$\lambda = 1. \tag{15}$$

From (14) and (15) it follows that the solution ϕ of the functional equation (1) is in the form (4).

Remark 1. *If we replace in Theorem 1 the continuity assumption by*

(i) ϕ is continuous at a point,

(ii) ϕ is bounded from above on an interval,

then the Theorem 1 holds true.

Proposition 2

The following results is generalization of Theorem 1.

Theorem 2. *If a nonconstant and continuous function $\phi : R \to R$ with*

$$\phi(a) = \phi(-a) = a, \tag{16}$$

where a denote a constant real number and $a \in R^+ \setminus \{1\}$, is a solution of equation (1) for all real x, y then it is the absolute value function (4).

Proof. Let $\phi : R \to R$ satisfy the assumption of this theorem. Similarly as in the proof of Theorem 1 we have (12). Since $\phi(a) = a, a \in R^+ \setminus \{1\}$, we get $\lambda = 1$. From this it follows that

$$\phi(x) = x, \quad x \in R^+. \tag{17}$$

Now, putting in (1) $x = y = t$ or $x = y = -t, t \neq 0$ we have $\phi(t^2) = \phi^2(t)$ or $\phi(t^2) = \phi^2(-t)$, respectively. Thus $\phi^2(t) = \phi^2(-t)$. Hence $\phi(t) = \phi(-t)$ or $\phi(t) = -\phi(-t)$. This and (16) yield that the function ϕ is an even function in R. Hence by virtue of (17) the function ϕ is the absolute value function.

Proposition 3

In this part we will show the method of defining the function given by (4), using a solution equation (2). We start with following.

Lemma 2. *If the function $\phi : R \to R$ have the inverse function, then the function (6) with $\lambda = 1$ is a solution of equation (2).*

Theorem 3. *If an even function $\phi : R \to R^+ \cup \{0\}$ is a solution of equation (2) and the restriction of this function to the interval $[0, +\infty)$ have the inverse function, then ϕ is given by the formula (4).*

Dowód. Let an even function $\phi : R \to R^+ \cup \{0\}$ satisfy equation (2) and the assumption of this theorem. Then $\phi(x) = \phi^{-1}(\phi(x)), \quad x \geq 0$, where ϕ^{-1} is the inverse function of ϕ. Hence we get $\phi(x) = x, \quad x \geq 0$. Because ϕ is an even function, formula (4) holds. □

Proposition 4

Now we will define the absolute value function using the solutions of the functional equation (3).

Theorem 4. *If an even function $\phi : R \to R$ satisfy the condition*

$$\phi(1) = 1, \tag{18}$$

is the solution of equation (3), then ϕ is given by the formula (4).

Proof. Putting $x = 1$ in (3) we have $\phi(a \cdot 1) = a \cdot \phi(1)$, where a is a positive real number. From this and (18) it follows that

$$\phi(a) = a, \quad a > 0.$$

Putting in (3) $x = 0$ we have $\phi(0) = \phi(a \cdot 0) = a \cdot \phi(0)$. Hence, $\phi(0) \cdot (1 - a) = 0$, $a > 0$. Because $a > 0$, we get that $\phi(0) = 0$. Therefore, we have the condition (17). Because ϕ is an even function, the formula (4) holds.

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ELECTRONIC TEXTBOOK IN LMS MOODLE

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Abstract. Moodle is the software used as a tool for on-line distance learning as well as a support for face-to-face teaching. The article outlines the structure of electronic textbook for LMS Moodle and the ways in which it can be utilised in training prospective elementary mathematics teachers under conditions of Prešov's Faculty of Education.

1. On-line Electronic Textbook

An Electronic textbook can be developed on-line (accessible via net, most frequently www) or offline (accessible on CD or DVD medium). On-line access to curricular resources is made possible by software environment referred to as LMS (Learning Management System). LMS indicates a software package which enables development, management and delivery of contents of electronic courses and textbooks. It offers tools for communicating, testing, valuation, administration and archiving of academic results. There is a range of open source LMS: Moodle, LRN, ATutor, Bodington, Claroline, Dokeos, OLAT, Sakai Project and VClass. Currently, in the area of higher education the most popular e-learning platform is Moodle (Modular Object-Oriented Dynamic Learning Environment). It is a software package used for supporting education in both full-time and distance learning, offering on-line courses accessible on www. More detailed information on Moodle can be obtained on http://moodle.com/.

When developing the electronic textbook we drew on the methodology of M. Turčáni [6]. Based on its principles, students were given access to studying texts presented in the form of lecture. Each chapter is introduced by formulation of study unit objectives. Explication (theoretical part of the chapter) looks upon real life situation and is elaborated on the basis of it. The chapter ends with a list of recommended further references.

One of the key elements of the self-study course is providing student a feedback on their progress in the subject area. This part of education is elaborated in the electronic textbook by means of the following study units: EXCERCICES, TESTS and DISCUSSION FORUM. The main idea of the electronic textbook is not merely to supplant traditional print version textbooks but, employing hypertext links and multi-media files, offer the learners new and attractive environment for stress-free way of self-study that each learner volunteered for when taking up electronic support of e-learning via Moodle. Study unit EXCERCICE contains the list of recommended tasks with instructions for their solution. Study unit TEST adopted the form of self-corrective test. The outlined form of control provides students with feedback on the level and quality of their study, with respect to the problem area they dealt with, having an advantage of being accessible on-line. Tutor does not have to laboriously correct and revise each test, yet s/he can get an overview of overall success rate and the progress in each test item. Tutor is informed of the number of students who took the test, time and proficiency level, and is able to follow those participants who in their pace of study have not arrived to the test yet (could be warned of time lag). Each student, after submitting the elaborated test, is immediately informed about the level of acquired knowledge and skills. When assessing each task's item solution students are presented with further recommendation on how to remove flaws in tackling particular problem area. Apart from recommendations resulting from the solutions, students can make use of DISCUSSION FORUM as well. It provides space for on-line discussion with fellow students and tutor. It is important though, to formulate correctly the problem area, so as tutor and other participants can properly react on it. Such discussion is then, to a degree, a replacement for consultation or seminar session; regular instruments in the full-time form of study.

2. Experience with Electronic Textbook

In the 2006/2007 academic year we monitored part-time students of Pre-school and Elementary Education. Among our primary focus was students' interest in electronic support of learning mathematical disciplines. The following courses were surveyed: Introduction into Study of Mathematics and Creating Early Mathematical Concepts I. These disciplines are taught in the first year, scheduled as 1 unit (45 min.) of lecture and 2 units of seminar per week.

The content of Introduction into Study of Mathematics builds up on mathematical curriculum of basic and secondary school. It aims at refining mathematical phrasing and building homogenous and comprehensive terminology.

Creating Early Mathematical Concepts I concerns knowledge of geometry in the perspective of developing geometrical concepts of pre-school aged child, and designing activities aimed at introducing and reinforcing basic geometrical concepts. B. Tomková [5] informs in more detail on one specific methods of working applied in this course.

In 2006/2007 academic year the number of full-time students enrolled for the introductory mathematical discipline taught in winter term was 180. Electronic support of Moodle was utilised by 87 students (48,3%) of which 43 (23,9%) delivered us self-corrective tests. In the part-time form of study 99 students enrolled for the course of Introduction into Study of Mathematics. 85 students (85,9%) opted for an electronic support of Moodle of which 57 (57,6%) performed self-corrective tests.

Creating Early Mathematical Concepts was taught in the ensuing summer term. The number of students enrolled for the course in both forms of study was 249 (162 full-time student and 87 part-time students). Almost all students of both forms of study registered for Moodle environment support. Preliminary test was delivered by of 152 full-time students (93,8%) and 76 part-time students (87,4%).

From the increased interest in exploiting the electronic textbook in the second term of study in its both forms we can assume that provision of electronic support of study in the outlined modification was tenable and beneficial for students.

As for the form of consultations in Moodle environment students could use synchronous form of communication (chat) or asynchronous form (discussion forum, internal e-mail). However, the interest in such form of discussion was almost negligible. Only 3 full-time students (1,7%) and 16 part-time students (18,4%) resorted to it. This finding indicates greater dependency of part-time students on more detailed consulting of the examined topics.

Upon completion of the second term, the students were given an option to comment on their utilisation of e-learning support during their first academic year in the form of questionnaire. The questionnaire was administered among the students of full-time and part-time forms of study. 157 full-time students and 84 part-time students joined the survey.

One of the survey items was listed in order to get overview of the variety of information sources used by students during their study of mathematical disciplines. The survey's outcomes are presented in the following table.

The above survey has proved that students of both forms of study make use of more traditional information sources during their training in mathematics. Dominant source of information was direct learning contact (lecture and seminar). Use of printed textbook was less frequent than utilisation of its on-line version in Moodle. This statement is also supported by our monitoring of students access to Moodle environment. We identified students need for further consultations of their study related problems. In this respect, students contacted their fellows, colleagues specialising in mathematics (in-service elementary teachers, part-time students) and family members (under Other Sources section).

Information Source	Mean Use of Particular Information Source	
	Full-time Students	Part time Students
Seminars and Lectures	90,45%	97,62%
Textbook	70,70%	86,90%
Electronic Textbook in Moodle	78,98%	96,43%
Discussion with Fellow Students	68,15%	80,95%
Consultation with Tutor	10,83%	14,29%
Consultation with In-service Teacher from Elementary or Secondary Stage	9,55%	26,19%
Other Sources	8,28%	23,81%

The second question was formulated as follows: "Had I been given an option to utilise all indicated information sources, in which order would have I used them?Śtudents were asked to rank the information sources on the basis of their importance to studying mathematical disciplines. The following table indicates the sources offered and mean order of their preference on the scale of 1 to 7.

Information Source	Mean Order of Information Source Preference	
	Full-time Students	Part time Students
Seminars and Lectures	2,08	1,67
Textbook	2,47	2,63
Electronic Textbook in Moodle	2,99	2,64
Discussion with Fellow Students	4,06	4,60
Consultation with Tutor	4,13	4,58
Consultation with In-service Teacher from Elementary or Secondary Stage	5,76	5,48
Other Sources	6,50	6,40

The above survey has proved that for the purpose of mathematical training of prospective elementarist it is appropriate to combine it with electronic support accessible through the net. E-learning as a supporting element of face-to-face teacher training thus opens more room for delivering an efficient mathematical education.

3. Conclusion

In the course of their study, students have accepted the provision of lectures with hypertext links to multimedia applications, animations, web sites and other software products (e.g. Geonext for learning Geometry). Such accompaniment (or its complex replacement) of lectures and seminars is a suitable tool for stimulation of study. Self-corrective tests serving as an immediate feedback on quality of study were also frequently employed. Analogous experiences with Moodle as an electronic support of study are also presented in [1], [2], and [3].

Moodle is a convenient tool for designing methodological materials for in-service teachers. Electronic textbook is a resource offering broader scope of utilities than printed textbook. Traditional textbooks on Didactics of Mathematics usually do not contain samples of pupils' solutions. However, electronic textbook of didactics of Elementary Mathematics can provide enough space for analysing samples of pupils' solutions of particular tasks [4]. Its potential is also in transferring knowledge to larger number of recipients.

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HOW MANY AND WHAT KIND OF STOOLS CAN BE BUILT BY A CARPENTER? – MEANING HOW PEDAGOGICS STUDENTS SOLVED CERTAIN PROBLEMS

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Abstract Uncommon mathematical problems play an important role in childrens' education in mathematics. These exercises inspire creativity in children and help them develop a sense of divergent thinking. Pedagogics students, as future teachers, must not only recognize the value of such mathematical problems, but must also be able to solve them. This article is a presentation of the skills of the students in this regard.

1. Introduction

An important part of integrated education is to form a person who is creative and has the mental disposition to enable him to come up with many ideas. His traits should include fluency, flexibility and original thinking, as well as the ability to analyze, synthesize, generalize, compare, classify, deduct, abstract, define, step-by-step problem solve and so forth. We care about creating a pupil who is open and adaptable. We can reach this goal by encouraging solving text mathematical problems with many solutions.

What makes these problems so valuable is that there are a very specific problematical situation and the ability to solve them will be of practical use, the solutions becoming a pattern (paradigm) that will come in handy in any difficult situation (problematical).

Solving problems (as long as it is self-reliant) can be a splendid learning experience, teaching both creativity and criticism, work-organizing, searching for effective problem solving strategies, the ability to code and present solutions, self-control of progress and elimination of mistakes.

2. Research

How pupils deal with problem solving and if they become creative in the progress depends greatly on the teacher's style of teaching. The success of his students is dependant on the teacher's own knowledge and pedagogic skills. The teacher should be characterized by a creative attitude and should give his students as many occasions as possible to solve open and atypical problems. If the teacher has trouble with solving certain problems, he definitively should not assign his students to solve them.

I have decided to assess the competency of pedagogics teachers of early education in the regard of solving atypical problems.

In January of 2007, I conducted tests among 163 teachers, who had received bachelor degrees (from different institutions) in preschool and early school pedagogics and had started master degrees in preschool and early school pedagogics in The Pedagogics Academy in Cracow.

Each one of them was asked to solve the following problem in writing:

Problem: A carpenter builds stools with 3 and 4 legs. He has 30 legs for the stools. How many and what kind of stools will he be able to build?

The mathematical model of this problem is a diophantine equation $3x + 4y = 30$, where x is the number of three-legged stools and y is the number of four-legged stools. The equation has three solutions: $(10; 0)$, $(6; 3)$, and $(2; 6)$.

It is worth to discuss the amount of solutions taking into account the following questions:

[1] Is the answer $(10; 0)$ correct? It does not include any four-legged stools.

[2] Must the carpenter make use of all of the legs? The content of the problem does not mention that. Therefore, it is worth to take into consideration the solutions, which will involve a smaller than 30 number of legs. In this situation the mathematical model of this problem is the equation $3x+4y = n$, for $n \leq 30$, which, together with the $(x; y) = (0; 0)$, has 48 solutions. In this situation the number of unused legs is any whole number from 0 to 27 or the number 30. There cannot be 29 nor 28 legs, because a stool cannot be built with neither 1 nor 2 legs. The following table shows all solutions for $n > 25$. The solutions have been numbered (the first row) with numbers from 1 to 13, which will be discussed in the later part of the article.

nr solution	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	etc.
x	10	6	2	7	3	8	4	0	9	5	1	6	2	etc.
y	6	3	6	2	5	1	4	7	0	3	6	2	5	etc.
n	30			29		28			27			26		etc.

[3] If not all the legs are to be used, does it make any sense to leave more than 2 legs? If there are 3 legs left, they can be used to build another thee-legged stool and none will be left. If there are 4 legs, a four-legged stool can be built and a three-legged one, leaving only 1 leg. When 5 are left ... and so forth.

[4] Can a carpenter build stools having only legs? The content of the problem does not specify the presence of other indispensable in the construction of stools elements such as, for example, the seat.

Let us take notice that, depending on the answers to the former questions, the number of solutions changes. It is therefore difficult to state if the student gave all the answers or not. It is expected that the teachers understand that to solve this problem means to discuss all of the possible variants and find an answers in each of these specific cases.

None of the tested students presented all of the answers for n from 0 to 30. The highest number of solutions was 8. They were all of the cases for $n = 28, 29, 30$, which are the answers marked from 1 to 8. The students were evidently looking for the largest number of stools. From every 3 or 4 legs they "made" more stools.

Analyzing the work of teacher-students I took notice not only of the number of correct solutions, but also of the following a) how the tested students understood the content of the problem, b) what the term "solving the problem" meant to them, c) what strategies were used to find solutions and d) how they presented their results.

3. Test results

3.1. Understanding the content of the problem

All tested students understood the content of the problem very well and the question therein. Some expressed doubt if the problem is properly formulated because:

- *It was not explained if all legs must be used.*
- *There was no information instructing that there must be as many stools as possible.*

- *No stools can be made without seats!*
- *They thought the problem should be formulated more precisely.*

3.2. What does "solving the problem" mean?

The following table illustrates a summary of the answers taking into account the number of solutions and the type of solutions (numbered solutions). In the table the solutions with the help of illustrations are separated for ones given simply arithmetically (without illustrations). The analysis of the facts compiled in the table allow one to suspect how the tested students understood "solving the problem". This understanding may vary greatly.

<table>
<tr><th colspan="2">Type of solution</th><th>Numbered solutions</th><th>With illustration</th><th>Without illustration</th><th>Together</th><th>Summary</th></tr>
<tr><td colspan="2">Trivial: dividing $30:3=10$ & $30:4=7$ s 2</td><td>1 and 8</td><td>0</td><td>5</td><td>5</td><td>5</td></tr>
<tr><td colspan="2" rowspan="3">Only one possibility</td><td>1</td><td>0</td><td>1</td><td>1</td><td rowspan="3">8</td></tr>
<tr><td>2</td><td>3</td><td>0</td><td>3</td></tr>
<tr><td>3</td><td>4</td><td>0</td><td>4</td></tr>
<tr><td rowspan="4">Two possibilities</td><td rowspan="3">One trivial $(30:3=10)$ & one „special"</td><td>1 & one with spare legs</td><td>5</td><td>1</td><td rowspan="3">25</td><td rowspan="4">36</td></tr>
<tr><td>1 & 2</td><td>4</td><td>0</td></tr>
<tr><td>1 & 3</td><td>15</td><td>0</td></tr>
<tr><td>$n=30$ & $x\neq0$ i $y\neq0$</td><td>2 & 3</td><td>11</td><td>0</td><td>11</td></tr>
<tr><td colspan="2">Three possibilities — all solutions for $n=30$</td><td>1 & 2 & 3</td><td>35</td><td>9</td><td>44</td><td>44</td></tr>
<tr><td colspan="2">Four possibilities: all for $n=30$ & $30:4=7$ spare 2</td><td>1 & 2 & 3 & 8</td><td>14</td><td>2</td><td>16</td><td>16</td></tr>
<tr><td colspan="2">More than 4 solutions</td><td>1 & 2 & 3 & one with spare legs</td><td>7</td><td>2</td><td>9</td><td rowspan="2">11</td></tr>
<tr><td colspan="2">Almost all with spare legs smaller than 3 (one possibility missing)</td><td>from 1 to 8 without one example</td><td>1</td><td>1</td><td>2</td></tr>
<tr><td colspan="2">All for $n=28, 29, 30$</td><td>from 1 to 8</td><td>1</td><td>8</td><td>9</td><td>9</td></tr>
<tr><td colspan="2">Strange calculations (including correct ones) but without an answer</td><td></td><td>0</td><td>4</td><td>4</td><td>4</td></tr>
<tr><td colspan="3">Summary</td><td>118</td><td>45</td><td></td><td>163</td></tr>
</table>

a) There were some (5 people), that, without reflecting on the different possibilities of building stools, divided $30 : 3 = 10$ and $30 : 4 = 7$ spare 2 answering that the carpenter can build 10 three-legged stools or 7 four-legged ones. For them, "solving the problem" means operating arithmetically on numbers and giving the results as solutions.

b) There were also those for whom "solving the problem" involved finding the one correct answer (8 people). One can suspect that these people either do not see other possibilities or (which is more likely) associate solving a problem with finding only one correct answer.

c) For many the term "How many and what kind of stools will he be able to build?" implied that there were many possibilities and give more than one answer but not all of them. Two possibilities were given by 36 people, in which for 11 of them they were all none-zero solutions of the $3x + 4y = 30$ equation. I will elaborate on this group in my next point.

d) For many, the important part of solving the problem is to find and present "all" of the possibilities and that was what they did (64 people), although for some "all" meant those in which:

   – There would be no legs left and both types of stools would be made;
   – There would be no legs left and not both types had to be made;
   – Legs would be left, but less than necessary to build a stool (nr from 1 to 8; 9 people).

   The various understandings of solving the problem were expressed in comments:

   $30 : 3 = 10$ *If he were to build only three-legged stools, yet he has to make them with 3 and 4 legs!;*

   $30 : 4 = 7$ spare 2 *He could build 7 four-legged stools, but he would have 2 legs left, so that cannot be;*

   The first comment indicates that the student looks only for solutions in which $x \neq 0$ and $y \neq 0$, disregarding other solutions, the second indicates, that the student looks for possibilities in using the total number of 30 legs (no unused leg can remain).

e) There were some who in solving the problem tried to find and present many possibilities, many even wanted to give "all" the possibilities. Often, in the course of looking for solutions for $n = 30$ they noticed that there can be a different situation ($n = 28$ or 29) which they presented as another possibility. In 16 of those works there was an additional solution (apart from solutions nr 1, 2, 3) being the result of dividing 30 by

4 meaning solution nr 8. In the remaining 9 works, apart from solutions 1, 2 and 3, there are also cases in which 1 leg or 2 legs are left, yet they are not the only possibilities of that kind. In 2 works there are up to 7 solutions (without 1 out of the 8 for $n = 28, 29, 30$).

f) None of the tested students tried to find all of the solutions meaning all of the solutions to the $3x + 4y = 30$ equation, for $n \leq 30$. There were however some that found solutions for $n = 30$ explaining for example:

   This problem has many solutions. Every answer (when not more than 30 legs are used) will be correct as long as there will be enough seats.

   This sort of comment is evidence of the awareness of many solutions, however, most likely, the tested student either did not feel like searching for them, or, even more likely, did not know how to do it.

3.3. Solving strategies

In the works of the students one can discover many solving strategies. Some use arithmetical methods, other prefer illustrations. Others first perform an illustrated simulation, then confirm their liability with appropriate calculations and vice-versa: first they find a solution through arithmetical methods, then use an illustration. Some illustrations are very realistic, other schematic. Most commonly, 30 lines and dots were drawn and grouped in threes and fours. Some presented their results in tables. Others subsequently subtracted the number 3 or 4 from 30 and analyzed the divisibility of the result by the other of the two numbers. Nonetheless, sometimes miscalculations occurred, which did not allow to obtain all solutions even for $n = 30$.

In many works one could observe a "fairness" in splitting an even amount of legs among the type of stools. The latter students would apply a form of symmetry being the division of 30 legs into 2 equal parts ($30 : 2 = 15$) and building from each part one type of stools. Another form of this kind is to build an equal amount of three and four-legged stools.

Among works were those that presented ready solutions without a trace of any sort or arithmetical calculations or illustrations. It is not out of the question that these solutions were guessed or found through trial-and-error method without presenting the method.

In a few solutions a number axis was employed on which waves of 3 units and 4 units were applied. Such an illustration on a number axis allows one to notice the common multiples of the numbers 3 and 4 (being 12 and 24) and, moreover, that each of the 12 legs (there are two 12s) can be used to build either 3 four-legged stools or 4 three-legged stools. From the remaining 6 legs one can build for example 2 three-legged stools. The analysis of such an illustration allows one to find many solutions.

3.4. Answers

Some tested students, after finding "all" – in their own estimation – of the solutions to the problem, answered in full sentence by listing all of the solutions that they found, thus closing the problem solving process. Others answered by adding short sentences to each found solution. There were even some that would not formulate answers and only underline obtained results. Finally, it seemed some "worked for themselves". After finding the solution, they felt no need to formulate any sort of answer plainly because they found the answers already.

5. Summary

The tested students showed varied abilities in dealing with the aforementioned problem. Out of 163 people merely 9 gave all of the solutions for $n = 28, 29, 30$. No one gave solutions for the remaining n.

There were miscalculations in many of solutions given by the students, which I chose not to present because of the length of this article. Some had trouble writing the proper formula for dividing with a spare. They were not able to manage to find solutions. They limited themselves solely to dividing. Or to present a guessed idea. In their search for solutions they were very often rather clumsy. After finding a solution, they were not always capable of presenting it.

Nevertheless, many tested students were resourceful and creative. Many used illustrated simulations. They used their problem solving strategies rationally and, even though they did not find all of the solutions, those strategies could be used in the search for successive solutions.

The tested students are indeed qualified educators and work in schools, at the same time they decided to start complementary mathematics studies. One can hope that those who have difficulties in solving atypical problems will better their abilities in this regard. Otherwise I doubt they will be able to achieve the goal I mentioned in the beginning. If the teachers will not be creative themselves, it is difficult to expect them to teach creativity to their own students.

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MATHEMATICS MADE POPULAR: A CHANCE FOR BOTH PUPILS AND TEACHERS

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Abstract. The contribution reports on an untraditional presentation of mathematical activities for elementary school pupils, which are being prepared as a part of grant focused on developing pupils' interest in mathematics and change of their attitude to mathematics as a school subject. Solving non-standard tasks, competitions, games and manipulative activities provide pupils, teachers and parents with a chance to change their perception of school mathematics.

1. Introduction

Motivation is seen as an important precondition of efficiency of mathematical education by didactics of mathematics. It is a key element in pedagogical practice as it assigns subjective sense to learning activities of pupils. Usually it is seen as a result of interactions between pupil's personality, teacher's personality, classmates and subject matter. The role of elements of motivation such as teaching tools, didactic games and non-standard tasks has been stressed in a number of recent works (for an overview with respect to educational reality see *Skalková, 2005*).

In the contribution I report on activities performed at elementary schools in the Olomouc region. When solving a *Playful mathematics* grant we performed a set of activities, which aimed at presenting mathematics in a way other than as a boring and uninteresting subject to pupils and their parents.

2. Motivation: a constructivist background

The issue of motivation in mathematics has become very important due to the comprehensive reform of the system of elementary school curricular documents. So far pupils have been required to acquire a prescribed content of knowledge, information, activities and values. However, the new curriculum does not treat the specific prescribed content acquisition as a primary educational aim. It is rather a partial aim or a means of accomplishing a more general educational goal – the ability of every member of the society to make use of and apply specific results of his/her education in practical real-life situations. The importance of the educational content is not diminished – the focus of interest is rather shifted to something on a higher level. It is the pedagogical constructivism that is one of the most important trends following the requirements of the new educational paradigm. This wide range of attitudes attempts to adapt the means of education (especially methods and forms of teaching) to natural ways of pupils' learning.

Therefore, we assume that motivating pupils to be active is the primary task for the teacher. Once the teacher has been successful, the constructivist cognitive process of the pupils has been started – the pupils start creating their own images and building their own structure of pieces of knowledge. The pupils' internal world becomes a place where comprehension processes take place, images occur and concepts crystallize. This process of construction is influenced by previous knowledge, skills, experience and mental structures (cognitive maps) already acquired by the pupil.

This is based on respecting individual characteristics of the learners, i.e. especially their pre-concepts and individual experiences and learning styles. Implementing constructivist approaches is not trivial, as it requires conditions, which encourage and support pupils's activity. It is obviously the teacher – facilitator who is the cornerstone of the constructivist teaching. The teacher helps pupils to create new pieces of knowledge and prepares various sources of information for pupils' comprehension.

Mathematics for tomorrow's young children should become an environment for developing their personality. The idea of humanization, in which the school equals a service to children and a tool in their development, the center of which are affective components of learning (*Crowl et. al., 1997*), is of key importance in this respect. Developing the personality of a child is seen as education in the broadest sense of the word. Children are not objects of lecturing but subjects of their own learning. (*Wittman, 1997*) A teacher in school is to develop students' know-how, their ability to reason as well as to encourage their creative thinking. (*Polya, 1966*) This enables changing concepts, forms and methods of teaching mathematics so that the teachers could teach mathematics in a creative and interesting way and could become agents of a new and challenging class environment. At the same time, parents can learn at least part of what their children know – to see the constructivist oriented teaching mathematics based on intersubject integration (ópen classes"). They can see the challenges and experience of their children, however, on condition that they come and share the experience with their children (*Kafoussi, 2006*).

3. *Playful Mathematics* project characteristics: problem, goals

The *National programme of research II* project of the Ministry of Education, Youth and Sports named *Research on new methods of creativity competitions for the youth focused on motivation in scientific area, especially in mathematics, physics and chemistry* is a chance to apply efficient instruments of motivation in teaching mathematics while following the basic constructivist principals. When solving one particular task of the project (preliminary name *Playful mathematics*, manager B. Novák) we were inspired by the above-mentioned ideas and tried to confront them with the actual elementary school practice. Our experiment focused on creation and support of and research into educational efficiency of a number of activities: school mathematical competitions, projects, events for parents and public. The project is aimed at various elementary school target groups: mathematically talented pupils as well as ąverage"pupils (focus on raising and developing their interest in mathematics) or special needs pupils. The events are prepared in order to give pupils (even the less mathematically talented ones) a chance to acquire new mathematical experience and especially to let them get to know mathematics as something else than a boring subject – as an environment for personality development, interesting experimenting and discoveries. Reflection of the participants' view is very important in this respect – the participants are welcome to subsequently give their comments on both the content and the form of the event.

4. *Playful Mathematics* project characteristics: methods, ways of realization

A sociological research performed on a big sample of pupils (645 pupils aged 12, incl. 423 elementary school pupils, 222 pupils of 8 year grammar school, 276 boys, 369 girls) studied current interest of pupils in mathematics and natural sciences and their views of social status and applicability of these branches of science. The average grade (grading as in schools, 1 = the best, 5 = the worst) of mathematics was 2,04. Its popularity as a subject turned out to be quite good – 5th place out of 14 school subjects.

We aim at creating the basic framework of new competitions and other activities and advertising them. The activities include seminars on didactics for students and teachers, afternoon workshops on games, competitions, etc.

When applying and performing the activities we rely on co-operation of the researchers (a team of 8) with elementary school teachers taking part in the project. Tutoring students and PhD students is another important part of the project. Between November 2006 and May 2007, 12 events took place at 8 schools with more than 1000 pupils as participants. Such activities offer possibilities to:

pupils / students to be given space for interesting experiments and discoveries, activities connected with everyday experiences, projects,

teachers to change the approach, forms and methods, to be able to teach mathematics in an inspiring and interesting way, to help in creating the new climate, challenges for both them and the pupils,

parents to get to know part of what their children learn, find out that mathematics need not be boring and uninspiring formula training, however, they must come, see and share experience with their children.

The events are preceded by a systematic preparation of pupils during classes. The pupils are trained in tasks interpretation and presentation, arguing skills when defending one's way of solving the task, communication skills when co-operating on solving the task (pair work, group work, class work). This develops a number of competences: *competence to learn, citizen competence, work competence, competence to solve problems, communicative, social and personal competences.*

Mathematical activities performed in the project could be categorised as follows:

a) *games*, e.g.: sudoku, crosswords, board games, computer games, brain teasers

Didactic game require that the learning material is used in uncommon situations; otherwise the students won't be challenged to develop analytical and mathematic skills. Each game has to be prepared according to the principles of the theory of didactical games and must have special phases, which children have to go though in order to learn something new in mathematics. We believe that the most important feature of the game is that children acquire their knowledge without the didactical influence of the teacher as they work on their own. Researches report that regardless of the subject matter pupils working in small groups tend to learn more of what is taught and retain in longer than when the same content is presented in other instructional formats. This pedagogical sense of games could be characterized as follow (*Novák, 1995*):

- motivation of pupils, making them active,
- possible use of the game for making students work on their own, competitions (of individuals, groups etc.),
- development of counting skills.

Some examples of games include:

- *surprising assembling* – geometrical jigsaws or assembling solids without gluing, tantangram type or Columbus' egg type brain teasers
- *agic paper* – origami (water lilly in blossom, box, dog, etc.),
- *matches type brain teasers* (move the matches so that …),
- *pyramid puzzles* – number pyramids, clusters and other number based tasks,
- *"funny"tasks* – aimed at logical thinking developement,
- *ȩstimate the number of"type tasks* – e.g. of beads, beans or other small objects in closed jars, words on a book page or on a handwritten sheet of paper.

b) *activities connected to everyday life (projects)*

Some examples of such activities include:

- *Building a town* project, where pupils place themselves into a position of a citizen who wants to build a house. They have to consider financing (i.e. savings, loans, bank, jackpot, etc.), choosing the suitable plot (thus e.g. count the area), design, project, construction or buying the material (i.e. consider discounts, prices, etc.)
- *Treasure hunt* project, where pupils have to find the hidden treasure (such as sweets, small gift objects). The treasure can be found after solving a number of everyday life tasks including measuring, number and volume estimates (*How much / how many?* tasks), calculator calculations, navigating through the labyrinth, etc.

c) *unusual mathematical problems*, e.g.: *Kangaroo problems* (such as the following ones).

- On the left side of the Main Street one will find the housenumbers 1, 3, 5, ... , 19. On the right side the housenumbers are 2, 4, 6, ... , 14. How many houses are there on Main Street?
  a) 8 b) 16 **c) 17** d) 18 e) 33

- Four people can sit at a square table. For the school party the students put together 7 square tables in order to make one long rectangular table. How many people could sit at this long table?
  a) 14 **b) 16** c) 21 d) 24 e) 28

- Six weights (1g, 2g, 3g, 4g, 5g and 6g) were sorted into three boxes – two weights in every box. The weights in the first box weigh 9 grams together and those in the second box weigh 8 grams. What weights are in the third box?
  a) $3g$ **and** $1g$ b) $5g$ and $2g$ c) $6g$ and $1g$ d) $4g$ and $2g$ e) $4g$ and $3g$

- Four crows sit on the fence. Their names are Dana, Hana, Lena and Zdena. Dana sits exactly in the middle between Hana and Lena. The distance between Hana and Dana is the same as the distance between Lena and Zdena. Dana sits 4 meters from Zdena. How far does Hana sit from Zdena?
  a) $5m$ **b)** $6m$ c) $7m$ d) $8m$ e) $9m$

5. Our experience

Except for spontaneous reactions during and after the events a space is given to feedback from pupils, parents and teachers. A feedback questionnaire had been prepared, its evaluation will be performed before the project completion:

a) Evaluate the following statements (use the ++, +, 0, -, -- range):

[1] I enjoy mathematics.
[2] I like solving non-traditional mathematical tasks.
[3] I like solving brainteasers; I play chess or other board games.
[4] I like working with a computer.
[5] I found today's tasks simple.
[6] I solved the tasks successfully.
[7] What I was asked to do was new and unusual.
[8] I enjoyed being a part of a team and helping each other.
[9] On the whole, I liked the event.
[10] I would welcome another event like this one.

b) Choose and write the names of five activities, which you took part in, and sort them from the best one to the worst one:

c) Write your own assessment of the event.

At the same time, both children and their parents responded on a board titled "What I liked most / least". The following are sample reactions of parents: "We liked all activities because one has to think. Even if not successful, one has to think about the task.ór "I liked the smart children who prompted me when I was at a loss."

Most pupils enjoy their being both solvers and "first-aid staff", i.e. those who give and evaluate the tasks. Some sample reactions include:
Vojta: "I liked that there were so many customers at our post. The sweets almost disappeared!"
Vendulka: "I liked folding paper – I finally learned how to do things!"
Jana: "It was nice, it was superb!"
Mirek: "I liked it all! I hope there will be another event like this!"

During the events the teachers are also able to uncover some characteristics of their pupils or social relations in their classes. One of the teachers pointed out that: "Terezka is a rather quite, thoughtful and not so active a child. Yet when her parents were present, she was a leading personality of her post. Ondra, otherwise an average pupil, was most devoted. Jana seemed to be a good organiser, Radek, a rather introvert person, was very happy when he solved the task on his own – a quite adequate reaction on one's success. This is typical of Radek – he is very quiet in lessons; he does not speak even when at a loss – he tries to ignore his problems. When being approached, he reacts fearfully. However, he rewards even a partial success by a great joy and start of work. Yet he is not always able to continue on his own. This explains his joy after solving tasks, which were not problematic for other pupils. Lucka, quite weak in mathematics, could manage her post not only as an organiser but also as a 'professional' giving 'scientific' advice. Furthermore, she was able to show her readiness to help."

6. Conclusion

We tried to show at least some possibilities offered by the *National programme of research II* project at Faculty of Science and Faculty of Education, Palacký University in Olomouc. We aim at increasing motivation of pupils for mathematics, improving perception of mathematics as an interesting school subject and making pupils learn mathematics. Our ambitions include utilizing non-traditional forms as a suitable tool for making mathematics more popular, forming positive attitude of pupils to mathematics and improving the overall class environment. Subsequent reflection and evaluation is important. We have learned that pupils as well as parents and general public like the events. We believe that this is caused by the fact that the events give everybody an unconventional view of mathematics, increased their interest in studying context and relations in problem solving. As far as motivation is concerned, this has enormous importance. We are happy to see pupils' interest and enthusiasm, especially that fair play rules were never broken. This gives us the feeling of having done a good and meaningful thing.

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THE REALISATION OF SELECTED ATTRIBUTES OF 'FUNCTION' USING THE PROJECT METHOD

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Abstract. The notion of function plays a crucial role in teaching mathematics. Why is this important issue so problematic for students? It is worth noticing that precise specification of this notion took place relatively late, in 19th century. This is why it is so important to attempt students' active participation in defining and understanding the notion of 'function'.

I was encouraged to use the project method in my work by Marie Kubinova's article 'The activating role of project in teaching mathematics'. This method, in a wide thematic spectrum, is realized in Mathematics and Didactics Department, which functions as part of Pedagogical Department of Charles University in Prague. The Mathematics Department of Rzeszow University has cooperated with the above mentioned Department in research on mathematics didactics for years.

The method of discussing the notion of 'functions' and their basic attributes proposed in this study is based on experimental approach to this issue. It is a method of teaching based on students' closer, every-day experience, who carry out certain ventures, wider than the traditional homework. It turns out, that by using techniques which correspond with students' world and dealing with a real problem, we can make 'function' quite a pleasant notion.

The project method I suggest to realize the above mentioned issues allows for shifting the load of recognition from the teacher as an 'informer' to the student. After research conducted in one of Rzeszow secondary schools it was possible to form a conclusion that active approach to a subject is an effective impulse motivating students to action. Such approach contributes to integral recognition and development of student's communicative skills, which perfectly harmonizes with the project method.

You say I'll forget,
you show I'll remember,
I experience I'll understand
(Chinese proverb)

1. Explaining the function notion using the project method

My students find mathematics extremely difficult. It is seen as a complicated, if not impossible to understand subject. Such negative opinion has been expressed by over 90 per cent of the students I worked with in one secondary school in Rzeszow. Struggle against this stereotype is a real challenge for an active mathematics teacher. Unfortunately, the teachers themselves are to blame as they present mathematics as a subject understood only by the chosen few. Teachers pay attention primarily to the knowledge students need to complete competency test and not to students' cognitive ability. I have therefore decided to help my students see mathematics as a tool for creative work, useful in the world around us.

In the school I work for it is a tradition that during the second lesson in a new school year students take a test checking the knowledge from the lower level (lower secondary school). 65 students did the test. The tasks which involved the knowledge of the 'function' notion caused the greatest problems. Only one student managed to accurately match the function $f(x) = x - 1$, $x \in N$ with the right graph and specifying the attribute (field, set of values, monotonicity, the zero of the function) given by the formula $f(x) = 2x + 1$. This result was an impulse for me to think about it and to make an attempt to change this situation.

Functions play a basic role in teaching mathematics. Why then does such an important notion cause so many problems. Based on the assumption that in learning a new issue it is vital who and how will present its content, we can claim that students' problems are a result of the fact that teachers do not always present the message in a clear way. This assumption is consistent with the rule of didactic parallelism, which should play an important role in the choice of content and the teaching methods on particular levels of education. How to effectively teach function?

I was encouraged to work using the project method by Maria Kubinowa's article *Activating role of project in teaching mathematics.* This method is used, among others, in the Pedagogical Department of Charles University in Prague, which has cooperated scientifically with the Mathematics Institute of Rzeszow University.

If we assume that the best way to learn mathematics is to discover by your own action then it is certainly worth considering working with the project method.

The choice of subject is an important aspect of the project method. The choice should be based on the ability and interest of students, the teacher's needs and, if possible the school environment. Next we set the time limit, project presentation form and the evaluating criteria. My students had the choice of two alternative subjects:

I. Write down the air temperature at a set time in the same place.

II. Every morning (right after waking up for 14 days) assess your mood on the scale from 0 (-) to 10 (+). The meaning of the scale is for you to decide.

The students' task was in particular:

1. To present the results of their observation on the right grid, graph, in a table and in the set of ordered pairs.

2. Specify (if possible):

a) domain and set of values

b) zero of a function

c) monotonicity

d) added value, negative value

e) periodicity

f) the highest and the lowest value of the function

g) parity and odd parity

From the very beginning the students knew how they will be assessed. They were given the criteria of detailed assessment, which contained the following elements:

a) accurately done single way of describing function - max. 2 points

b) correctly specified attribute - max. 3 points

c) mistakes in formal record (using mathematical symbols) - 1/3 points as well as components of the final assessment, divided into the following elements:

a) correctness and completeness of data - 20 per cent

b) the data presentation method - 30 per cent

c) the class assessment - 20 per cent

d) the teacher assessment - 30 per cent

The Project was supposed to take 18 days (from 9th October to 27th October 2006). The table below contains the detailed work schedule:
William H. Kilpatrick believes that *The project method is a philosophy of independent learning* and as such develops very precious abilities such as. Planning long-term work, the choice of optimal method of solving problems:

a) work division

No.	The tasks	Terms
1.	Introducing students with the issue connected with the project (specifying the subject, the method of collecting information, the rules of presentation and evaluation)	9.X.06.
2.	Preparing the plan of work connected with the project in clearly specified terms:	
	a)collecting and writing down	from 10.X. to 23.X
	b)consultations with the teacher	13. i 20.X. 14.15-15.00
	c)preparing the data and preparing presentation in a hand-written or computer-typed	from 23.X. to 26.X.
	d)presentation of the results of the work in front of the class, summing up the experience	27.X. (2 classes - each 45 minutes)

b) cooperation in a group, discussion, shared responsibility for the effects of work

c) using different sources of information (experiment, literature, internet, interview)

d) processing data (also using computer science technology, computer programs)

e) planning and doing experiments

f) defining, classification and creating its criteria

g) forming conclusions from data and observation

h) forming new research questions

i) presentation of results.

Carrying out projects based on function created an opportunity to make attempts connected with defining and classifying objects, argumentation and practicing the ability of preparing in writing and presenting he obtained results. The students came against numerous difficulties, even though working using this method initially seemed easy. I discovered that during consultations, when questions indicating problem with the understanding of function appeared:

a) Should points on the graph be connected?

b) How to formally record monotonicity intevals?

c) Can there be two highest or lowest values?

d) How is point placed on axis OX and earlier and the following point?

e) Above it then is the first one the zero of the function?

f) Will the odd parity of the function result from its specification?

g) How to record the set of ordered pairs?

h) What should be marked on axis OX: the time or the result of the measurement?

The presentation of the attributes the students noticed caused fewer problems. The presentations were made in a variety of forms. The presentations differed in size and the technique used - there were pictorial posters, written work and multimedia presentations.

2. The basic rules of the project method

Teaching using the project method is an alternative for formal teaching. Project teaching correlates with other subjects. It helps to obtain knowledge. The teacher assumes a role of a guide or a friendly observer, hence their role is described as progressive. It is undoubtedly more difficult than a traditional role. However, it is more beneficial from a didactic point of view. To conclude, the teacher is an organizer of the teaching process, supports students in the choice of possible actions and solutions.

It has to be stressed that the project method has numerous features which distinguish it from other teaching methods. Mostly interdisciplinary nature, independence, the possibility of forming conclusions, extending research mathematics and predicting the future. The project method facilitates subjective treatment of the student, it takes into account their interests, abilities, needs and aspirations. It stimulates the emotional development of the student, group work skills, familiarises with responsibility for one's own and group actions. The project method helps to perceive mathematics as a form of creativity, in which the world around us plays an important role. The students can carry out even unconventional ideas, surprising their teacher. My students showed highly creative attitude in specifying the levels of mood, using internet emoticons, verbal description or drawings (*it's bad but I'm still alive; weak, oh, no ...*). Similar final conclusions were reached: *My mood was getting better when it was weekend and I did not have to go to school; On Friday it was worse, because I had to use my energy reserves to get up to go to school; Wednesday was awful - test in English.*

The additional advantage of the project method is that it clearly shows two basic assessment functions: classifying and diagnostic. To realise both the teacher must carefully prepare the criteria of assessing the particular stages of work and types of students' activity. The possibility of students' self-assessment, not present in the traditional system of assessment, is an important element.

3. The effects of using the project method

The method proposed in this work for teaching functions and their basic attributes is based on experimental attitude to this issue. It is a method based on students' closer, every-day experience, who carry out certain ventures, broader than the traditional homework. It turns out that using practical solutions corresponding with student's world, dealing with a real problem, we can make function quite a pleasant notion. This is what happened with my students, which can be proved by the fact that the test in function results were very good: 52 per cent of students received mark good (4,0) or higher. What is more, the method helped to:

a) Suggest a not only formal look at mathematics,

b) Decrease the fear before mathematics class,

c) Mathematics as a tool for describing the phenomena surrounding us,

d) Forming such features as: responsibility, diligence, punctuality, independence and self-discipline,

e) An opportunity to form arguments allowing for accepting or rejecting an opinion,

f) The ability to react to other students' arguments (particularly during the presentation),

g) The ability to prepare in writing and present materials,

h) Multi-aspect understanding of mathematical terms on the basis of one's own experience,

i) Making attempts to define and classify objects,

j) Facilitating constructive attitude in teaching and learning mathematics, including the ability to form conclusions, interpretation and finding cause and effect relationships,

k) To notice students' outstanding, other abilities.

To sum up, I'd like to collect my students' most common problems and dilemmas, appearing during the realisation of the project.

a) Specifying the set of arguments and the set of values (in some cases - the day of the weak became dependent variable,

b) The problem understanding assignment idea (2 people),

c) Incorrect intuition connected with the understanding of the zero of the function - if there is no line, which intersects axis OX,

d) Problem coding the subset of the set of arguments (transferring the record on arguments from he finished set) - 40 per cent of students,

e) Joining points on graph (because that's how they showed it on TVN - student's defending argument, after making an incorrect graph).

The projects were finished with evaluation in the form of a short questionnaire, containing the following questions:

a) What have we managed to achieve?
b) What new things have we learned?
c) What value does it have for them?
d) What could have been done better?
e) Is it worth working using this method in the future?

The students have all said that working using this method has been very pleasant and they are willing to take new tasks of this kind. The students took this job voluntarily. It has not been obligatory, according to Kilpatrick, who said that students should not be forced to take unwanted tasks, because as unwanted they will not give long-lasting and satisfactory results.

The research helped to see that the effective impulse motivating the students to work are all types of activity. This attitude contributes to complex acquisition and development of communicative skills, which is compatible with the project method. This method was initially used at the beginning of the 20th century, and the aim was to make it possible for students to independently obtain knowledge and check their skills in every-day situations, instead of purely theoretical knowledge.

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PROBLEMS TEACHER'S PRACTICE FORMING MATHEMATICAL ACTIVITY AND CREATIVITY OF THE GIFTED PUPILS

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1. Introduction

The words of P. J. Taylor (2003), who states that *mathematics and mathematics education, and teaching to a high standard, are the keys to solving the world's existing problems and planning for the future* have inspired this work. The subject of the work, **Problems teacher's practice forming mathematical activity and creativity of the gifted pupils**, is submerged in the issue of one of the contemporary trends in researching the methodology of teaching maths, which is called: **activity and creativity in teaching mathematics - theory, diagnosis and methodology, prospects**. This work is the extension and some kind of modification of the previous work of mine, see: A. Pardała (2006). In this work I refer to certain synthesis of the knowledge to that point, see: A. Pardała (2003, 2004, 2006), and I also articulate one of its' aspects - the crucial importance of teacher's intervention on activity and creativity of a student, who is solving a mathematical problem. And then I synthetically present findings of research, assessing teacher's impact on stimulating a gifted student, in particularly which was done for doctoral thesis by E. Śmietana (2005). In summery of my work, I put forward some remarks and final reflections related to mathematical activity and creativity in gifted students' education.

2. The aim and tasks of the work, understanding its basic concepts

The aim of the work is: 1) outline the current state of research and experience in the scope of stimulating a pupil's activity and creativity who is just solving a mathematics problem, 2) point out the extent and directions of how to use the teacher's intervention in the process of solving a problem by a gifted pupil.

The task of the work is to try to find answers to the following questions: 1) How advanced are current experiments and research into mathematical education of gifted students? 2) What are the didactic recommendations to improve the practice of forming a pupil's mathematical creativity.

In my work I make references to my synthesis of the knowledge of the present state of research into creativity in teaching mathematics and school and teacher experience in this respect. Here, I refer to the results found in the literature and hitherto completed Polish research works. I suggest the following list of the concepts found in this work: mathematical creativity, mathematics teacher's intervention, see: A. Pardała (1984, 2004, 2006), and its types, see: E. Śmietana (2005). **The very term "intervention"** explains what kind of situation is meant here. It is as follows: a pupil has just got a problem and is solving it possibly all by himself (cooperation with the whole class or a group of pupils is also possible). However, at same moments, the teacher joins in by asking questions, making suggestions, comments, gestures, reflections, etc. Each activity of that kind is called the teacher's intervention in the problem solving process.

3. Formation of creativity in mathematics teaching and the prospects of mathematics education

The document, *50 Years of CIEAEM: Where we are and where we go, Manifesto 2000 for the Year of Mathematics*, contains, among other things, some diagnoses and views on the evolution of notions, experience and intentions to improve mathematics teaching. Part II of the document, Where we go?, focuses on some problems, ideas and directions of the future work of those concerned with: 1) development of mathematics for all and its popularization, 2) revitalization of the consciousness of democratic society and winning its support for a preferential treatment of mathematics education, 3) modernization of the existing base for mathematics education taking into account its position in the modern mathematized world. And the core of the diagnosis of *Where we go?* is the following:

1) A re-evaluation of the aims of general education has taken place; from a universal education for the elite to an education for all. This immediately implies a change in the aims of mathematics education and its mission which must: a) ensure an understanding of "mathematization processes"in society, b) create a clear and critical assessment of the role mathematics plays currently and its application in social environment. Further on, the authors of the document ask questions defining this scope of research, some of which are:

- How can both mathematics teaching and learning be represented not only as an introduction into some great ideas of our culture but also into the criticism of their content and application?
- What kind of research on mathematics didactics could contribute to the creation of a new outlook on mathematics teaching practice?
- How to make society aware of the fact that mathematics teaching develops responsibility and gives free rains to a democratic vision of introducing new forms of social contact, communication and dialogue?

2) Views on mathematics education are of dichotomic, bipolar nature. On one hand, mathematics is still one of those school subjects which causes strong anxiety, aversion and feeling of incompetence and which is difficult and senseless for most pupils. Hence they consider themselves "mentally handicapped"in this field and doomed to failure. O the other hand, for some parents, pupils and politicians mathematics requires particular aptitude and is considered to be only for the chosen.

Hence "mathematical abilities", ą talent for mathematics", a natural gift for mathematical thinkingór ą natural interest in mathematicsąre seen as something very rare among pupils or in society. And this makes mathematics a natural factor of social selection, which only aggravates a feeling of anxiety and aversion to it. Further on, in the above mentioned document , there are quite definite views on the didactic aspect of the problem: **mathematics as a means of social selection**. In my opinion, one can find a strong thesis formulated here: as long as the social focus is on "the talented", the majority will not be properly educated. In particular, the authors pose some crucial investigative questions, some of which are:

- *Should we keep the highly selective framework and methods of education, but give up the privileged position of the subject as part of the core of general education? Or do we seek to keep mathematics at the core of the curriculum but find ways of teaching the subject to all students? How to overcome the limitations of this dichotomy?*
- *The notions "mathematical ability", "individual differencesąnd the "gifted pupiląre ideologically collective constructions based on convictions or prejudices, as a possible vehicle of purpose and interest. Moreover, the prejudice of "mathematical giftednessżeadily associates itself with other hereditary features such as gender and ethnicity - how can we act against that?*

The above outlined ideas and trends of updating mathematics teaching and stimulating mathematical creativity correspond to the experience gained by a French association, Math Pour Tous, described in the work by L. Beddou and C. Mauduit (2001). In this concept the teaching of mathematics is based on investigative activity and dedication of as many pupils and students as possible. Here, one tries to follow or imitate some behaviours, patterns and principles characteristic for scientific inquiry, such as: discovering by a sking questions; learning by research; stimulating creativity and imagination; appreciating the importance of error in learning; learning how to listen to, get across and exchange ideas, etc.. An academic teacher is a supervisor, puts forward a number of tasks and problems whose solution would not make use of the already ącquired knowledge". The teacher running such a workshop is obliged to get the work of particular pupils or groups of pupils (twin groups) to progress. The results must be presented on-line or at a conference. The pioneers of the concept are G. Polya, I. Lakatos, I. David, E. Marchisotto, Z. Krygowska and others. The unique nature of the concept and the educative action "Math en Jeans"(Math Pour Tous) consists in its being addressed to all the pupils and students concerned and not only to their elite - i.e. would-be research workers or professional mathematicians. Besides, the teacher and the pupil (pupils) begin from the same level. The pupil, getting an open problem, has an impression of doing new things and demonstrates his emotional attitude by saying: "I have solved", "I have found". Here is an example of the problem - **Conway's Sofa**, solved during this educational activity:

Let us consider a corridor consisting of two parts, either 1 metre long, at an angle of 90° to each other. Along the corridor we want to carry a sofa represented by an undeformable flat figure S. An example of the figure may be a square with each side 1 m long. What is the largest possible area of a sofa with area exceeding 1 m , which can be carried along the corridor? What can be said about the problem when the corridor consists of several parts? What happens when the angles at which *the corridor turns are not right angles?*; see: L. Beddou and C. Mauduit (2001), p. 24. At the beginning young people find the problem rather difficult: how to mathematically describe the situation and the movement of the figure?; how to construct examples of the figures or sets of figures meeting the conditions set out?; is there any relation between the shape and the surface plane of the figures and vice versa? Concretization or extension of the problem seems only too natural here. A successful attempt at solving it takes some mathematical skills and activities as well as further studies. One can notice the participants change their attitude to the problem and its solution as: 1) it is necessary to substantiate such things as: "it is obvious that ... ", or "it can be seen that ... "; 2) one has to be open to the reasoning of the others; they are or may be right; 3) one must be aware of the extent of the obtained solution (partial solution, solution for a set of figures and description of their properties, knowledge of only some theorems useful for finding the solution, etc.). The below mentioned research papers on mathematics didactics are also related to the problem of how to form pupils' mathematical creativity. R.A. Utiejewa (2001) reminds us that "mental energy"forms the basis of man's abilities although the concept of ability is not unequivocally described even today. The authoress calls up L. Terman's (1959) research work which confirmed the idea that the main distinguishing mark of ability is intellect characterized by its seven components including a logical-mathematical one. Besides, contemporary psychologists are unanimous that one should distinguish creative mathematical abilities from exceptional mathematical abilities (mathematical genius) which occur at most in 1% of the pupils. R.A. Utiejewa tested 2000 Russian pupils aged 10 - 15 and found out only 8% of this population has mathematical talent. And her model of differentiated mathematics teaching in the case of creativity endowed pupils assumes spotting them early (in the fifth form at the latest) in order to offer them the right conditions ensuring the optimum development of their mathematical potential in their school environment (class, school). The main purpose of such concept is to individually form scientific and mathematical activities of creative pupils taking into account: motivated activity of both the pupil and the teacher, teaching through setting problems and individual attention to each talented pupil. The basic organizational features are: "group workór "individually differentiated"work. **The authoress of the concept also gives four indispensable conditions of its successful realization: 1)** special mathematics programmes focused on the interests and creative aptitudes of the pupils which should be constantly updated considering individual capabilities (also work pace) of the pupils; **2)** important methodical changes in the teacher's work and teaching and a change in his relation with mathematically talented pupils both during the lesson and while organizing their individual study; **3)** special programmes to prepare a would-be mathematics teacher during his university education for how to professionally work with talented pupils; **4)** providing the teachers working with gifted pupils with all the necessary educational aids (literature, guide books, course books, collections of problems, periodicals, etc.). In the summary of the article she emphasizes an important aspect of creativity pedagogy: working out programmes, together with their relevant methodology, for mathematically talented pupils.

The work of E. Jagoda, D. Panek and A. Pardała (2001) corresponds to the above. The authors point to the growing impact of the media technology and the

Internet on the state of education in the world and the forms of pupils' development and education. For this reason tele-education can be a successful means of work with a talented pupil. And further, the paper also signals some shortcomings affecting creativity pedagogy and gifted pupils because the didactics for gifted pupils has not been consolidated and the knowledge of school and teaching practice, i.e. practically functioning programmes, their realization, methodics of identifying, educating and developing mathematically gifted pupils is not sufficient. That may be an encouragement to carry on research on mathematics didactics, and enrich it. To prove it the authors żevealęxamples of the methodics of work with gifted primary and secondary school pupils. The evidence of the efficiency of the methods are: **1)** an early detection of mathematical interests of the pupils, their willingness to demonstrate that they are capable of noticing something in mathematics or in a maths problem that others can not see; **2)** correct interaction between teacher and student, i.e. the most brilliant students can be a mathematics teacher's helpers (his subject assistants); **3)** a relevant selection of the teaching material and problem material, forms of educating and developing talented pupils (mathematics societies are preferred) and making them more active (usually through mathematics competitions); **4)** work timetable and a programme of the mathematics society suitable for talented pupils.

In the Rzeszów region, where I live, for example a man of pedagogical success with respect to work with the mathematically talented pupil is A. Bysiewicz, a mathematics teacher in M. Kopernik Secondary School, Krosno. He is the teacher of Jarosław Wrona, the winner of the silver medal in 2002 in the XLIII International Mathematics Olympiad in Glasgow. Another successful educationist is W. Rożek, a teacher of mathematics in KEN Comprehensive School Complex, Stalowa Wola. He is known as the teacher and tutor of a large group of finalists of Mathematics Olympiads. One of them is Tomek Czajka, the prize-winner of the domestic and international final of a Mathematics Olympiad. What is more, Tomek Czajka was the mainstay of the three-person representation of Warsaw University that won, in March 2003, the world championships in computer programming in Beverly Hills.

Here is W. Rożek's work timetable for a secondary school mathematics society intended for gifted pupils. Mathematics activities are carried on in two groups. The first one includes first form pupils and those from upper forms who have just decided to enlarge their mathematics knowledge. Here I have introduced basic notions related to the content of Olympic problems. In particular, they are as follows: 1) Number theory: number congruence, small Fermat theorem, 2) Problems related to chess-board colouring , 3) Dirichlet's drawer theorem , 4) Inequalities and means, 5) Geometry: lines, circles, polygons, Tales arcs, points and lines in a triangle, geometric transformations. In the other group including pupils from upper forms the knowledge is enlarged with such notions as: 1) Power of a point in relation to the circle, 2) Inversion, 3) Brioncham theorem, Cevy theorem, Menelaos theorem, 4) Monotone sequences and inequality proving, 5) Viete formulas, 6) Functional equation, 7) Recurential equations.

Another form of support and formation of pupils' creativity in the Rzeszów region, generally, formation of creativity in mathematics teaching is the affiliation of mathematics school societies and mathematics competitions in the district (e.g. H. Steinhaus Mathematics Competition in Jasło) by the Rzeszów Section of the Polish Mathematics Society. Selected staff of mathematicians - members of the Section and academic teachers of Rzeszów colleges-offer essential and educational help to the members and tutors of the societies. They also organize seminars, workshops and mathematical competitions for them.

Another supporter of the development of pupils' mathematical creativity is the School Superintendent Office, which organizes seminars for the executive staff and teachers, e.g. **Education and care of a gifted pupil**, in order to demonstrate and popularize the local experience and achievement in this respect. It is that kind of activity resorting both to past experience and looking ahead, carried on by headmasters and outstanding teachers of not only mathematics but also other subjects that forecasts good progress in spotting, educating and developing mathematically gifted pupils and students.

The work of S. Grozdev (2003) synthetically presents reflections and Bulgarian experiences in the field of stimulation of students' mathematical activity and creativity. The author reveals that the background of the methodology of work with students at the stage of the preparation to domestic and international Mathematics Olympiads is based on the implementation of advices of H. Freudenthal, J. Piaget, H. Poincare and Bulgarian educator I. Ganchev. The first of the mentioned above states that *in mathematical teaching the taught one should pass through the following stages: first stage - instinctive rediscovery, second - conscious application, third - formal definition*, see: H. Freudenthal(1973). With reference to the first stage, J. Piaget says that *a complete acquire of knowledge occurs in the rediscovery process only, which needs creation of problem situations.* Following H. Poincare, however, to create means to distinguish and choose. And further S. Grozdev admits that successes of the Bulgarian students in the International Mathematical Olympics result mainly from the following actions:

1) *Our special attention to the żevisionąctivity is connected with the so called a hierarchic approach for investigating and systematizing of students' cognitive activities in the preparation for Olympiads. The żevisionąctivity, which is discussed in the present note, is concretized in individual reading through personal notebooks and is in a direct relation with "keeping a notebookąctivity. The latter activity is a result of the above mentioned search, collection, investigation and systematization of topics, methods and problems. In its turn the żevisionąctivity influences 9up along the vertical) successful problem solving, creativity and scientific research.*

2) The main task in the preparation of gifted students for a successful participation in Mathematics Olympiads is to stimulate their cognitive and will for individual work and research. Some people say that is very simple to become a scientist.

4. Results of research on teacher's intervention

Now I would like to outline the results of empirical research on teacher's practice of forming pupils' activity and creativity based on E. Śmietana's (2005) doctor thesis.

E. Śmietana (2005) - a secondary school mathematics teacher - in his doctoral thesis deals with the effect of the teacher's intervention on the mathematical activity of a talented pupil solving a problem. He carries out an individual teaching experiment in a group of secondary pupils and then proves the thesis: *the teacher's distracting interventions increase the mathematical activity of the pupil solving the* *problem and sometimes an indispensable help in solving an untypical mathematics problem*. The author formulated four aims for the experiment: "1) finding effective teacher interventions in the process of solving a mathematical problem by a mathematically talented pupil, 2) identifying the activity blocks that occur in the process, pinpointing their causes and suggesting the ways of eliminating them, 3) analysis of the pupil's behaviour while solving the problem after the teacher's distracting intervention and a description of his mathematical activity, 4) evaluation of the effect of the distracting intervention on the solution of the problem", ibid, p. 378. He achieved these goals applying, among other things, a method of stimulating the process of solving a mathematics problem (an algebra or geometry one) through properly prepared scenarios of the intervention. The impact and effectiveness of the intervention is precisely analysed and described in view of the revealed activity aspects of the pupil or his activity blocks. The behaviour dynamics of the tested pupils is clearly presented in specific blocks diagrams. The interesting conclusions which E. Śmietana has drawn from the qualitative didactic analysis should be further verified in more numerous group, at different levels in the working conditions typical of a mathematics teacher. The core of his conclusions is as follows:

1) Opening interventions appeared to be useful and effective in the methodics of solving mathematics problems. They resulted in the pupil's being receptive to the areas of the applied knowledge and triggered the association that helped to solve the problem, for example, the pupil associated the basic problem with an equivalent one, which caused a change or extension of the pupil's knowledge.
2) Opening interventions have the following characteristics: a) they do not always properly stimulate the pupil's mathematical activity (they may arise a feeling of being distracted, which hampers the activity necessary to solve the problem or confuses the pupil how to use the knowledge that is still new or "unfriendly"), b) they are not necessarily effective in the case of all types of mathematics problems and all mathematically gifted pupils, c) they can occur at any stage of a problem being solved and frequent occasions.

3) In the process of developing the skill of solving mathematics problems little attention has yet been paid to divergent (distracting) interventions, causing divergent thinking that, in the case of a talented pupil, positively increases his mathematical activity. The research subject of E. Śmietana's is in the same line as A. Pardała's doctoral dissertation (1984) which I describe as: **Didactical problems of the teacher's intervention in the process of teaching mathematics to pupils**. E. Śmietana successfully introduces some terms from the analysed literature into his research scope defined by the subject and aim of the research. For example, he takes such terms as convergent thinking and divergent thinking from psychological literature, see e.g.: J.P. Gilford (1978), whereas the notion: the teacher's intervention, its forms and kinds, understood as convergent intervention which consolidates the pupil's knowledge necessary to solve a mathematics problem, comes from mathematics didactics and becomes the object of his research, see: A. Pardała (1984). The result of his investigation is enriching the practice teaching a gifted pupil and introducing new terms: *opening intervention* and *divergent intervention* into mathematics didactics, which helps describe the process of solving a mathematics problem by a gifted pupil.

5. Summary, remarks and final conclusions

Taking the above into consideration and historically analyzing the evolution of the education and development of pupils' abilities and strategies of educating exceptionally talented pupils, one can distinguish **four types of didactic and organizational activities** that can be recommended: **1)** making the development of gifted pupils faster, which means increasing the pace of their teaching and learning, **2)** providing them with a greater amount of knowledge, i.e. expanding the scope of the subject taught, **3)** offering them a more advanced knowledge but only slightly more advanced than the level of their current knowledge and personal expectations, **4)** forming pupils' creativeness in the process of teaching a given subject.

All the above suggestion being accepted, the very process of educating gifted pupils resembles a realization of a sequence of consecutive organizational and educational tasks and the formation of pupils' creativeness (also mathematical creativeness) is its target. Such a system of education of gifted pupils can be also adapted to talented students, however it takes: 1) multilevel teaching and varied teaching content, 2) fully qualified staff supervising their development.

The concept of forming the pupil's mathematical creativeness in mathematics teaching is integrally connected with the idea of developing creative mathematical activities of the pupils which should be realized, according to M. Klakla (2002, pp. 48-49), through activities regarding intellectual, didactic and evaluation aspects, with the teacher responsible for planning and guidance. M. Klakla's concept (addressed to secondary school pupils) consists of two steps i.e. teaching as well as developing elements of creative mathematical activity and creative mathematical activities of particular pupils and appropriately selected teaching programme and unique methodology of attaining a particular kind of creative mathematical activity. This is also the result of the selection of the problems and initial examples, multistage problems and paradigmatic examples.

The effectiveness of the formation of the pupil's creativeness in mathematics teaching and work with a talented pupil is also conditioned by a useful cooperation with the school, mathematics teacher and his experts (methodics advisors, mathematicians and mathematics educators, educationists and creativity psychologists). At the school level, a solidly prepared and well organized programme of work with the gifted pupil and in particular with a mathematically gifted pupil should be indispensable. Such a programme should include the accepted concept of education and development of gifted pupils and make use of long-standing experience of a given school as well as the break-through solutions of the Society of Creative Schools and the Association of Active Schools based on the pedagogy of individual differences. While creating the programme, the following targets can be defined: **1)** attracting and spotting talented or exceptionally gifted pupils, **2)** abandoning the class-lesson system in favour of other forms of encouragement and work with pupils who are exceptionally mathematically gifted, **3)** special forms of work with the pupil or pupils who have a particular talent for mathematics.

Special forms of work ensuring the effectiveness of the formation of the pupil's mathematical creativeness include: 1) extra activities, i.e. those performed outside lesson hours adapted to the individual needs and interests of mathematically gifted pupils, **2)** mathematics societies whose members come from forms of different levels and where the motivation is the intention to broaden mathematical knowledge and exchange of ideas, **3)** an individual teaching programme for a particularly talented pupil who was a success in mathematical competitions or in the Olympiad, **4)** September meetings and cooperation (at school or by correspondence) between former Olympiad participants from a particular school or region and their younger schoolmates interested in taking part in a Mathematics Olympiad or individual contacts between the latter and research workers, mathematicians who once attended the same school, **5)** participation of exceptionally talented pupils in the activities (meetings and workshops, lectures and seminars) organized by the mathematics departments (Institutes, Faculties) at school for higher deduction or confiding mathematically gifted pupils in the care of the mathematicians from those colleges, 6) participation in educational activities organized by the National Found for Children and contacts between the most talented young people in this country and the leading representatives of Polish mathematics and science.

The above examples of international and domestic experiences from building the mathematical activity and creativity of students confirm the existence of many forms and possibilities. Additionally, they reveal the related methodics and problems of teachers' practice. Those examples are some kind of proofs, which enrich the practice of stimulating mathematical activity and creativity of students. Moreover, they confirm that it is not just about looking for only one optimal and effective way of their creation. It is rather about **healthy competition** that gives school, as well as students wider vision for the reached level of mathematical activity and creativity. Following P.J. Taylor (2003) *activities related to competitions can also give a student a rare opportunity to meet students from other schools, broadening their perspectives and making new friends. Activities such as camps and mathematical ćirclesćan very much enrich the student's interest and motivation in mathematics, potentially propelling them to a level far beyond the classroom, and since mathematics is such a big subject, not prejudicing the material they will learn in the school in the following year. Giving a student access to the right books and journals can also achieve the same result.*

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MENTAL MANIPULATION WITH A NET OF A SOLID

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Abstract. The contribution deals with selected options for development of space imagination of pupils as an important competence, which can support role of mathematics in development pupil's personality. We talk about tasks, in which space imagination is implemented during mental manipulation, when pupil creates a solid from a net of solid in a playful manner.

Introduction

In the Framework education program recently introduced in the Czech Republic play major role key competences of pupils. In am opinion, some of these key competences can be developed using mathematical education. One of these important competences is a geometrical and mainly spatial imagination. Its level can be satisfactorily developed in mathematical education at all grades.

It is connected to quality of teachers of mathematics in the field, for example in the fact, that they will not omit spatial imagination task, that they will on they own will deal with spontaneous stereometry tasks.

Task situations

I tend to verify experimentally, which kinds of tasks can be used for development of spatial imagination of basic school pupils, especially in the early grades. These should be tasks not requiring knowledge of projection methods, but utilize active spatial orientation of pupils and mental manipulation with objects. These are tasks, which can be solved without direct relation to the explained matter and introduced as a exercises of relaxation tasks.

One of the options is a task situation, when the pupil only in his mind manipulates with a net of a solid and creates a solid from it. In our case it is incomplete net of a solid ("inside", without one side), which is introduced as a net of a room in the shape of cube into which we look through the missing side and try to orient ourselves in mind, it means to say, which side is the upper, which is the bottom etc. The only point for orientation is the drawn door on one of the sides. The pupil is required to complete the net with a ćeiling light", ćarpet"in the floor, "windowąnd ćaseón side walls.

Explanation and rehearsal:

The pupil has into an incomplete net of a cube without one wall, where the door is drawn only, fill in pictures of objects in the way, that after make-up of the solid these are correctly placed. The pupil is required to make-up the net only in his mind with mental manipulation. After filling the objects into the net he can verify the correctness by making the net up.

Fill: **into the room net:**

On the floor is carpet,
on the ceiling is light,
on the side walls positioned
is door, window and case.

For example:

corresponds to:

Examples of given incomplete nets:

In the case of a net of a block it is possible to require orientation of more objects:

Carpet from the door   window   light   case

The other option is a task situation, when pupil in his mind creates a model of a solid from given net, on which the óuter"walls have various symbols and looks for a corresponding solid. Since the task was given also to the youngest pupils, nets and models of houses were used instead of solids. The pupils are required to orient the net and say, which of the walls can be seen on front, on right or left.

Explanation and rehearsal:

Which cube can you make-up from this net?

A B C D

Solution: B

Pupils then solve various level tasks in two different variants: correspondence of cubes and houses.

The task are designed to explore, how the pupil deals with "mental"make-up of a net of a solid, it means mental manipulation with the net, how fast and how successfully determines the right solution. After writing down the solution pupil can verify his imagination.

Examples of tasks:

Which house does not origin from the net?

Solution: B

Which house does not origin from the net?

Solution: B and C

Which house is origin from the net? *Which die is origin from the net?*

Solution: B and C

Conclusion

The research confirms that basic school pupils can develop their spatial imagination and that development of spatial imagination can be started already in the early grades. It shows that these tasks are suitable to be used in the mathematical lessons also in cases, when geometry is not the main matter of the lesson.

Is it sought-after, that future teaches of both mathematics and elementary school met frequently similar tasks and methods, which can support development of spatial imagination.

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GEOMETRIC CONSTRUCTIONS AS PROBABILISTIC SPACES CONSTRUCTIONS

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Abstract. The study proposes a visualization of the discrete probabilistic space idea as well as its construction.

1. The discrete probabilistic space and its production

Definition 1-1 [THE PROBABILITY DISTRIBUTION OVER A SET-CLASS] Let $s \in \mathbb{N}_2$ and $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_s\}$. *The distribution of probability over the* Ω *class* is any function $p : \Omega \longrightarrow \mathbb{R}$, which is non-negative and such, that:

$$p(\omega_1 + p(\omega_2 + p(\omega_3) + \cdots + p(\omega_s) = 1.$$

If $p(\omega_1 = p(\omega_2) = p(\omega_3) = \cdots = p(\omega_s) = \frac{1}{s}$, we call the function p *the classical distribution of probability over the* Ω *class.*

Definition 1-2 [THE DISCRETE PROBABILISTIC SPACE] Let Ω be any s-element class, and p be the distribution of probability over that class. Let $\mathcal{Z} = 2^{\Omega}$. We shall consider a function $P : \mathcal{Z} \longrightarrow \mathbb{R}$, where

$$P(A) = \begin{cases} 0, & \text{when } A = \emptyset, \\ p(\omega), & \text{when } A = \{\omega\}, \\ \displaystyle\sum_{\omega \in A} p(\omega), & \text{when } A \text{ is a class of at least two elements.} \end{cases}$$

It is not very difficult to prove, that a trio $(\Omega, \mathcal{Z}, P)$, which evolved from a pair (Ω, p), is a probabilistic space in terms of its axiomatic definition (see [13], p. 124). Such a trio, or – which is the same – pair (Ω, p) we call *the discrete probabilistic space.*

Definition 1-3 [THE CARTHESIAN PRODUCT OF DISCRETE PROBABILISTIC SPACES] Let us assume, that (Ω_1, p_1) and (Ω_2, p_2) are discrete probabilistic spaces, that $\Omega_{1-2} = \Omega_1 \times \Omega_2 = \{(x, y) : x \in \Omega_1 \wedge y \in \Omega_2\}$ and

$$p_{1-2}(x, y) = p_1(x) \cdot p_2(y) \text{ for every } x \in \Omega_1 \text{ and } y \in \Omega_2.$$

We shall call a pair (Ω_{1-2}, p_{1-2}) *the Cartesian product of the* (Ω_1, p_1) *and* (Ω_2, p_2) *discrete probabilistic spaces* and we shall note it as $(\Omega_1, p_1) \times (\Omega_2, p_2)$. The product $(\Omega, p) \times (\Omega, p)$ shall be called *carthesian square* and noted as $(\Omega, p)^2$.

It is easy to prove, that the Cartesian product of discrete probabilistic spaces is a discrete probabilistic space as well.

2. Figures with grids as geometric presentations of discrete probabilistic spaces

Definition 2-1 [A GRID IMPOSED ON A FIGURE WITH A POSITIVE AREA] Let F mean a set of those figures on a plane, which have an area. If $A \in F$, $m_2(A)$ means the area of the A figure, and $int(A)$ means its inside.

Let us assume, that $S = \{1, 2, 3, \ldots, s\}$, where $s \in \mathbb{N}_2$, and that
(i) $F \in \mathcal{F}$ and $0 < m_L(F) < +\infty$,
(ii) $\forall j \in S : [F_j \subset F \wedge F_j \in \mathcal{F} \wedge m_L(F_j) > 0]$,
(iii) $\forall j, k \in S : [j \neq k \Longrightarrow int(F_j) \cap int(F_k) = \emptyset]$,
(iv) $F_1 \cup F_2 \cup \ldots \cup F_s = F$.
The class $\Omega = \{F_j : j \in S\}$ we call *a grid imposed on the figure* F, and each element of that class (each figure F_j) we call the mesh.

The Tangram (see [16] and Fig. 1) was created by imposing on a square a grid of triangular meshes T_1, T_2, T_3, T_4, T_5 a square mesh K and a parallelogram R. Those meshes are called *tans*. If the base square's area is 1, the function p_T, which attributes an area to each tan, is a distribution of probability over the class $\Omega_T = \{T_1, T_2, T_3, T_4, T_5, K, R\}$. So a pair (Ω_T, p_T) is a non-singular discrete probabilistic space. Defining the p_T function is a geometrical task.

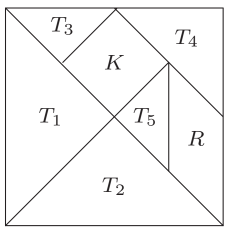


Fig. 1 The Tangram of seven tans

To define a discrete probabilistic space we need to impose on a square that has an area of 1 a grid consisting of figures that have an area.

■ [A PROBABILISTIC SPACE GENERATED BY A GRID IMPOSED ON A FIGURE WITH A POSITIVE AREA] Let a class $\{F_j : j \in S\}$ be a grid imposed on a figure F with a positive area, where $S = \{1, 2, 3, \dots, s\}$ and $s \in \mathbb{N}_2$. Function $p : \{F_j : j \in S\} \longrightarrow \mathbb{R}_0$, where

$$p(F_j) = \frac{m_2(F_j)}{m_2(F)}, \qquad \textbf{(1.0..12)}$$

is a distribution of probability over the class $\Omega = \{F_j : j \in S\}$. A pair (Ω, p) is called *a discrete probabilistic space generated by a grid imposed on a figure F*.

Figure 2 shows three probabilistic spaces generated by grids imposed on three figures having a positive area. To define each of those spaces as a pair (a class and a distribution of probability over that class) means to do a task of counting the areas of certain figures.

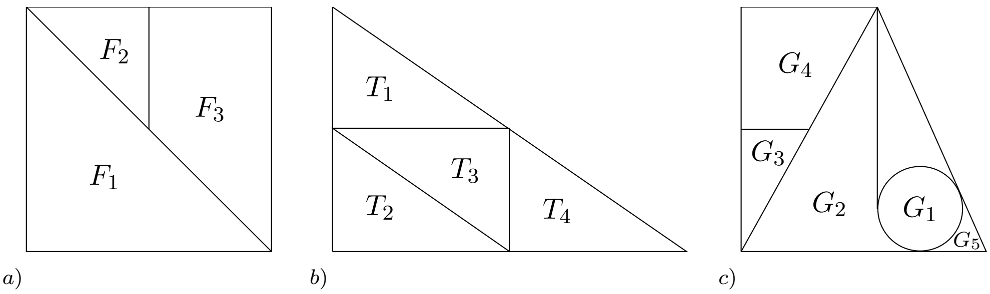


Fig. 2 Figures with finite grids as a presentation of finite probabilistic spaces

The probabilistic space generated by a grid on Figure 2a) is a pair (Ω_a, p_a), where $\Omega_a = \{F_1, F_2, F_3\}$ and $p_a(F_1) = \frac{1}{2}$, $p_a(F_2) = \frac{1}{8}$, $p_a(F_3) = \frac{3}{8}$.

3. Equivalence through splitting versus isomorphism of discrete probabilistic spaces

Definition 3-1 [ISOMORPHIC PROBABILISTIC SPACES] Probabilistic spaces (Ω_1, p_1) and (Ω_2, p_2) are *isomorphic* or *equivalent*, if there is a bijection g from the class Ω_1 to Ω_2 such, that

$$\forall \omega \in \Omega_1 \forall \varpi \in \Omega_2 : [\varpi = g(\omega) \Rightarrow p_2(\varpi) = p_1(\omega)].$$

We say that bijection g *determines isomorphism* and *saves probability.*

Definition 3-2 [FIGURES WITH GRIDS EQUIVALENT THROUGH SPLITTING] We say, that figures F with a grid $S_F = \{F_j : j \in S\}$ and G with a grid

$S_G = \{G_k : k \in S\}$ are equivalent through splitting if there is a bijection g from the class S_F to S_G such, that for every $j \in S$, if $G_j = g(F_j)$ the mesh G_j is an image of the mesh F_j in an isometry that is a rotation, a translation or a combination of both.

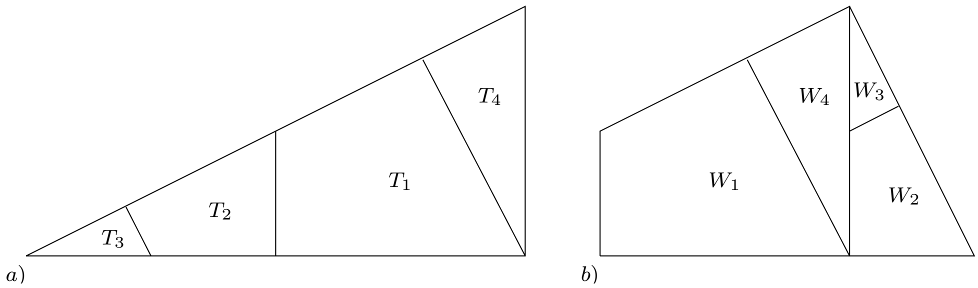


Fig. 3 A square K with a grid and a triangle T with a grid as figures equivalent through splitting

■ A square K with a grid of $\{K_1, K_2, K_3, K_4\}$ and a triangle T with a grid of $\{T_1, T_2, T_3, T_4\}$ from Figure **??** are figures equivalent through splitting. The bijection g is function $g(K_j) = T_j$ for $j = 1, 2, 3, 4$. The proof of that fact is a geometrical task. Its idea may be suggested by certain procedures done on those figures' models.

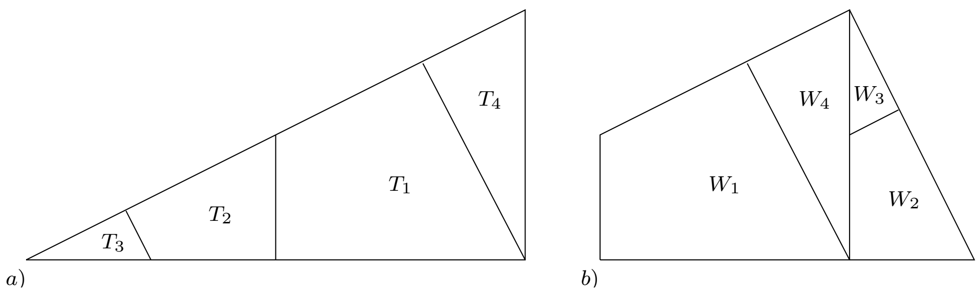


Fig. 4 Figures with grids equivalent through splitting

If two figures with grids are equivalent through splitting, the probabilistic spaces generated by grids imposed on those figures are isomorphic.

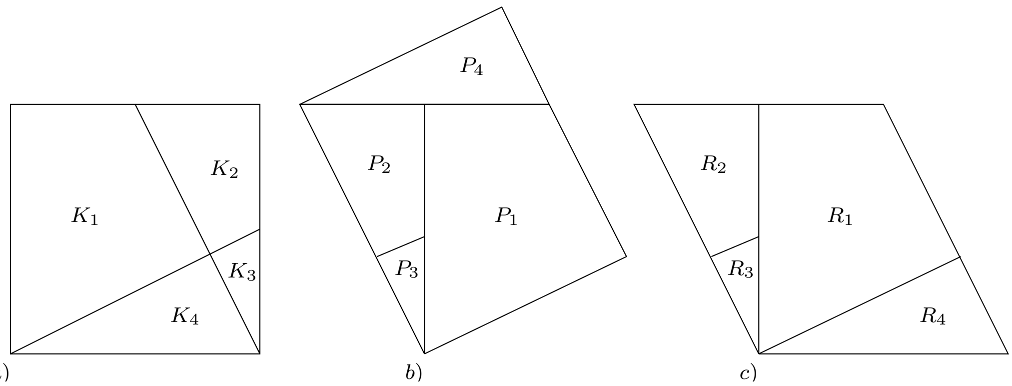


Fig. 5 A square K, a rectangle P and a parallelogram R with grids as figures equivalent through splitting

■ Figure 5 shows three figures with grids, each two of them are equivalent through splitting. Each of the figures generates a discrete probabilistic space. Each two of those spaces are isomorphic. Defining each of them as a pair (Ω, p) is a geometrical task of calculating areas of certain figures.

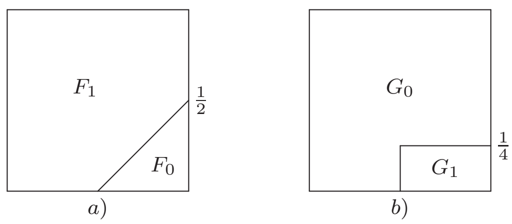


Fig. 6 Two isomorphic probabilistic spaces as geometrical objects

Two probabilistic spaces generated by grids on Figure 6 are isomorphic, but figures with those grids imposed are not equivalent through splitting. The bijection g, which defines the isomorphism is a function from the class $\{F_0, F_1\}$ onto the class $\{G_0, G_1\}$ determined by a formula $g(F_j) = G_{1-}$ for $j = 0, 1$.

4. A random discrete experiment and its probabilistic model – a model as a probabilistic space

□ [A RANDOM DISCRETE EXPERIMENT] A random discrete experiment is such an experiment (real or thought-of), that its effect is random, the set of results is not larger than countable and for each result we can *a priori* state or *a posteriori* estimate its probability (see [4], pp. 16–17 as well as [8], p. 13).

□ [A MODEL OF A RANDOM DISCRETE EXPERIMENT] The set of results of a random experiment has at least two elements and at most countable. A discrete probabilistic space of (Ω, p) is called *a model of a random discrete* *experiment* δ, if Ω is the class of all the possible results of the δ experiment and a function p associates every result the probability of the experiment δ ending with this result.

■ [A MODEL OF A COIN TOSS] Let us code the results of a coin toss with numbers: 0: `heads`, 1: `tails`. A model of this toss is a probabilistic space of (Ω_M, p_M), where

$$\Omega_M = \{0, 1\} \quad \text{and} \quad p_M(0) = p_M(1) = \tfrac{1}{2}.$$

■ [A MODEL OF THROWING A DICE] Let us code the results of throwing a dice with a number of stitches that show. A model of that throw is a probabilistic space of (Ω_K, p_K), where

$$\Omega_K = \{0, 1, 2, 3, 4, 5, 6\} \quad \text{and} \quad p_K(j) = \tfrac{1}{6} \text{ for } j = 1, 2, 3, 4, 5, 6.$$

A classical probabilistic space is a model of a random experiment only in case, when all the results of the experiment are equally possible.

5. A tangram of a discrete probabilistic space as its special geometrical presentation.

The probability associated to a result of a random experiment is, in a way, a kind of its measure, which can be interpreted (and visualized) as a figure's area.

□ [A TANGRAM OF A PROBABILISTIC SPACE] Let (Ω, p) be a non-singular discrete probabilistic space (so p should be a function with positive values). We interpret the elements of the Ω class as mashes of a grid imposed on a square with an area of 1 in such a way, that for every $\omega \in \Omega$ the number $p(\omega)$ is the area of the ω mash. The square with such a grid we shall call the tangram of the (Ω, p) probabilistic space.

If the (Ω, p) pair is a probabilistic space generated by a grid imposed on a square having the area of 1, that square with a grid is simultaneously the tangram of that space.

Every non-singular discrete probabilistic space (Ω, p) can be shown as its tangram. It is a measured (and so geometrical) presentation of that space (see [14]). In the same time it is a didactic way that allows visualization of such abstract probabilistic issues as an event and its probability.

■ [A TANGRAM OF A $K_{1\to 8}$ DICE THROW MODEL] The result of throwing a $K_{1\to 8}$ dice, which has numbers from 1 to 8 on its sides (its schedule is shown in Figure 7), is the number showing on the upper side after the dice falls.

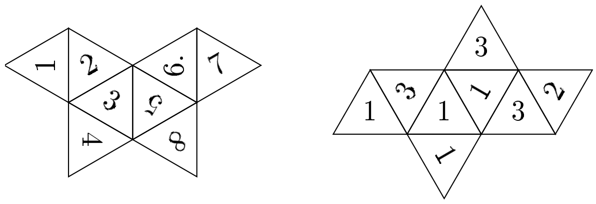


Fig. 7 Schedules of two eight-sided dice

The model of a $K_{1\to 8}$ dice throw is a probabilistic space (Ω_8, p_8), where

$$(\Omega_8 = \{1,2,3,4,5,6,7,8\} \quad \text{and} \quad p_8(j) = \tfrac{1}{8} \text{ for } j \in \Omega_8.$$

Three tangrams of that classical probabilistic space (Ω_8, p_8) are shown in Figure 8.

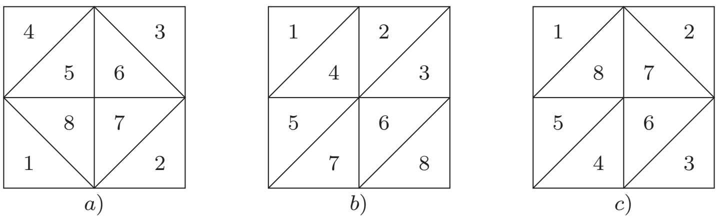


Fig. 8 Tangrams of a $K_{1\to 8}$ dice throw model

The model of throwing a $K_{11112333}$ dice, the schedule of which shows Figure 8*b*, is a probabilistic space of (Ω_8^*, p_8^*), where

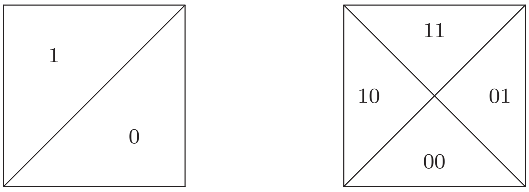
$$\Omega_8^* = \{1,2,3\} \quad \text{and} \quad p_8^*(1) = \tfrac{4}{8},\ p_8^*(2) = \tfrac{1}{8} \quad \text{and} \quad p_8^*(3) = \tfrac{3}{8}.$$

The probabilistic spaces (Ω_8^*, p_8^*) and (Ω_a, p_a) (the space generated by the grid shown in Fig. 8*a*) are isomorphic. The bijection that states the isomorphism is a function $g : \Omega_a \longrightarrow \Omega_8^*$, where $g(F_j) = j$ for $j = 1, 2, 3$.

Constructing of a tangram of a probabilistic space generated by a grid imposed on a positive-area figure may be accompanied by actual manipulations on models of the figure and the grid. Those real manipulations (similar to doing a jigsaw puzzle) may suggest all the imagined ones as parts of constructing a tangram as a mathematical object.

6. Models of random experiments with a coin and their tangrams

■ A tangram of a coin toss, i.e. an (Ω_M, p_M) probabilistic space, is shown in Figure 9a.



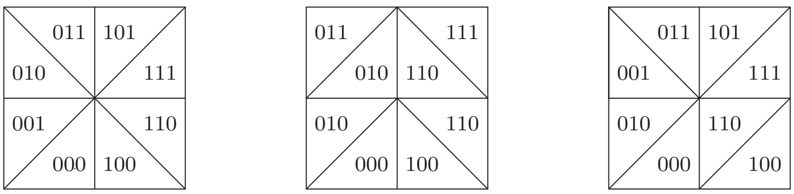
a) *b)*

Fig. 9 Tangrams of a single and double coin toss models

■ [A MODEL OF AN n-TUPLE COIN TOSS] The result of an n-tuple coin toss is an n-element variation of a $\{0,1\}$ class. Its j element is the result of the j^{th} toss. All the results of this experiment are equally probable (because in each single toss heads and tails are equally possible to show) and there are 2^n of them, so the model of an n-tuple coin toss is a probabilistic space of (Ω_{nM}, p_{nM}), where

$$\Omega_{nM} = \{0,1\}^n \quad \text{and} \quad p_{nM}(\omega) = \tfrac{1}{2^n} \text{ for every } \omega \in \{0,1\}^n.$$

Note, that $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$, so $(\Omega_{nM}, p_{nM}) = (\Omega_M, p_M)^n$. The probabilistic model of an $n-$tuple coin toss is an n^{th} carthesian power of a single coin toss. A tangram of a double coin toss is shown in Figure 9b. Three tangrams of a triple coin toss are shown in Figure 10. Each of them is a square with a grid, that is equivalent through splitting to each of the tangrams in Figure 8. Apparently, we can simulate a triple coin toss with a single throw of a $K_{1\to 8}$ dice.



a) *b)* *c)*

Fig. 10 Tangrams of a triple coin toss

Let g be a function from the $\{1,2,3,4,5,6,7,8\}$ class (that is the set of possible results of a $K_{1\to 8}$ dice toss) onto the $\{0,1\}^3$ class (i.e. a set of results of a triple coin toss) that is stated as follows:

k :	1	2	3	4	5	6	7	8
g(k):	001	010	011	100	101	110	111	000

This function g defines the isomorphism of two probabilistic spaces: a model of a $K_{1\to 8}$ dice toss and a model of a triple coin toss. At the same time it is a „dictionary" to translate the result of a $K_{1\to 8}$ dice toss to the result of a triple coin toss. If the result of a $K_{1\to 8}$ dice toss is k, the result of a triple coin toss is $g(k)$.

A tangram of a probabilistic space, which is simultaneously a model of a multi-step experiment may be created in stages by dividing a unitary square. Let us assume, that vertical dividing lines mean odd stages of the experiment and horizontal dividing lines mean even stages. With this assumption the tangrams of single, double and triple coin toss look as it is shown in Figure 11.

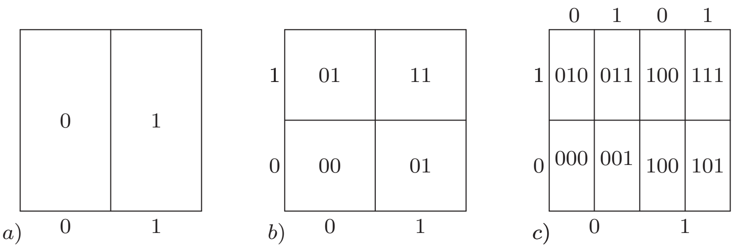


Fig. 11 Tangrams of single, double and triple coin toss

7. A tangram of a Cartesian product of probabilistic spaces – the probability of an event as an area of a figure

Figure 12 shows schedules of three dice: dodecahedronal K_{12}, hexahedronal K_6 and octahedronalK_8 .

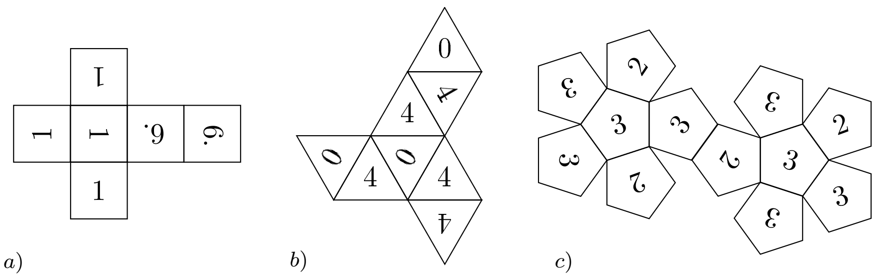


Fig. 12 The schedules of the three dice

A throw of each of them is a random experiment. Its result is a number that shows on the upper side of a dice when it falls. The K_6, K_{12} and K_8 dice are props in a lot game. There are two players, each has a dice. The one who gets a bigger number after this throw wins the game.

We will prove that in such a game the K_{12} dice gives a player a better chance to win than a K_6dice gives to his rival. The K_{12} dice is *better* than the K_6. We note this fact with a symbol $K_{12} \gg K_6$. We can see the arguments in Figure 13. It is about geometrical means of organizing the stages of counting.

Let us consider three probabilistic spaces:
— (Ω_{6-8}, p_{6-8}), which is a model of throwing the K_6 and the K_8 dice,
— $(\Omega_{8-12}, p_{8-12})$, which is a model of throwing the $K_8$and the K_{12} dice,
— $(\Omega_{12-6}, p_{12-6})$, which is a model of throwing the K_{12} and the K_6 dice.
Figure 13 shows a protocol of constructing the tangrams of those three probabilistic spaces. If the (Ω_j, p_j) pair models the throw of the K_j dice and the (Ω_{j-k}, p_{j-k}) pair models the throw of two dice: K_j and the K_l $(j = 6, 8, 12)$, then $(\Omega_{j-k}, p_{j-k}) = (\Omega_j, p_j) \times (\Omega_l, p_l)$.

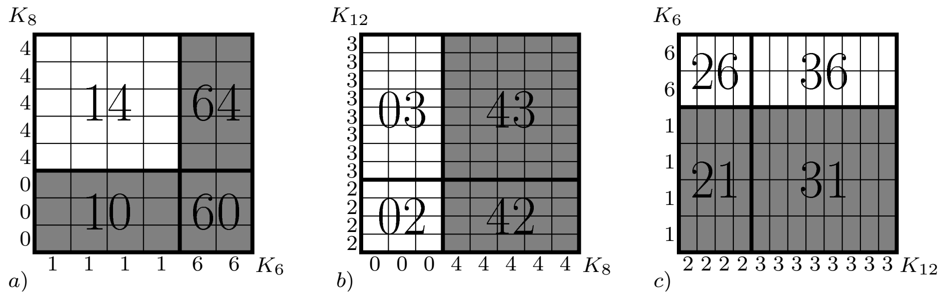


Fig. 13

Let us assume that in the game described above one of the players throws the K_6 dice, and his rival throws the K_8 dice. The blackened part (i.e. the set of blackened meshes of the tangram) in Figure 13*a* shows the event

$A = \{$*the* K_6 *dice shows a bigger number then the* K_8 *dice*$\}$.

The probability of this event happening is the area of the blackened part.

The white part (the set of the white meshes of the tangram) in Figure 13*a* shows the event

$B = \{$*the* K_8 *dice shows a bigger number then the* K_6 *dice*$\}$.

The area of the blackened part is bigger then the area of the white one, so $P(A) > P(B)$, that is the player who throws the K_6dice has in this game a better chance to win than his rival having the K_8 dice. And so it is $K_6 \gg K_8$.

Figure 13*b* shows, that $K_8 \gg K_{12}$ and Figure 13*c* shows that $K_{12} \gg K_6$. In case of the K_6, K_8 and K_{12} dice

$$K_6 \gg K_8 \quad \text{and} \quad K_8 \gg K_{12} \quad \text{and} \quad K_{12} \gg K_6.$$

So the relation of $\gg$ is not transitive in the set of $\{K_6, K_8, K_{12}\}$ dice. Among those dice there is no „best" one. For each of them there is a „better" one between the two others.

In this reasoning a tangram represents a probabilistic space, a set of meshes – some geometrical figure – is an event, and the area represents the probability of the event happening.

Intransitivity of the $\gg$ relation in the set of $\{K_6, K_8, K_{12}\}$ dice is a kind of a paradox. However, this fact may be easily transferred to the basis of reality. Let us assume, that the player was offered to choose his dice first. His rival would choose his dice from the two left. The intransitivity of the $\gg$ relation causes the decision of taking advantage of the right of priority is not a rational one.

Epilogue

This work shows:
– how to inspire translation of mathematical contents from the symbolic to iconic language and vice-versa;
– how to include geometrical means into stochastic reasoning;
– how to reduce counting probabilities to counting areas of figures.

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TEACHER'S STUDIES STUDENTS DIFFICULTIES CONCERNING THE GENERALIZATION OF THE CONCEPT OF THE RIEMANN INTEGRAL

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Abstract. One of the most fundamental concepts of the mathematical analysis is the Riemann integral. For a teacher of mathematics the concept of the integral is important because of the connections with the Jordan measure which is considered in the elementary geometry. Besides the Riemann integral the course of mathematical analysis includes multiple integrals, line integrals and surface integrals. In this paper we present the results of our research concerning the difficulties of students in noticing mutual connections between different kinds of integrals.

1. Introduction

One of the most fundamental concepts of mathematical analysis is the Riemann integral. It has found many important applications in both mathematics and related sciences, for example physics. In the syllabus of mathematical analysis that has been effective in the Pedagogical University of Cracow during recent years the integral calculus comes in the first and third years of studies, with the reservation that for the first year it involves integration of functions of one variable and for the third year - integration of functions of several variables. The third-year curriculum additionally includes multiple, oriented and non-oriented line and surface integrals as well as the concept of an integral with respect to a measure, and in particular the concept of the Lebesgue measure and integral. The order of familiarizing students with these concepts can be different. Some lecturers follow the differential calculus of functions of several variables with the theory of the Lebesgue measure and integral and only then do they proceed to multiple, line and surface integrals. With this approach it is possible from the general theory of an integral with respect to a measure to specify properties of this concept in a specific model, which is the space $\mathbb{R}^n$ with the Lebesgue measure therein specified.

The modified teaching *standards for fields of study*, which were published by the Central Council of Higher Education on its website on 19^{th} February 2007 (www.rgsw.edu.pl/files/active/0/matematyka20070210), provide with reference to Mathematics that the curriculum of first degree studies should cover the differential and integral calculus of functions of one variable and several variables. According to the publication such generalizations as an integral with respect to a measure are moved to the curriculum of second degree studies.

The concept of the Riemann integral, referred to at the beginning of the article, is an important item in the curriculum of a teacher's field of study, since what matters for a teacher of mathematics is its connection with the concept of the Jordan measure, which is covered in school at a predefinition level. Please note that, if an integrand is nonnegative in the interval P, then its integral on this interval is equal to the area of a *curvilinear trapezium* formed by the graph of the function f, the x-axis and its perpendicular lines which intersect the interval boundaries, and so it is the Jordan measure of a certain area.

While defining the above-mentioned integrals it is the so-called *integration process* that comes to the foreground. It consists in considering a normal sequence of partitions of a set (a segment, a curve arc, a regular area) in $\mathbb{R}^n$ that an integrand is specified on, which is subsequently followed by constructing sequences of lower sums and sequences of upper sums, or alternatively a sequence of approximate sums*. The common limit of sequences of lower and upper sums, as long as it does not depend on a normal sequence of partitions, is called a (definite, line, surface) integral of a given function on a given set. Such integral can also be defined as a limit of a sequence of approximate sums, as long as it does not depend on a normal sequence of partitions and the way of selecting intermediate points from particular domain subsets.

In order to define an oriented line and surface integral, orientation on a curve or a surface needs to be specified first and followed by applying the integration process for the properly determined integrands. For this reason the concepts in question rank among those analogous to the concepts of the relevant non-oriented integrals and, therefore, are not the subject matter of this paper.

*A lower (upper) sum is defined as a sum of products of function minima (maxima) for each subset that the function domain has been divided into and measures of relevant subsets. An approximate sum is defined as a sum of products of function values in any point of each subset that the function domain has been divided into and measures of relevant subsets.

Long-standing observations of students during their classes on mathematical analysis point to the fact that the above-referenced subject matter is not popular with them. The reasons for it can be the following:

- too high complexity of calculations needed to perform the integration process;
- areas of integrals' application that are remote from the interests of students of mathematics;
- not properly shaped spatial imagination of students;
- failure to perceive analogies between different types of integrals and lacking skills of generalizing these concepts.

In order to find an answer the question relating to students' difficulties in generalizing the concept of a definite integral, a questionnaire was developed and conducted among third-year students of mathematics. The answers to the questionnaire questions as provided by research subjects were subsequently analyzed. Additionally, the attitude and behaviours of the students included in the research were observed during their classes on mathematical analysis.

2. On certain generalizations of the Riemann integral

The ability to generalize and perceive generalizations is an important component of mathematical activity, essential to study mathematics (Krygowska, 1986). It is discussed in didactics of mathematics in the context of concepts and theorems. While developing new concepts or theorems there are two ways that can be followed, which H. Siwek defines as follows:

- *from detailed examples to general concepts and thus to formulating theorems and proving them on a high level of generality,*
- *from general concepts and theorems of a given theory to examples and counterexamples which reflect definitions and to detailed theorem cases* (Siwek, 2005, p. 290).

These could be briefly described as *bottom up* and *top down* approaches. This paper regards the first type and, due to its size, focuses on the activity of generalizing concepts. Generalizing theorems will be the subject matter of the next article. In the context of developing concepts the *bottom up* approach can be organized in two ways. A. Z. Krygowska describes them in the following way: *Generalization of mathematical concepts by a student himself (...) can* *be developed in such a way that the student either discovers a superiority relationship between two concepts he is already familiar with or consciously and deliberately constructs a concept superior to the one he is already familiar with* (Krygowska, 1977, p. 93). In didactics the former is known as generalization *through recognition*, while the latter is called generalization *through construction*. Both types have been presented during the lectures on mathematical analysis.

Multiple, non-oriented line and surface integrals are determined similarly to the Riemann integral of a function of one variable on an interval $[a, b]$. In fact the diversity of these integrals originates from increasing the dimension of the space where the integrand domain occurs. Please note that a function of two variables has a subset of the set $\mathbb{R}^2$ as its domain. It this is a rectangle, i.e. the Cartesian product of two intervals contained in the $\mathbb{R}$ or a plane curve understood as a homeomorphic interval image, then, by applying the *integration process*, referenced in paragraph 1, a double integral on the rectangle or a non-oriented line integral is obtained. This is an example of generalization *by recognition* as students learn all these concepts independently from one another and then their attention is called to the superiority relationship between these concepts, which is performed by analyzing the space dimension and the manner of constructing concepts.

If a function f has a regular set A as its domain, i.e. a set whose boundary consists of a finite number of curves of $y = y(x)$ or $x = x(y)$, then, in order to define a double integral of the function f on the set A, the following construction is performed:

- the set A is inscribed into the rectangle P whose sides are parallel to the axes of the coordinate system,
- a new function g is defined which is equal to the function f on the set A and takes zero on the set $P \setminus A$.

In this case a double integral over the set A of the function f is called a double integral of the function g over the rectangle P. This can be taken an example of generalization *through construction* as the approach which leads to a new concept (here to the concept of a double integral of the function f over the regular set A) is based on the previously known concept, i.e. the concept of a double integral over a rectangle. The above-described construction is performed on purpose so as to refer to a situation which is already known to the students.

The theory of an integral in the space $\mathbb{R}^3$ considers functions which can have plane areas, curves or regular surface sheets as their domains. By applying the above-mentioned *integration process* independently for each of these functions one can reach the concept of a triple non-oriented line or surface integral. In this case again we can talk about generalization *through recognition*.

These different ways of generalization were brought to students' attention during the lectures and classes. The aim of these didactic techniques was to draw students' attention to the connections between relevant concepts, to point out which concept is a generalization of which one. For this reason they were meant to facilitate overcoming difficulties related to the concepts of particular integrals.

3. Research description and result analysis

The research was conducted in the Pedagogical University of Cracow in the academic year 2005/2006. It included 30 third-year students of mathematics.

In order to work out how students perceive different types of integrals and connections between them they were asked to express their own opinion on definitions and properties of integrals, to indicate these fragments of their definitions that they find difficult or to make a list of applications of relevant integrals (including the content of the tasks which included such integrals).

While analyzing the research results it became evident that the students found the non-oriented line integral and multiple integrals as the easiest, whereas the oriented and non-oriented surface integrals as the most difficult. The reason for might be the fact that the non-oriented line integral is a generalization of the Riemann integral, since, if a plane curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ is contained in either axis of the coordinate system, then a non-oriented line integral over such curve is reduced to a definite integral of a function of one variable. What arises from this is that the students might have noticed this generalization. Another reason why the students recognized the non-oriented line integral as the easiest might be not too high complexity of required calculations, which actually boil down to calculating only one single integral.

The students deemed multiple integrals as more difficult than the non-oriented line integral. Some of them wrote that the fundamental difficulty in applying these integrals for different geometrical problems is the lack of a properly shaped spatial imagination. This is proved correct by observations made during the classes, which show that a great number of students could not imagine surfaces denoted by such equations as $z = x^2 + y^2$ or $z = x^2 - y^2$. Their difficulty in dealing with such tasks might originate from algebraization of geometric problems. There is no doubt, however, that problems of spatial imagination are a significant barrier that needs to be overcome while solving tasks.

The students found surface integrals as the most difficult concept. As pointed out in paragraph 2, the non-oriented surface integral is a generalization of the multiple integral specified in a plane area in the case of a surface sheet in $\mathbb{R}^3$, but this connection was not always perceived.

The questionnaire also asked the students to specify which parts of the definitions of the above-mentioned integrals caused them special difficulty. The analysis of their responses points to a few reasons:

a) Not understanding a normal sequence of partitions of a relevant set as well as the manner of selecting intermediary points and constructing a sequence of approximate sums;

b) Not understanding concepts crucial to the definition of an integral; concepts of a normal area and a regular surface sheet were mentioned here.

Please note that the first type of difficulties was pinpointed also by the students themselves when the understanding of the concept of a single integral was being researched (Powązka, 2007). The second type of difficulties probably results from an inability to visualize patterns describing boundaries of areas on which integration is performed. The research showed that 20 respondents failed to notice connections between relevant types of integrals. With regard to the non-oriented line integral only three research subjects noticed that it is a generalization of a single integral and with regard to the surface integral only two persons indicated that it is a generalization of a double integral. Worth mentioning is also the fact that one of the research subjects found a non-oriented line integral to be a generalization of an oriented line integral. Two other students formed a similar conclusion with regard to surface integrals.

Although this article deals with problems related to concepts and issues concerning application of theorems will be discussed in the next article, it is worthwhile hinting that difficulties in calculating integrals (and therefore application of relevant theorems) have a negative effect on understanding them. For example, as the analysis of the research data reveals, the difficulty in calculating multiple integrals may lie in the application of the Fubini's theorem and the necessity to calculate at least two single integrals. This task is more effort-consuming than calculations required for a non-oriented line integral. Sometimes it also becomes necessary to apply the change of variable theorem in a multiple integral, which admittedly simplifies calculations but requires a proper transformation of the integration area. However, finding the correct transformation can be difficult for a student whose spatial imagination is not properly shaped. Yet, it does not seem true to state that the complication of calculations significantly hinders the perception of connections between concepts.

4. Final notes

The research and observation of the students allow constructing some hypotheses concerning the students' perception of the concept of a single integral in the context of generalizing. At the same time, the hypotheses express difficulties which the research students had while applying the indicated concept. The research results seem to point to the following:

[1] The students identified the concept of a non-oriented line integral with the Riemann integral; they did so most probably because of the similarity of symbols used for both integrals, but they did not always use these denotations correctly.

[2] The research subjects associated non-oriented surface integrals with the symbol of a double integral (this was observed in the part of the research on theorems).

[3] The respondents showed poor spatial imagination, which caused significant difficulty in describing the integration area.

[4] It seems that generalization through recognition was easier for the students than generation through construction.

In the context of the last hypothesis it is worth to quote Z. Krygowska: "It is not difficult (...) to notice that generalization *through recognition* is something psychologically different from generalization *through construction*"(Krygowska, 1977, p. 94). It would be worthwhile performing further research in order to determine whether the last hypothesis is really correct in the context of the concept of an integral. Moreover, interesting would be researching whether generalization *through recognition* is a more complicated process for students than generalization *though construction* also in the context of other mathematical concepts.

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EXPERIENCE WITH TEACHING COURSE OF "FUN MATHEMATICS"

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Abstract. The programs of study offered by the Faculty of Education of the University of Prešov contain in its undergraduate level also courses under the category of Recommended Optional Courses. The author in the paper presents her experience with teaching Fun Mathematics. The content of the course is adjusted for the Moodle software environment to be utilised in e-learning.

1. Introduction

One of the aims of mathematical training of undergraduate students of the Faculty of Education of The University of Prešov is in developing their positive attitude towards mathematics. It seems urgent due to the fact that in academic year 2005/2006 only 25% of freshmen students in the Faculty of Education came from grammar schools (B. Tomková [4]). Moreover, most of them have rather negative attitude to the discipline.
When designing the contents of mathematical courses our colleagues at the department strived to respect the profile of a prospective graduate - holder of Bachelor degree in one of the following Programs of Study:
Preschool and Elementary Education;
Preschool and Elementary Education of Socially Disadvantaged Groups (Teacher's Assistant);
Preschool and Elementary Education of Psycho-Socially Disabled;
Education of Mentally Handicapped.
Graduate of the above programs of study can be employed as:

- Educator in pre-school facilities,
- Educator in school clubs and leisure time centres,

- Teacher's assistant,
- Educator in special education facilities.

Characteristics of the mathematical courses listed in particular programs of study are given by Scholtzová [3].

2. The Course of "Fun Mathematics"

A course titled *Fun Mathematics* is listed under the block of recommended optional courses in all of four undergraduate programs of study. The course is offered in the second year of study in both summer and winter semester. Its goals include mastering the mathematical tasks which belong to the so called recreational mathematics within the scope of primary education practised especially during out-of school activities. The course follows the pattern of dividing contents into the themes. The content is thus composed of eleven thematic areas:

[1] Mathematical Games.
[2] Algebragrams.
[3] Puzzles. Cryptography.
[4] Mathematical Crossword. Painted Crosswords.
[5] Tasks in Grid.
[6] One-Line Drawings.
[7] Puzzles with Matches (Sticks).
[8] Tangram. Pentamino
[9] Magic Squares. Sudoku.
[10] Maze. Labyrinth
[11] Mathematical Competitions.

The structure of contents varies in particular themes. Each chapter usually starts with an introductory text in order to give student an overview of the problem area. It is followed by the characteristic features of particular types of activities; some of them are complemented by notes concerning history of developing the given tasks. Substantial part of each theme (chapter) is completed by references to relevant literature or other sources in the form of links to www pages which can provide students with task samples. Students thus are able to work independently in selecting and solving tasks of recreational mathematics up to their aptitude. The course structure designed in Moodle environment enables students treat individual themes independently, satisfying their interests and needs. This way a room for individualised time management of studying is opened - either through retrieving information from web pages or studying relevant literature.

Elaboration of seminar work on any of the above themes is a requirement for successful completion of the course. The seminar assignment should contain the collection of suggestions for task that could be used in extracurricular activities with elementary stage pupils.

The course of *Fun Mathematics* was taught for the first time in academic year 2006/2007. 15 full-time students and 44 part-time students enrolled for it in winter semester while the number of students enrolled in summer semester increased at 29 and 58 respectively.

Students' feedback regarding both the contents of the course and its modification for Moodle environment was positive in most cases. For illustration we present some of responses indicated in students' written reflection on the taught course.

- *I think the course is meaningful as we have obtained adequate inspiration for practice.*
- *I personally liked the course very much. The themes in Moodle are well presented enabling quick access to necessary information. I enjoyed the lessons much, got acquainted with many new fun tasks while learning mathematics from yet another side. I learned how to solve many tasks which seemed troublesome for me in the past.*
- *I learned about new logic games and puzzles which can be presented to children. Mathematics is not only about solving assignments but also a game.*
- *Recreational and fun course - something I did not expect a Mathematic can be.*
- *I would recommend this course to everybody.*
- *I think that the course fulfilled the expectation of "fun" mathematics since it was not about academic but really enjoying and fun mathematics. The Moodle environment is well arranged enabling everyone to find what is sought for. I assured myself that mathematics is no only boring calculations.*
- *I met with tasks which I had not been able to solve, yet thanks to the course I learned something new. The themes were aptly transformed to Moodle.*

3. Conclusion

Presented views prove that the course of *Fun Mathematics* has partially contributed to an increase in undergraduate students' interest in mathematics. It has been shown that "for the purpose of mathematical training of prospective elementarist it is appropriate to combine traditional forms with electronic courses accessible through the net" (M. Mokriš [1]).
uggested themes which are accessible in the electronic format can, apart from university study, also be utilised in working with pupils after lessons in school club or mathematical circle or leisure time centre. Some of the themes taught in the course are accessible on the web page designed to addresses the issues of mathematical education in primary stage of education [6]. There are also suggestions for methods of working with gifted children in mathematical circles.

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PRESENTATION SOFTWARE AND ITS USE IN TEACHING MATHEMATICS

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Abstract. The contribution is focused on different possibilities of using presentation software in teaching mathematics. Interesting topics which students presented through seminars are offered for muse in contribution. Contribution is further supplemented by demonstrations of students' solutions of concrete problems which were the topics of seminars from didactics of mathematics, with a view to various methods problem solving.

1. Introduction

Presentation software makes it possible to fit becomingly pupils to the education and fully apply constructive approach not only of teacher, but especially of pupils.

Constructivism is able to appear in many directions. Next enumeration, which we cannot consider in no case as unchanging and entire, open up various directions orientation to usage presentation software:

- personal generation problemsitem
- reformulation problems
- search different methods solving engaged problem
- formulation of partial problems, which lead in finale to solution of engaged problem
- analysis mistaken solution of pupils
- processing concrete thematic units, interpretation new subject matter

Usage presentation software makes it possible to concentrate in bigger measure on individual work with pupils. It makes possible to teacher to fit connection of different thematic units, because teacher is able composedly get and think out and prepare testing exercises, usage motivational materials for interpretation new subject matter and so on. In best case it is possible to use this technology in connection with interactive board, which opens up other options for usage other various doctrinal techniques.

2. Presentation software

Usage different doctrinal technology makes it possible fully to apply constructive approach in teaching. Therefore the seminar work which concerned on preparation lesson was engaged to students of teaching. Seminar work had to be processed with use of power point and was presented in the end of students' period before others students.

Students at the same time used presentation software in many directions:

- motivation to new subject matter
- entry new subject matter
- mathematical limbering - up
- written verification - tests
- presentation different methods solving of given problem - lucidity, comparison

Usage of presentation software further shows profitable sideline e.g. for these reasons:

- lucidity entry
- clear visibility - fall off problems with wet table, with continuous clearing the black board
- facilitation bigger particular access to pupils - communication with pupils
- clear and well - considered entry to the exercise book
- running detection text
- possibility setting timing or hand operating according to needs pupils and teachers
- bigger space on table
- possibility return to previous societies
- refilling information - running information communication
- connection with authentic scenes - in - fillings - short video, photographs,
- preservation attention pupils

3. Exhibits students' subjects

In the following parts we set up exhibits students' subjects for usage presentation software

3.1. Mathematical limbering - up

Student dichotomized the given problem into the partial parts. Pupils have ourselves find way, how to reckon all triangles. After symposium above solution it is shown them correct solution. Solution isn't submitted all of a sudden, but is offered in parts, how follows from ilustrations of single coloured figures, that are detected step by step by teacher.

Solution:

In the end of discussion whole solution is exposed to the pupils. Colored resolution and suitable signs are used.

3.2. Demonstration of different methods solving

In this case student/teacher offers various methods solving, let us say he can summarize all topics. Especially usage of presentation software is in case of graphic problem solution, when emphasis on accuracy in construction of graph is given. Graph itself is escorted by commentary and accurate record (remark), which then pupils have in their exercise-book. For design graph in following demonstration student availed the program Derive 6.

Setting:

From two seats of distant 784 km went out at the same time against to each other two cars. They met after eight hour. Calculate out the speed of cars, if you know, that rate of the 1st car is about 10 km/h greater than the speed of second one?

Graph solution

1st car: speed ...	x [km/h]
2nd car: speed ...	x + 10 [km/h]
Distance ...	s_1= 8x [km]
Distance ...	s_2 = 8(x + 10) [km]
We know:	$s_1 + s_2$ = 784 km. So s_2 = 784 -s_1.
Then	784 - s_1 = 8(x + 10).

Denote:

$$f:\ s_1 = 8x, \quad g:\ 784 - s_1 = 8(x + 10).$$

We will search solution on intersection graphs of these functions.

The graphs of functions $f:\ y = 8x$ and $g:\ 784 - y = 8(x + 10)$.

By measuring we can find out, that graphs of functions f and g intersect in point $x = 44$. The speed of the 1st car is then 44 km/h; the speed of the 2nd one is 54 km/h.

3.3. Demonstration progress construction of constructional exercises

Upon this instance student showed progress construction triangle. Individual steps are operated by hand. On top teacher can becomingly connect calculation of lengths of sides of triangle and subsequently to show construction of triangle. In this case teacher has more time for individual work with pupils.

There are no problems with complicated manipulation with drawing helps. Construction is guided absolutely exactly and no fears of possible inaccuracies are there. Next pictures are demonstrations of calculation and construction.

Remark:

Presentation makes it possible to lead construction in consecutive steps place usage drawing needs on table.

3.4. Motivation, interpretation of new subject matter

In this part we demonstrate, how students align with usage power point to interpretation new subject matter.

- Presentation software makes it possible to use directly pictures from textbook or by other sources to explain given problems.

- At loading new subject matter it is possible to use motivate pictures and information

- Real problems, using the different subjects matter

3.5. Practice subject matter, testing pupils

Presentation software makes it possible to in the appropriate manner get job file to practice subject matter. Pupils have not e.g. problem with finding out an exercise in textbook according to hints teaching, with recording text exercise (setting) according to dictation teacher etc. Exercise may be in addition detected step by step hand operating by teacher or timings presentation in case testing.

4. Conclusion

In contribution only some topics which were used by students at processing concrete subjects were pictorial. Presentation software makes it possible to teacher to apply constructive approach in teaching. It is possible in appropriate way to compare various method solving. It is possible to present new subject matter, to use it also to practice subject matter or testing pupils. At usage this method teacher has at the same time bigger opportunity for individual work with pupils. Therefore it is very useful to join students in generation fit power point presentation through didactical and selective seminars.

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THE ROLE OF INTERACTION BETWEEN STUDENTS IN THE PROCESS OF DISCOVERING THE REGULARITY

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Abstract In teaching mathematics, interactions between the teacher and the student and among students play a vital role. Through making students formulate and defend their points of view we develop in them their self-control. Thanks to it during solving problem a child is more responsible and conscious of what s/he does. Necessity of verbalization of executing activities and explanation of using procedures show that pupils are able to notice new things. The verbalization forces to look at the own work from a different perspective. In this paper I present a part of my research concerning discovering the regularity by 9-years old children. In this research I focused on mental process and interaction between the students.

1. Introduction

Discovering and perceiving regularity by students is the important problem present in the world trends in teaching mathematics. In many countries, teaching mathematics is closely connected with the rhythm and the regularity. We can find references (for example [3], [7]) to the description of research concerning discovering and generalization noticed rules.

Searching regularity is an extremely effective method during solving mathematical problems; it is a strategy for solving tasks. As E. Swoboda says in [5]: " ... perceiving the regularity is a desirable skill. Activities during which the child has to perceive the regularity and act according to the rule are stimulating its mental development. These activities are the basis of mathematical thinking for each level of mathematical competence." One of the theories which says about development of mathematical knowledge based on discovering the regularity is TGM theory described by M. Hejný in [1], [2].

The child learns/develops its mathematical knowledge through building its own cognitive structures, a web of interrelationships, mental "maps". Accumulated experience enables to create so-called a data set used by a child to build up its mathematical knowledge. The essential factors which help a child to develop its mathematical knowledge are interactions with its environment, particularly during the teaching-learning process (e.g. during mathematics classes). It is present during teacher-student interaction and student-student interaction, as well. The best way to activate these interactions is group work in cooperating teams.

Reflection is very important to appear. Reflection about our experience is a perfect starting point of understanding the world. Everyone creates his own 'rules' and mental models which we try to apply in order to understand and use our experience of mastering the knowledge of our environment. The reflection appears when we have to manifest our ideas. Expressing our thoughts we look for the appropriate form of words. Reflection does not appear automatically among 7-11 children. Therefore a conversation during the cooperation with students is an opportunity to recognize their mental processes while solving their tasks.

2. The aim of the research

I have been dealing with perceiving regularities and applying discovered rules by students for some time. Presented investigation is the part of the series of research concerning perceiving regularity by students on different levels.

The aim of this research was getting the answers for the following questions:

- Will 9 years old students be able to perceive mathematical regularity and if yes – what is their thinking process about solving the task in which they have to discover and use the mathematical regularity?
- Will pupils be able to collaborate during solving the task?
- To what degree will the common work have an effect on the way of solving the task and discovering the regularity and using them in the task?

Fig. 1. John's puzzles – first five figures.

3. Description of the research

The research was carried out in May 2006. Six 9 years old pupils (5 girls and one boy) from the third grade of primary school took part in this research. The research was carried out during the additional classes and participation was voluntary. Pupils worked in pairs. Each group received two sheets with the task - one for each pupil. Pupils worked for the whole lesson (45 minutes). Pupils' work was being videotaped. The teacher's role was limited. He answered the questions and monitored students' work. In case of student's problems with the task the teacher asked an appropriate directing questions.

The task was as following:

John creates figures from white and black circles. He has already created 5 figures. How many circles of each color has he used for each figure? How many circles will he use to create the next one? Write your results in the table.

No of figures	**No of black circles**	**No of white circles**
1		
2		
3		
4		
5		
6		
7		

4. Results

In this paper I want to present the fragment of work which belongs to one of pupils' pair – Ala and Piotrek. At the beginning Ala and Piotrek worked separately. Each of them read the task in a low voice and was trying to solve it. After reading the task Ala claimed that she did not understand it. Piotrek tried to explain this task to Ala:

1. A: I do not understand this.
2. N: Read aloud.
3. A: [she is reading the task in an undertone]
4. N: Piotrus, and do you understand? Have you read?
5. P: Yes
6. N: well, explain it to Ala.
7. P: [he is showing next figures] I 'm adding two black circles and one white . . . . No . . . I'm adding two black circles . . .

Piotrek trying to explain the task to Ala in the same time he started to interpret it. He was giving the relations which he perceived in the task to his friend. Ala has not seen those so she was not able to understand what Piotrek was talking about. Piotrek sees the structure of set of figures. For him these figures are arranged in the sequence with clear connection among next figures. Despite he talks about white circles at the beginning, he focuses on the black ones. Maybe he could not verbalize the connection occurring for white circles and the connection for black was easy to express by means of numbers.

The teacher joins in the next part of dialogue. Only after reading the text together and discussing it thoroughly Ala could understand the task. After analyzing the task together they started to solve it. At the beginning, the students did not cooperate, despite the fact they were obliged to. Although, each of them worked independently, the way of solving the task was practically identical: they were counting circles of each figure in turns (black and then white) and they put the results into the table. Despite the fact that Piotrek said the rule for black circles, he did not use the table till the 5th level. They drew their own figures using the picture of the 5th figure to complete the 6th and 7th line of the table (there were no appropriate pictures for them). Piotrek drew black circles, using the rule 'add two black' consequently; he was counting them and completing the table with the results. Then, he was completing the picture with white circles, drawing them along the left, lower side. Ala drew the whole figure and then she was completing the table.

Ala was the first to complete the table. When Piotrek finished they started checking each other the results.

8. A: [she is looking into Piotrek's solution] Why do you have 30 here? [she is checking up her results] It should be 36 here.
9. P: [he is checking his picture, counting circles again and correcting his result] oh, yes, you are right. (. . .) Why do you have 14 here? [he is showing in Ala's table]
10. A: I don't know. Maybe I made a mistake. Explain it to me.
11. P: Because it is addition here . . . [he is pointing at additional black circles to the figure no. 4] Because this two should be added [he is showing on utmost black circles]
12. A: Ok.
13. P: Well . . . which is it? [he is looking at the table] seven, well [he is counting black circles from figure no. 5 in an undertone] 1., 2, . . . 13
14. A: Thirteen blacks.
15. P: [he is counting white circles in one row] 1, 2, . . . 6 times [he is counting white circles in one column] six, so there are 36.
16. A: [she is observing Piotrek and trying to paint the fig. no. 5 as he]
17. P: [he is monitoring Ala's work]
18. A: But here is something which does not match

19. P: Because here you should have one more [he is drawing a circle]
20. A: Oh, yes [she is counting once again]

After filling the table pupils started to check up their results. It was not a typical cooperation but some kind of solidarity and being responsible for a group. In fact, they were working together at the begining, but they were aware that they create a group and as a team they would present their results. Ala and Piotrek are partners. None of them assumes that his/her result is correct and the partner is wrong. They suggest that a different result could be present in the specific unit of the table; they try to justify why it is in this way according to them. When they feel they could make a mistake they ask their friend to explain the applied rule. It is typical of students not to give ready and correct results; they explain the rule to the friend and let him/her reach the correct conclusion.

The way of counting white circles (15) given by Piotrek is not the result of arithmetical operation connected with discovery of arithmetical relation; it is the operation of counting the set which elements are arranged in six rows of 6 .

When the students finished completing the table, the teacher suggested to think what the next figure will look like.

21. T: What can you say about these numbers which you wrote here? [he is pointing at the column 'the number of white circles']
22. A: That they are the results of multiplication. 1 times1, 2 times2, ... 6 times 6 [she is looking at the table and is reading next products]
23. N: Piotrek, are Ala right?
24. P: I do not know.
25. A: Look, 1x1, 2x2, 3x3 [she is pointing at the first number from the column "number of figure" and next at the appropriate number from column "number of white circles], ... , 6x6, 7 times 7

The geometrical aspect of the task was more important for Piotrek. He noticed the mutual position of the circles in following figures and the rule according to which every next figure is being created: you should draw 2 circles to black circles- one above and one below; you should draw one row and one column to white circles. Moreover, white circles are arranged in a square n rows with n elements.Piotrek used mainly geometrical relations and referred to the pictures justifying his result.

Ala did not notice geometrical relations. She started to analyse the following numbers in the columns of the table and on their basis she discovered some relations. Ala in her work used arithmetical relations, justifying her opinion she referred to such relations. Observing individual students work while completing the table made an impression that both of them think in a similar way; they discover and apply the same relations. The fragment of their conversation given above shows that their behavior (what could be observed) was practically identical but the ways of thinking were different. It was visible especially while 'extending' the table.

Both of them extended the table till the 15th line. Piotrek started completing the extended table from the column with 'the number of black circles', and Ala started from the column with 'the number of white circles'. Both of them were applying his/her own rule.Ale completed the extended table using only arithmetical relations. Piotrek completing the column with 'the number of white circles' was trying to use geometrical relations. However, he noticed that this way is time-consuming and it is easy to make a mistake. As a result, he used the Ala's way.

At the end of the classes the teacher asked the students what they could say about the puzzle from the task.

26. N: What can you say about the John's puzzle about white and black circles?
27. P: For the white ones there was a multiplication 0x0, 1x1, 2x2 [he is reading next products]. And for the black ones there was an addition 2, here and here [he is showing in the picture]
28. A: That is 1+2 is 3, 3+2 is 5
29. A+P: [they are counting together and showing the next numbers in the table] 5+2 is 7

Common discussion shows that such directed work resulted in creating common strategy which students used in their further work. At first, it was Ala who talked about multiplication and Piotrek about addition; the roles has changed here.

At present, Piotrek is the one who gives the rule for white circles 'multiply the number of the previous figure'. Students accepted the friend's rule and were giving it as functioning universally. The common work is clearly visible.

5. Resume

Pupils are able to perceive mathematical regularities. They cope with tasks, in which such regularities should be discovered and used in further work. Pupils can collaborate/cooperate.

The presented fragment of research shows child's different mental ways, despite the external way of proceeding is identical.

If the students'conversation did not take place, it could be said that they were solving the task identically and were thinking in the same way. It turned out, that different things were important for them in the task. Both of them noticed different relations and structures present in the task. During individual work each of them collected different experiences connected with the task and created own isolated models. During common work these 'isolated models' started to interfere what resulted in creating common 'generic model' connected with the task. This model enabled to generate further examples for extending the table till the 15th line.

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ABOUT DEFINITION OF A PERIODIC FUNCTION

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Abstract. In this paper we consider various definitions of a periodic function and establish connections between them, in particular, we prove equivalence of some of them. In papers and textbooks one can find different definitions of a periodic function. This raises the question which of them are equivalent.

1. Periodicity of a function $f : X \to \mathbb{R}$ in a domain X $(X \subseteq \mathbb{R})$

For a function $f : X \to \mathbb{R}$ $(X \subseteq \mathbb{R})$ one can meet with the following definitions of periodicity*:

Definition 1.
A function f is periodic in a domain X (in the α-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T \neq 0}{\exists} \underset{x \in X}{\forall} [x \pm T \in X \wedge f(x+T) = f(x)] \qquad [1,10]$$

Definition 2.
A function f is periodic in a domain X (in the β-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T > 0}{\exists} \underset{x \in X}{\forall} [x \pm T \in X \wedge f(x+T) = f(x)] \qquad [4]$$

*To distinguish particular definitions of periodicity we use terms: periodicity in the α-sense, in the β-sense and in the γ-sense.

Let us prove that definitions 1 and 2 are equivalent.

Theorem 1.
An arbitrary function $f : X \to \mathbb{R}$ $(X \subseteq \mathbb{R})$ is periodic in the α-sense if and only if it is periodic in the β-sense.

Proof:
It is obvious that periodicity in the β-sense implies periodicity in the α-sense.

To prove the inverse implication let us assume that a function f is periodic in the α-sense. According to Theorem 1 there exists T_1 such that

$$T_1 \neq 0, \tag{1}$$

$$\underset{x\in X}{\forall}[x \pm T_1 \in X \wedge f(x+T_1) = f(x)]. \tag{2}$$

As $T_1 \neq 0$, then $T_1 > 0$ or $T_1 < 0$. In the case $T_1 > 0$ from (2) we have:

$$\underset{T>0}{\exists}\ \underset{x\in X}{\forall}[x \pm T \in X \wedge f(x+T) = f(x)].$$

Assume additionally that $T_1 < 0$. Let also $x \in X$. Then on the basis of (2) we have $x - T_1 \in X$ and $f[(x - T_1) + T_1] = f(x - T_1)$, whence it follows that $f(x - T_1) = f(x)$. Introducing notation $T_2 = -T_1$ we obtain: $T_2 > 0$ and $f(x+T_2) = f(x)$ for arbitrary $x \in X$. Furthermore, $x \pm T_2 \in X$. Therefore,

$$\underset{T>0}{\exists}\ \underset{x\in X}{\forall}[x \pm T \in X \wedge f(x+T) = f(x)].$$

If $T_1 > 0$ or $T_1 < 0$, then

$$\underset{T>0}{\exists}\ \underset{x\in X}{\forall}[x \pm T \in X \wedge f(x+T) = f(x)].$$

Hence, on the basis of definition 2 a function f is periodic in the β-sence, which proves the statement.

In view of Theorem 1, to characterize periodicity in a domain X $(X \subseteq \mathbb{R})$ we can use both definitions 1 and 2. Using complete induction one can prove that:

Theorem 2.
If f is a periodic function (in the α-sense) with the primitive period T in a domain X and $x \in X$, then $x \mp nT \in X$ $(n \in N - \{0\})$.

The following definitions are also connected with the notion of periodicity:

Definition 3.
A function f is periodic in a domain X (in the γ-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T\neq 0}{\exists} \underset{x\in X}{\forall} [x+T \in X \wedge f(x+T) = f(x)] \qquad [3].$$

Definition 4.
A function f is progressive periodic (“forward”-periodic) in a domain $X \Leftrightarrow$

$$\Leftrightarrow \underset{T>0}{\exists} \underset{x\in X}{\forall} [x+T \in X \wedge f(x+T) = f(x)] \qquad [5].$$

Definition 5.
A function f is regressive periodic (“backward”-periodic) in a domain $X \Leftrightarrow$

$$\Leftrightarrow \underset{T<0}{\exists} \underset{x\in X}{\forall} [x+T \in X \wedge f(x+T) = f(x)].$$

It is easy to show that

Theorem 3.
The following implications are true for particular types of periodicity:

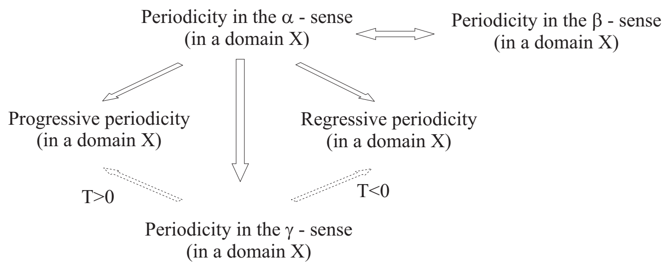

Fig. 1.

There are progressive periodic functions and regressive periodic functions which are not periodic in the α-sense. For example, the function $f(x) = \sin x$ (with the primitive period $T = 2\pi$) in a domain $X = \langle \frac{\pi}{4}, +\infty\rangle$ is a progressive periodic one, whereas the function $g(x) = \cos x$ (with the primitive period $T = -2\pi$) in a domain $X = (-\infty, \frac{\pi}{3}\rangle$ is a regressive periodic one (Fig. 2) [10].

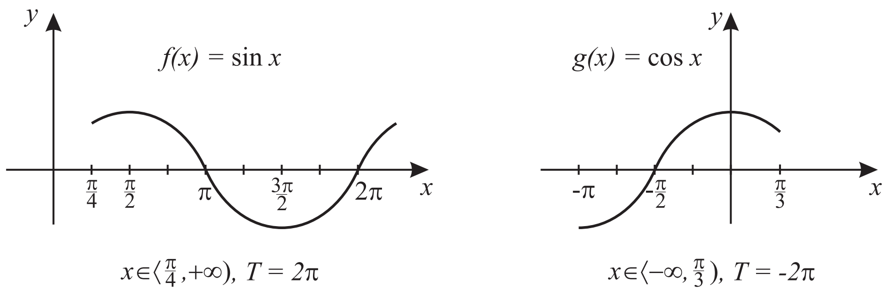

Fig. 2.

2. Periodicity of a function $f : X \to \mathbb{R}$ in a domain $\mathbb{R}$

For a function $f : \mathbb{R} \to \mathbb{R}$ the following definitions of periodicity are considered:

Definition 1*.
The function f is periodic (in the α-sense) $\Leftrightarrow$

$$\Leftrightarrow \underset{T\neq 0}{\exists} \underset{x\in\mathbb{R}}{\forall} [f(x+T) = f(x)] \qquad [2, 6, 8, 9].$$

Definition 2*.
A function f is periodic (in the β-sense) * $\Leftrightarrow$

$$\Leftrightarrow \underset{T>0}{\exists} \underset{x\in\mathbb{R}}{\forall} [f(x+T) = f(x)] \qquad [5].$$

Theorem 4.
A function f in a real domain is periodic in the α-sense if and only if it is periodic in the β-sense in this domain.

The proof of this theorem is similar to the proof of Theorem 1. On the basis of the abovementioned theorem periodicity of a function in the domain $\mathbb{R}$ can be characterized by both the definitions 1* and 2*.

*In the textbook [7] periodicity of a function in a domain $\mathbb{R}$ is characterized as follows: A function f is periodic (in the β-sense) $\Leftrightarrow \underset{T>0}{\exists} \underset{x\in\mathbb{R}}{\forall} [f(x) = f(x \pm T) = f(x \pm 2T) = \cdots = f(x \pm kT)]$, where k is an arbitrary integer.

3. The notion of a primitive period

The notion of a primitive period for a periodic function is usually defined as follows: if there exists the least positive number T satisfying the condition $f(x+T) = f(x)$ (for arbitrary $x \in X$), then it is called the primitive period of a function $f : X \to \mathbb{R}$.

In the paper [10] it was noted that "if there exists the least positive period or the largest negative period, then the larger of these two numbers which exists (when two numbers do not exist at the same time) is called a primitive period".

With this definition of a primitive period, one can also consider a primitive period for progressive and regressive periodic functions.

It should be emphasized that there are periodic functions which do not have a primitive period. As an example we can consider the following functions (compare [1]):

$$f(x) = c \qquad (c = \text{const}), \qquad x \in \mathbb{R};$$

$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is a set of all rational numbers.

An arbitrary nonzero real number constitutes a period of a function f, while an arbitrary nonzero rational number constitutes a period of a function g.

A periodic function in a domain X ($X \subseteq \mathbb{R}$) which does not have a primitive period can have a domain bounded above as well as bounded below.

A progressive periodic function $f : X \to \mathbb{R}$ having a primitive period is not a function with a domain bounded above, whereas a regressive periodic function having a primitive period is not a function with a domain bounded below.

By the way, it should be mentioned that apart from periodic functions (having one period) there are also functions having two and more periods. For example, a two-periodic function $f : \mathbb{R} \to \mathbb{R}$ having two periods 1 and $\sqrt{2}$ is defined as follows:

$$f(x) = \begin{cases} 0 & \text{for numbers } x \text{ of the form } m + n\sqrt{2}, \\ & \text{where } m \text{ and } n \text{ are integers,} \\ 1 & \text{otherwise.} \end{cases} \qquad [8]$$

Functions with many periods were studied by Polish mathematician A. Łomnicki (1881–1941). Extensive information about functions with many periods can be found in the paper [8].

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GEOMETRY AT PRIMARY SCHOOL

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Abstract. Geometry is an essential part of primary stage mathematics curriculum. Its syllabus and performance standards exactly define what a pupil should master after completing each year of primary stage of education. In our survey we mapped real outcomes of mastering key terms from geometry by pupils after their completion of primary stage. The survey also includes a comparison of views held by both primary education teachers and secondary junior stage teachers of mathematics on some issues of concern when teaching geometry in primary & junior school age.

1. Starting points for survey

Teaching mathematics in primary school builds up on the pupils' experience of mathematical character gained during pre-school age. Analogically, knowledge and skills gained by pupils in the course of the primary stage of education form essential fundaments on which further mathematical instruction is built. Mathematics in primary stage is divided into the three basic strands of learning: arithmetic, algebra and geometry. Primary school aged pupils are given early access to some key terms from geometry. Yet, these notions are not defined by the exact terminology of scientific mathematics but rather by language more adequate of their age, most often circumlocutory. Pupils should have formed their own conception of some geometrical terms (Prídavková, 2007). Math curriculum in the respective years provides for the following hours (number of lessons) of geometry: year 1 – 9 lessons out of 132, year 2 – 20 lessons out of 165, year 3 – 25 out of 165 and year 4 – 25 out of 165. Both content and performance standards in primary Mathematics articulate core elements of contents for the particular year. Requirements for knowledge and skills operationalise objectives of the given area that should be appropriated by each pupil at the end of each year.

Year One

Contents: Geometric shape: triangle, circle, square, rectangle, cube, sphere, cylinder. Curve, straight line, open line, closed lines.
Requirements for knowledge and skills: Distinguish geometric shapes: triangle, circle, square, rectangle, sphere, cube and cylinder. Trace straight lines. Draw and distinguish between open and closed line.

Year Two

Contents: Point, line segment, straight line. Drawing lines and its segments with ruler. Units of length centimetre (cm), metre (m), length of distance.
Requirements for knowledge and skills: Identify points and denote them with capital letters. Draw and denote straight line and line segment. Denote segments which lie (do not lie) on the given geometric shape (straight line or line segment). Measure the length in centimetres. Draw a line segment of corresponding length.

Year Three

Contents: Measuring length of line segment. Drawing line segment of corresponding length. Conversion of length units. Comparison of line segments lengths. Drawing circle. Drawing triangle and rectangle on grid paper, denoting their vertices.
Requirements for knowledge and skills: Measure the length of line segment in millimetres and centimetres, the distance in metres, draw the line segment of corresponding length. Acquaint with the length unit kilometre. Convert units of length. Draw circle with given centre and radius. Compare line segments according to their length and notate the result of comparison by means of $<$, $>$, $=$. Draw triangle and denote its vertices and sides. Draw rectangle and denote its vertices and sides. Draw rectangle and square in grid paper and denote its vertices and sides.

Year Four

Contents: Drawing perpendicular lines. Performing addition and subtraction of the length of segments. Multiple of segment's length. Perimeter of triangle, rectangle and square. Conversion of length units.
Requirements for knowledge and skills: Draw a line perpendicular to another line (by means of right angle triangle ruler). Determine addition and subtraction of segment's lengths. Determine multiple of segment's length. Calculate perimeter of triangle, square and rectangle. Convert units of length. (Bálint, 1998).

2. Methodology of survey

Non-standardised test containing geometry tasks was administered to 230 pupils from the 5 basic schools at the time of completing year four. The pupils sample was proportionately representative of village and urban schools including both fully organised as well as small size integrated mixed age schools. Pupils were taking test during one lesson - 40 minutes to answer, which provided an adequate amount of time. The tests were then assessed on the basis of phenomenal analysis.

Another part of the survey was a distribution of questionnaire on teaching geometry in primary school among teachers in primary stage (93 respondents), and questionnaire on pupils' preparedness for geometry syllabus taught in the secondary junior stage among teachers of Mathematics in the corresponding stage of education (68 respondents).
Through comparative analysis the pupils' real achievement in the test was compared with the respective views held by teachers in both groups (stages).

3. Results of survey

The administered test was designed in order to determine the following range of knowledge of pupils aged 10: name and denote geometric figures, name and discriminate plane and space geometry, identify elements of circle and disc, draw perpendicular lines, draw plane geometric figures (square, rectangle), determine perimeter and area of polygons (polygons presented in grid structure).

The answers were assessed on the basis of phenomenal analysis. The most frequent errors were identified.

The questionnaires administered among the teachers of both groups were assessed in terms of agreement/disagreement between a) the views of the teachers and b) the real outcomes attained by pupils in solving geometry tasks (Table 1). A comparative analysis brings out the following findings:

1. Teachers of elementary stage expected solving the tasks by pupils on point, segment and line without any problem. However, the correct denoting of these figures has proved to be the least mastered item.

2. Teachers of both stages presumed that pupils would sufficiently master the contents of plane geometry figures and would be able to name them and identify their elements. The test results have proved that pupils appropriately mastered the identification of planar figures with the exception of circle/disc identification. The above two geometric figures were a bit more problematic to pupils, as they could not identify weather points belong to circle or disc, and interchanged the terms diameter and radius.

Table 1. Most Frequent Errors in Pupils Answers

Type of Error	**Number of occurrences**
1.1 incorrect identification of plane geometry figures (point, line, line segment, circle, disc)	53
1.2 incorrect denoting of geometric figures	194
Incorrect identification of plane geometry figures	
2.1 all figures	23
2.2 only circle/disc	82
2.3 only quadrilateral	34
2.4 only polygon	29
3.1 incorrect determining weather point resides on circle (disc)	111
3.2 incorrect marking of radius	90
3.3 incorrect marking of diameter	95
4.1 inaccurate drawing of lines (perpendicular)	57
4.2 incorrect (absent) denoting of line	137
4.3 incorrect answer on the given question (right angle)	156
5.1 inaccurate drawing of square or rectangle	12
5.2 incorrect - denoting of vertices	19
- marking of adjacent sides	84
opposite sides	65
- drawing of diagonals	44
Incorrect identification of 3D geometric figures	99
7.1 incorrect identification of geometric figure (polygon)	127
7.2 incorrect calculation of perimeter or area	108
7.3 missing unit or incorrect determining of it	115

3. Teachers' assumption that the contents of perpendicular lines and right angle would be appropriated on the desired level has not quite proved true. Pupils could draw a line perpendicular to another line passing through a given point but would forget to denote lines. The term right angle has not been appropriated on the satisfactory level; they would not mark the right angle on the picture.

4. The views of the teachers of elementary stage (slightly more positive) and the views of their colleagues teaching in the secondary stage (slightly more negative) on mastering key terms connected with square and rectangle varied.

Pupils' answers have revealed that they are not able to correctly identify adjacent and opposite sides.

5. Teachers of both stages expected the task with 3D geometric figures to be the most difficult. Although, this assumption has not proved true, pupils made mistakes, though, compared with the other tasks, this one was managed on the average level.

6. In many instances pupils interchanged the terms perimeter and area the planar figure. Due to the fact that this task is a part of the year four syllabus, the attained results were not satisfying.

Comparing the answers of the teachers from elementary stage and secondary junior stage of education we have recorded many consistent views on teaching geometry in primary school, yet some discrepancies in their views emerged:

I. In their answers on the question of the amount of time which curriculum allocates to teaching geometry in the first stage of basic school, most teachers indicated that it is sufficient. Yet more teachers of the secondary junior stage inclined towards an idea of extending the amount of time allocated for geometry lessons.

II. In the question on relevance or irrelevance of listing certain parts of geometry into the Math syllabus of the first stage of basic school, both groups of teachers unambiguously expressed that the proportion of geometry in Math curriculum is adequate. However, first stage teachers held some views differing from those of their colleagues: content items as area of square and area of rectangle (currently a part of the non-core, extending-the-breadth-of-study syllabus) should be taught in higher stages, more space should be reserved for practising polygons, and the item of angles should be returned back to the first stage. Math specialists from the secondary stage would conversely shift parallels back to the first stage syllabus.

III. On pupils' preparedness to progress from the first to the secondary junior stage of education in the context of mastering basic geometric terms the teachers held the following views:

Table 2. Pupils' Progress from the Primary to the Secondary Junior Stage of Education - Assessing Pupils by Teachers

Assessment Scale	**Elementary Stage Teachers**	**Secondary Junior Stage Teachers (Math Specialists)**
Without problems-excellent	14	0
Without major problems-well	51	40
With minor problems-adequately	28	28
With major problems-weak	8	0

4. Conclusion

The results of 10 year old pupils attained when solving geometry tasks have revealed that not all of them have mastered geometry syllabus on the desired level. The analysis of pupils' solutions has primarily brought out the details about the errors that pupils made when solving geometry. Comparing pupils' achievement in test with the views of teachers from the primary and secondary junior stages of education on the pupils' mastery level of geometry syllabus, some interesting findings have emerged. Real results attained by pupils did not always meet teachers' expectation. All relevant information gained from the survey intended for teachers - practitioners is made accessible on the web page www.matematickapointa.sk. Such form of feedback can in our view contribute to enhancing the quality of teaching geometry in primary school, so as the pupils after progressing to the higher stage can smoothly proceed with appropriating a more exacting geometry syllabus.

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A VIEW FROM ABOVE OR RATHER FROM BELOW?

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Abilities of every individual form the basis of spatial intelligence. They help him to perceive the visual world accurately, to transform perceptions, to manipulate perceptions in his mind and to modify his initial perceptions. They also enable the individual to create images in his mind from his own visual perception at the time when no external stimuli take effect. Spatial intelligence consists of a larger number of loosely connected abilities such as the ability to perceive things visually, draw a given shape, the ability to create mental images and to work with them and transform one shape into another.

Abilities of spatial perception of individuals of the same age are different. They can develop or inhibit themselves individually–they are mutually relatively independent. Practising this area stimulates their development. Spatial intelligence develops initially in terms of concrete situations. The ability to imagine various spatial orderings in one's mind appears with the outset of the formal operational stage.

We use solids to model various spatial situations. Children of pre-school age are able to recognize and name all the basic solids, which they use in different games.

In school they enhance the perception abilities further. In year six they learn to portray geometric figures onto a plane. Textbooks for pupils of this age most often use a representation called a 'free' parallel projection. While observing established rules, a clear and comprehensible picture emerges and the pupils are able to continue working with it. Another representation is drawing of solids by means of right-angled projections. It is possible to portray geometric figures onto a plane in different ways, which are dealt with in the subject descriptive geometry at a secondary school.

From now on, we will be concerned with primary school children. Different textbooks of mathematics use different pictures of solids. They are usually portrayed onto a free parallel projection, most frequently in the right view from above. We wanted to know how this fact influences the pupils' ability to perceive pictures of solids and to solve stereometric problems portrayed in different views.

In our brief contribution we present the results of a preliminary test research **A view from above or rather from below?** Pupils were randomly divided into four groups. They solved the same problems. However, solids were portrayed in a free parallel projection:

- right view from above,
- left view from above,
- left view from below,
- right view from below.

Example from the test

A piece of wire is placed in a cube. In the picture you can see the plan in view, the front view and the side view. Wind up the wire onto the cube and mark it in colour.

Hypothesis 1

Pupils are more successful while solving problems shown in the right view from above.

Results of testing hypothesis 1

	Number of pupils (boys/girls)	**Problem**					
		1	**2**	**3**	**4**	**5**	**6**
Test A right view from above	51 (27/24)	92.15%	43,23%	33,33%	12,77%	36,17%	31.91%
Test B left view from above	56 (32/24)	89.29%	37.50%	50.00%	12.00%	30.00%	28.00%
Test C left view from below	63 (32/31)	82.54%	26.98%	50.79%	7.69%	25.00%	23.07%
Test D right view from below	60 (36/24)	83.33%	35.00%	43.33%	6.00%	24.00%	10.00%

Test results confirmed that hypothesis 1 is valid. Most frequently, the right view from above appears in textbooks, which perhaps accounts for the fact that pupils succeed best in solving these tasks. Problem 3 in test A is the most difficult one because pupils need three particular mental processes to solve it. In other cases, they need only two of them, they use spatial thinking and memory much more. While selecting instructions to a problem, we need to take into account not only whether the problem is the right type of a problem but we also need to determine its level of difficulty so that the problems for the individual groups of tested students are comparable.

Hypothesis 2

Pupils in years 6 to 9 achieve better results.

Results of testing hypothesis 2

	Solved in total by	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
Year 6	64	70,31%	25,63%	35,93%	3,12%	12,50%	17,19%
Year 7	46	91,31%	23,04%	30,43%	4,35%	13,04%	17,39%
Year 8	59	91,53%	40,68%	54,24%	8,47%	35,59%	20,34%
Year 9	61	95,08%	42,62%	55,74%	16,39%	36,07%	24,59%

On the basis of the obtained results, we can see that that pupils in years six to nine are more successful. We believe this is due to the fact that these pupils have more experience solving similar problems and to the fact that they have better knowledge of maths while at the same time their abstract thinking is being developed faster.

Hypothesis 3

Right-handed pupils have comparable results with left-handed pupils.

Results of testing hypothesis 3

Year six	**Solved in total by**	**Problem**					
		1	**2**	**3**	**4**	**5**	**6**
Test A right view from above	1	0	0	0	0	0	0
Test B left view from above	1	1	1	0	1	0	0
Test C left view from below	1	1	0	1	0	1	0
Test D right view from below	1	0	1	0	0	0	0

Year seven	Solved in total by	Problem					
		1	**2**	**3**	**4**	**5**	**6**
Test A right view from above	1	1	0	0	0	0	0
Test B left view from above	1	1	1	1	0	1	0
Test C left view from below	2	2	1	1	1	1	0
Test D right view from below	2	1	0	1	0	0	0

Year eight	Solved in total by	Problem					
		1	**2**	**3**	**4**	**5**	**6**
Test A right view from above	1	1	1	0	0	0	0
Test B left view from above	2	2	1	2	1	0	1
Test C left view from below	2	1	0	1	1	0	1
Test D right view from below	1	1	0	0	0	1	0

Year nine	Solved in total by	Problem					
		1	**2**	**3**	**4**	**5**	**6**
Test A right view from above	1	1	0	1	0	0	0
Test B left view from above	1	1	1	1	0	0	1
Test C left view from below	2	1	1	0	0	0	1
Test D right view from below	1	0	0	1	0	0	1

Summary	Solved in total by	Problem					
		1	**2**	**3**	**4**	**5**	**6**
Test A right view from above	4	75%	25%	25%	0%	0%	0%
Test B left view from above	5	100%	80%	80%	40%	20%	40%
Test C left view from below	7	71.43%	28.57%	42.86%	28.57%	28.57%	28.57%
Test D right view from below	5	40%	20%	40%	0%	20%	20%

Tests solved by left-handed pupils were set randomly. There were 21 left-handers altogether (11 boys, 10 girls).

The test results are interesting – the left-handers achieve better test results in both the left view from above and the left view from below. Hypothesis 3 can be confirmed or refuted only on the basis of a sufficient number of left-handed pupils. This will be the subject of further investigation and only a longer-lasting study will yield more precise data.

Description of the invididual problems and their success rate

Problem 1) requires a basic knowledge of a free parallel projection and a representation of visible and invisible edges of a solid. The results suggest that this problem is very easy for pupils. It doesn't have to appear in a further study.

Problem 2) can be successfully solved by students with a well-developed spatial imagination and memory. The correct solution was plotting letters onto the cube with correctly turned letters according to the given net. I speculated about the success rate in case we considered only the correctly plotted letters. In such a case there would be 89 correct solutions out of 230 possible ones. Pupils had difficulty turning the letters in their mind.

Problem 3) uses a dice, pupils are reminded of the rule that the sum of the dots on the opposite sides always equals 7. Nevertheless, there are cases when some solutions are not in accordance with the rule. In view of the fact that a dice, an object used very often by children, was involved, we were suprised by just an average result. The problems were made more difficult by our deliberate usage of various untraditional nets of the cube.

Problem 4) this problem can be classified as a more difficult one because it assumes managing more demanding thinking operations. The same test was given to a group of university students – future mathematics teachers. Out of the total number of 18 students who tried to solve it, only 6 correctly solved the problem.

Problem 5) requires a correct understanding of the concept called a symmetry plane. The number of correct solutions increased significantly with a higher age of pupils. Solution analysis shows that some pupils have difficulty distinguishing identical and symmetric solids in space. These pupils tried to divide the cube into two identical parts. In such a case, there is an indefinite number of solutions. The position is unimportant with identical solids. However, symmetry is bound to a symmetry plane, i.e. the position is important. Corresponding points are at an equal distance from the symmetry plane.

Problem 6) is one of the more difficult ones. It would be probably much easier for pupils if it had opposite instructions. The section would be given and and they would have to draw it into the cube portrayed in a free parallel projection. Watching the pupils starting to make the paper models with excitement and interest proved extremely useful while finding the correct solutions.

The pupils encountered some types of problems for the first time, which influenced significantly the success rate. A very interesting result of all the testing is the hypothesis that the left-handed pupils can see figures in the left view from above or from below much better. The proposed hypotheses need to be confirmed by a further and more detailed investigation, i.e. not only in the form of a test with a pencil and a piece of paper, but through the pupils' real work with models.

	Solved in total by	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
Year 6	64	70,31%	25,63%	35,93%	3,12%	12,50%	17,19%
Year 7	46	91,31%	23,04%	30,43%	4,35%	13,04%	17,39%
Year 8	59	91,53%	40,68%	54,24%	8,47%	35,59%	20,34%
Year 9	61	95,08%	42,62%	55,74%	16,39%	36,07%	24,59%

Conclusion

Problems which appeared in views from below were solved much less successfully than in views from above. Solutions of problems in the right view from below caused a great problem. Most of the time only structural elements such as arches or eaves are protrayed in the in the view from below but such things do not appear in the pupils' textbooks very often. Based on the results of this study it is evident that views of a solid other than the right view from above should appear in materials for pupils more often. Appropriate changes of drawing solids in different views could substantially improve the pupils' spatial imagination. It would contribute to spatial thinking and to the developent of spatial memory.

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CONSTRUCTION PROBLEMS AND THEIR PLACE IN SECONDARY SCHOOL MATHEMATICS

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Abstract. A pilot study concerning construction problems in mathematics teaching at grammar schools and universities is described in this paper.

Currently, plane constructions are getting towards the centre of teaching again. It is very useful and necessary because they show a clear target to a student (e.g. what is to construct), they develop abilities of dialectical perception of relationship between theory and practice. They also serve as a convenient test method allowing a teacher to diagnose a quality of informal knowledge of students.

To determine the knowledge level of construction problems, three construction problems were submitted to students of the 2nd year-class of the grammar school, to another group of students of the 4th year-class and finally, to university students of teaching mathematics. The aim of this pre-research was to determine the level of students' abilities of solving construction problems and point to possible mistakes.

All three problems belong to the standard secondary-school plane geometry problems.

Problem 1: Construct a triangle ABC: $a : b : c = 2 : 3 : 4, v_a = 5$ cm

Problem 2: Construct a triangle ABC: $a = 7$ cm, $b + c = 12,5$ cm, $v_c = 6,5$ cm

Problem 3: Construct a triangle ABC: $c = 5$ cm, $t_a = 5$ cm, $v_a = 4,5$ cm

The results of the research were interesting in the point that no student of the three tested groups complied with all parts of the construction problem solution (analysis, construction, proof, discussion). Thus it was convenient to evaluate the test by means of the phenomenal analysis.

The results of the research are summarized in the following tables in which these symbols are used:

+ ... a part of analysis given, construction completed, number of solutions given (missing proof tolerated)
/ ... problem solved (constructionally), missing one ore more of prescribed parts of solution (most commonly analysis, discussion)
- ... problem solved but incorrectly
0 ... problem not solved)

Problem 1	2nd year-class of grammar school	4th year-class of grammar school	4th year-class of teacher training courses
+	16 %	0 %	7 %
/	80 %	4 %	67 %
-	4%	28 %	13 %
0	0 %	68 %	13 %

Problem 2	2nd year-class of grammar school	4th year-class of grammar school	4th year-class of teacher training courses
+	20 %	4 %	27 %
/	56 %	24 %	40 %
-	16 %	52 %	33 %
0	8 %	20 %	0 %

Problem 3	2nd year-class of grammar school	4th year-class of grammar school	4th year-class of teacher training courses
+	8 %	8 %	7 %
/	72 %	72 %	73 %
-	4 %	16 %	20 %
0	16 %	4 %	0 %

It is clear from the table that the most successful in all three problems were students of the 2nd year-class of the grammar school. They took the test immediately after going through plane geometry in mathematics lessons. The university students took the second place in the order and the graduates were the worst.
It follows from the results that even if the best group was formed by students of the 2nd year-class of the grammar school, their solutions were not complete (although we could expect the best results).
The students of all examined groups did not know the prescribed parts of solution, they did not state a discussion, they did not discern between construction analysis and construction procedure. Their graphics denotation was poor on numerous occasions.

We assume (and we will verify hereafter) that the reasons of incorrect solutions consist in the following factors:

[1] Insufficient preparation of students from basic schools.

[2] Geometry lessons are reduced by the teachers of mathematics or the presentation is intuitive.

[3] Unsatisfactory motivation of students by the teachers.

[4] Insufficient number of practice problems available.

[5] Number of geometry lessons reduced.

[6] Geometry lessons are shifted to marginal periods of the school-year.

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ON MISTAKES CONNECTED WITH DIFFERENTIATING

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Abstract. Many students have problems with solving tasks concerning the existence of the derivative of a function at a point. In this paper we discuss some of them.

1. Computing the derivative of some functions without testing their continuity

It is known that if a function is differentiable at some point x_0 of its domain, then it is also continuous at x_0. The converse is not true, i.e. a function that is continuous at x_0 need not be differentiable there. So, sometimes, before we start to differentiate, we should test the continuity of some functions.

Note, that $f'(x_0)$ exists if and only if $f'_-(x_0)$ and $f'_+(x_0)$ exist and they are equal. Then $f'(x_0) = f'_-(x_0) = f'_+(x_0)$.

Example 1. Test the differentiability of the given function f (Fig. 1):

$$f(x) = \begin{cases} ax + b, & \text{if} \quad x > 0; \\ x^3, & \text{if} \quad x \le 0, \end{cases}$$

where a and b are parameters.

For $x \neq 0$ we have:

$$f'(x) = \begin{cases} a, & \text{if} \quad x > 0; \\ 3x^2, & \text{if} \quad x < 0. \end{cases}$$

Sometimes, students try to use the following incorrect procedure to obtain the derivative $f'(0)$. First, they compute the right hand derivative and the left hand derivative at $x_0 = 0$:

$$f'_+(0) = \lim_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{a\Delta x + b - 0}{\Delta x} = \lim_{\Delta x \to 0^+} (a + \frac{b}{\Delta x}) = +\infty,$$

Fig. 1.

$$f'_{-}(0) = \lim_{\Delta x \to 0^-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(\Delta x)^3 - 0}{\Delta x} = \lim_{\Delta x \to 0^-} ((\Delta x)^2 = 0.$$

Since $f'_{+}(0) \neq f'_{-}(0)$, students draw the conclusion, that the derivative $f'(0)$ does not exist. It appears, that at first we should test the continuity of this function at $x = 0$. It is easy to show, that if $b = 0$, then the function will be continuous at that point. Indeed, we have $\lim_{x\to 0^+} f(x) = b$ and $\lim_{x \to 0^-} f(x) = 0$. So, there exists the limit $\lim_{x\to 0} f(x) = 0$, if $b = 0$. Since $f(0) = 0$ the continuity condition $\lim_{x \to 0} f(x) = f(0)$ is fulfilled, if $b = 0$. Now, we can compute the derivatives $f'_{-}(0)$ and $f'_{+}(0)$ of the continuous function

$$f(x) = \begin{cases} ax, & \text{if} \quad x > 0; \\ x^3, & \text{if} \quad x \leq 0. \end{cases}$$

In turn, we obtain that

$$f'_{+}(0) = \lim_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{a\Delta x}{\Delta x} = a,$$

and $f'_{-}(0) = 0$. The derivative $f'(0)$ exists if $f'_{-}(0) = f'_{+}(0)$, i.e. when $a = 0$. Thus, if $a = b = 0$, then the function f will be differentiable everywhere.

2. Stating the existence of the derivative $f'(x_0)$ by the equality at least one of the limits $\lim_{x\to x_0^+} f'(x)$ and $\lim_{x\to x_0^+} f'(x)$

It is known that there are functions for which the equality $\lim_{x\to x_0^+} f'(x) = \lim_{x\to x_0^-} f'(x)$ is true, but the derivative $f'(x_0)$ does not exist.

Example 2. Test the differentiability of the given function f (Fig. 2):

$$f(x) = \begin{cases} \arctan \frac{1}{x}, & \text{if} \quad x \neq 0; \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Fig. 2.

Then, using the usual rules for calculating derivatives we have

$$f'(x) = \frac{1}{1 + (\frac{1}{x})^2} \cdot (-\frac{1}{x^2}) = -\frac{1}{1+x^2}$$

if $x \neq 0$. We directly obtain that $\lim_{x\to 0} f'(x) = -1$, thus $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = -1$, however the derivative $f'(0)$ does not exist. It follows from the fact, that the function f is discontinuous at the point $x = 0$. Indeed, it is easy to see that $f(0) = 0$ while $\lim_{x\to 0^+} f(x) = \frac{\pi}{2}$ and $\lim_{x\to 0^-} f(x) = -\frac{\pi}{2}$

(see Fig. 2). The limit $\lim_{x\to 0} f(x)$ does not exist, so the continuity condition $\lim_{x\to 0} f(x) = f(0)$ does not hold.

On the other hand, if $x = 0$, we can compute the derivative $f'(0)$ by evaluating the limit

$$\lim_{\Delta x\to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}.$$

So, we see that

$$f'(0) = \lim_{\Delta x\to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\arctan \frac{1}{\Delta x}}{\Delta x} = +\infty$$

(we assume, that the continuity of this function at $x = 0$ was not tested earlier).

3. Stating the nonexisting of the derivative $f'(x_0)$ by the nonexistence at least one of the limits $\lim_{x\to x_0^+} f'(x)$ and $\lim_{x\to x_0^-} f'(x)$

There are functions, for which neither of the limits $\lim_{x\to x_0^-} f'(x)$, $\lim_{x\to x_0^+} f'(x)$, $\lim_{x\to x_0} f'(x)$ exist, but the derivative $f'(x_0)$ exists.

Example 3. For the continuous function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if} \quad x \neq 0; \\ 0, & \text{if} \quad x = 0, \end{cases}$$

the limits $\lim_{x\to 0^-} f'(x)$ and $\lim_{x\to 0^+} f'(x)$ do not exist, however $f'(0)$ exists. Indeed, the function f is continuous at $x = 0$, since $f(0) = 0$ and $\lim_{x\to 0}(x^2 \sin \frac{1}{x}) = 0$, hence we have the condition $\lim_{x\to 0} f(x) = f(0)$. Using the usual rules for calculating derivatives we have (for $x \neq 0$) $f'(x) = 2x \sin \frac{1}{x} + x^2 \cdot (-\frac{1}{x^2}) \cos \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. So, the limits: $\lim_{x\to x_0} f'(x)$, $\lim_{x\to 0^-} f'(x)$ and $\lim_{x\to 0^+} f'(x)$ do not exist. Computing the derivative of the function f at $x = 0$ as the limit, we obtain

$$f'(0) = \lim_{\Delta x\to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(\Delta x)^2 \cdot \sin \frac{1}{\Delta x} - 0}{\Delta x} = \lim_{\Delta x\to 0} (\Delta x \cdot \sin \frac{1}{\Delta x}) = 0.$$

In the case of the function g (below), we have very analogous situation

$$g(x) = \begin{cases} x^\alpha \sin \frac{1}{x}, & \text{if} \quad x \neq 0 \;\text{ and }\; 1 < \alpha < 2; \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Example 4. Let, the function f has the form

$$f(x) = \begin{cases} x^2, & \text{if} \quad x \in Q; \\ x^3, & \text{if} \quad x \in R \setminus Q \end{cases}$$

(Q denotes the set of rational numbers). It is obvious, that this function is discontinuous at points $x \neq 0$ and there are no limits $\lim_{x \to 0^-} f'(x)$ and $\lim_{x \to 0^+} f'(x)$. The derivative of the function f does not exist at any point different from $x = 0$. At the point $x = 0$, we have $f'(0) = 0$, since

$$\lim_{x \to 0, x \in Q} \frac{f(x) - f(0)}{x} = \lim_{x \to 0, x \in Q} \frac{f(x)}{x} = \lim_{x \to 0, x \in Q} \frac{x^2}{x} = 0,$$

$$\lim_{x \to 0, x \in R \setminus Q} \frac{f(x) - f(0)}{x} = \lim_{x \to 0, x \in R \setminus Q} \frac{f(x)}{x} = \lim_{x \to 0, x \in R \setminus Q} \frac{-x^3}{x} = 0.$$

Example 5. The function

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0 \end{cases}$$

is continuous at $x = 0$ (see Fig. 3).

Fig. 3.

For $x \neq 0$, we have: $f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$. It is easy to see, that the limit $\lim_{x\to 0} f'(x)$ does not exist, however we can not draw the conclusion, that $f'(0)$ does not exist. On the other hand, the expression:

$$\frac{f(0+\Delta x) - f(0)}{\Delta x} = \frac{\Delta x \sin\frac{1}{\Delta x}}{\Delta x} = \sin\frac{1}{\Delta x}$$

does not posses any limit, as $\Delta x \to 0$. Thus, the derivative $f'(0)$ does not exist. Similarly, the limits $f'_+(0)$, $f'_-(0)$ also do not exist.

Example 6. It is known that if the function f has got the continuous derivative at a point $x = x_0$, then it is also differentiable at x_0 and so the derivative $f'(x_0)$ exists (in that case, the condition $\lim_{x\to x_0} f'(x) = f'(x_0)$ holds). For example, the function

$$f(x) = \begin{cases} x^2 \arctan\frac{1}{x}, & \text{if } \ x \neq 0, \\ 0, & \text{if } \ x = 0 \end{cases}$$

is continuous at $x = 0$, since $\lim_{x\to 0} f(x) = f(0)$. So, the differentiability of that function at this point can be tasted. At points $x \neq 0$ we have the formula $f'(x) = 2\arctan\frac{1}{x} - \frac{x_2}{1+x^2}$. Thus $\lim_{x\to 0} f'(x) = 0$ and $f'(0) = 0$ (since the function f' is continuous at $x = 0$).

4. Incorrect calculating the derivative of the vectorial function of a scalar argument

If the vectorial function $\vec{a} = \vec{a}(t)$ has the form $\vec{a} = |\vec{a}| \cdot \vec{a}_0$, where $\vec{a}_0$ is the unit vector ($\vec{a}_0$ has the sense of the vector $\vec{a}$), then the correct derivative of the function $\vec{a}(t)$ is the following:

$$\frac{d\vec{a}}{dt} = \frac{d|\vec{a}|}{dt} \cdot \vec{a}_0 + |\vec{a}| \cdot \frac{d\vec{a}_0}{dt}.$$

Sometimes, the following incorrect derivatives of the function $\vec{a}(t)$ can be given:

$$\frac{d\vec{a}}{dt} = \frac{d|\vec{a}|}{dt} \cdot \vec{a}_0 \tag{1}$$

or the formula:

$$\frac{d\vec{a}}{dt} = |\vec{a}|\frac{d\vec{a}_0}{dt}. \tag{2}$$

The formulas (1) and (2) are true only in some cases. If the versor $\vec{a}_0$ is constant, (i.e. $\frac{d\vec{a}_0}{dt} = \vec{0}$), then the formula (1) holds (in this case, only the absolute value of the vector $\vec{a}$ is changeable). The second condition (2) holds, if the absolute value of the vector $\vec{a}$ is constant (i.e. $\frac{d|\vec{a}|}{dt} = 0$). Then, the direction of $\vec{a}$ is not constant. Sometimes, students or pupils forget about the fact, that in the given vectorial function $\vec{a}(t)$ of the form $\vec{a}(t) = |\vec{a}| \cdot \vec{a}_0$, the absolute value of the vector $|\vec{a}|$ as well as the unit vector $\vec{a}_0$ may depend on the argument t.

Example 7. For the vectorial function $\vec{a}(t) = 2t\vec{i}+t^2\vec{j}$, ($\vec{i}, \vec{j}$ are versors, $t \geq 0$) it holds: $\frac{d\vec{a}}{dt} = 2\vec{i}+2t\vec{j}$. Now, let the function $\vec{a}(t)$ is of the form $\vec{a}(t) = |\vec{a}| \cdot \vec{a}_0$. The unit vector

$$\vec{a}_0 = \frac{1}{|\vec{a}|} \cdot \vec{a} = \frac{1}{t\sqrt{4+t^2}}(2t\vec{i}+t^2\vec{j}) = \frac{2}{\sqrt{4+t^2}}\vec{i} + \frac{t}{\sqrt{4+t^2}}\vec{j}.$$

Thus, we obtain that

$$\vec{a}(t) = |\vec{a}(t)| \cdot \vec{a}_0(t) = t\sqrt{4+t^2}\left(\frac{2}{\sqrt{4+t^2}}\vec{i} + \frac{t}{\sqrt{4+t^2}}\vec{j}\right).$$

After all, the derivative of the function $\vec{a}(t)$ will be of the form:

$$\frac{d\vec{a}(t)}{dt} = \frac{d}{dt}\left(t\sqrt{4+t^2}\right) \cdot \vec{a}_0 + t\sqrt{4+t^2} \cdot \frac{d}{dt}\left(\frac{2}{\sqrt{4+t^2}}\vec{i} + \frac{t}{\sqrt{4+t^2}}\vec{j}\right) = 2\vec{i} + 2t\vec{j}.$$

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ICT TO ASSIST MATH TEACHING AT PRIMARY SCGHOOLS

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Abstract. Maths teaching has some specific features in comparison with other subjects taught. It is abstract and realistic, accurate and logical and it has its philosophy. Therefore teaching maths requires a more personalised approach, including highly motivating elements. This, in particular, applies at Primary Schools, as the attitude of children to maths is formed at this level of education. It is about the active involvement and creativity of children, which in turn requires a creative approach to teaching by the teacher. The teaching process can be suitably complemented by using up-to-date information and communication technology. In addition to professional teaching programmes developed by specialists, the teacher can apply a different approach by using commonly used software for developing his/her own teaching programmes.

1. Pupils, teacher and maths teaching software

Education and work with information nowadays, in the 3rd millennium, is nearly unthinkable without the use of modern IT and multimedia programmes. Today, children are born into and brought up in an environment where this technology and multimedia applications are common and perceived as a natural part of their lives. Similarly multimedia entertainment and education is quite natural to them. This is the reason why teachers, particularly in primary schools, must be aware that successful teaching must be, above all, interesting for the pupils and meet their expectations.

In maths teaching, there are many motivational teaching programmes, tools and numerous multimedia options. There is a wide range of software programmes available that support very complex tasks. For example equations, formulas, calculation of limitations, derivations, integrals, plotting graphs, modelling, data analyses, testing of hypotheses, construction of 2D and 3D objects, dynamic modelling etc. However, these programmes are neither cheap nor readily available and the normal PC users (teachers and pupils) are not trained in using them. Sometimes working with these programmes is closer to "developing" new software, which can have a discouraging effect on potential users. Unfortunately, these programmes are mainly targeted at the age group of 10+ years (secondary schools, high schools, colleges and universities).

However, in primary education the attitude of children to education is formed and maths has a very special position in this respect. This is most likely the reason why so many pupils and students have a very negative attitude to maths nowadays, which has very serious social consequences resulting in attempts to reduce the number of maths teaching hours in the education system.

2. Objective of using ICT in teaching

Alternative approaches, higher standards of visualisation and an immediate response to changes in parameters - these are the advantages of adopting IT in teaching, without mentioning its power to motivate pupils. There are many options and approaches for supporting maths teaching in our schools, but these activities are still centred around standalone activities of teachers - ICT enthusiasts - rather than being an educational platform supported by the management of the educational institutions and policymakers in order for it to become an official professional standard.

Let me reflect on this issue by presenting some very simple tools which may be used in maths and other classes taught in primary schools, as at this level of education the approach to teaching is more interdisciplinary and subjects are more intertwined. The objectives should be:

- To increase the popularity of maths as a subject
- To improve the quality of maths teaching and make it more attractive and motivating by allowing the use of IT for homework.
- To support active and efficient partnership between the teacher and the pupil in the educational process.
- To use modern teaching methods (interactive elements in teaching, team work and co-operation, e-learning).
- To develop logical and mathematical thinking of pupils and their interdisciplinary skills.

By choosing the right application, communication environment and the area of maths, which is the most suitable for them, and adoption of a creative approach to the teaching - all this can deliver better quality and a more personalised approach in teaching, which then becomes more attractive and innovative for the children. This can be achieved, particularly, through their active involvement in the learning process by allowing them to interact creatively with these electronic means. However, such an approach is much more demanding for maths teachers because it requires their involvement, creativity, courage and innovative methods in maths teaching, which can sometimes be risky compared with proven practice.

If we look at official "Framing Education Program of Basic Curriculum" for maths and ICT for primary schools, then it becomes clear that only a teacher with a great deal of imagination or an IT and maths enthusiast is able to combine these two subjects into a single formal document. The question is: How to persuade a primary school teacher to adopt this approach?

3. The user's application of electronic teaching tools

There are many options available for electronic support of the teaching process (some of these means are designed for children as young as three). One of them is to use off the shelf products (e.g.: "Virtual School - Math", "Mysterious Forest" and many others). These products require from the teacher a thorough preparation of the lesson in terms of using them. Although these "teaching" programmes are user friendly they are not well designed from the didactic point of view and very often they lack the necessary feedback, interactive input and other elements of both the teacher's and pupil's involvement. These products also tend to limit the children's creativity.

Another possibility is to have the teaching software developed by a specialist company according to the specification provided by the teacher. In this case the didactic objectives of the teacher are met, but the limitations are the high project costs and frequently experienced problems with the input analysis when the programme developer is unable to see the product with the teacher's eyes and the teacher does not know how to specify the requirements from the ICT perspective.

A preferred option is "in house" development of multimedia programmes. Their quality very much depends on the skills and capabilities of its author. If the ITC teacher is an "enthusiast" then he is able to develop his/her own projects matching his/her personal approach. These programmes have the best value, but very often they can only be used by the author or his/her closest colleagues.

The target group here are the teachers who already use some basic programmes such as MS-Office, Power Point, Presentation, Macromedia Flash, Movie Maker and other simple programmes that can be edited by the user. This is a group of multimedia users who can translate their teaching experience, methods and ideas into the multimedia environment, which allows them to adopt new approaches and teaching methods in a relatively simple user's environment. However, from the programme point of view, it is not a perfect solution but it has the advantage of universal application, the content can be quickly updated, it is interactive, generally understandable, can be easily distributed to other users including export to other information and communication systems.

It is important for an "amateur" developer of teaching tools to understand the key principles of creating a multimedia programme. In the easiest case it is the creation or transformation of graphical data, text or sound into digital format and their structuring and organisation according to a pre-defined scenario. This exercise does not require any deep technical understanding of ICT by the teacher, it is enough to be computer literate. The key interactive elements of the electronic teaching tool developed in this way are the hypertext links and use of various built-in functions such as the "if" function in Excel etc. This enables development of a template, which can later be filled with data resulting in an electronic teaching tool the format of which can be modified. In this way the teacher can achieve the intended objective, which is a creative approach to exercises on the part of the pupils.

This is an easy way to use "common" programmes in a less common manner and develop teaching materials, presentations, programmes, exercises, worksheets etc. The outcome is an electronic teaching tool using available software products (MS PowerPoint, MS Excel, MS Word, Macromedia Director, Macromedia Flash, Cabri Geometrie, Comenius Logo, Derive, SMath, off-line HTML etc.). This can help to bring a more personalised approach into maths teaching and increase the interest of pupils in maths by using modern methods of teaching and an innovative approach. Another important aspect is that everybody can work at a pace which suits him/her best when learning a new subject, developing logical thinking skills, creativity and the ability to work independently. It is also possible to use interactive links to other freeware products available in the Internet.

Using these teaching materials, and in some cases dedicated classrooms with interactive boards for example, makes it possible to replace the conventional frontal teaching methods with a more interactive approach, team work and co-operation. The software can be made available to pupils and teachers on the school website thus providing remote access to it. This will also improve the interdisciplinary relations between maths and other subjects and will help to fit maths better into everyday life.

When using "non-mathematical" programmes in maths teaching it happens that the direct solution must be by-passed and the approach that reflects the logics of the programme must be used, in other words, to push the computer to give the answers to the questions we are dealing with. It is possible to use many built-in functions, interlinks and process dynamics. Looking for alternative ways, a high level of visualisation and immediate response to changes in parameters - these are the advantages of using computers in teaching. Another efficient method is the development of teaching projects using HTML protocol for web presentations including off-line web pages in the form of e-books. At the moment this solution provides very good transferability and independence on the software platform and therefore it can be used at nearly all computers, as the Internet browser is a standard programme in every computer. Hypertext interactive navigation plays an important role in this solution. It is obvious that it is easy to use a multimedia presentation created in this way, it does not require any special computer skills and in combination with other methods, it can comprise more motivating elements, handle the traditional teaching contents in a more interactive way and present old "subjects" in a more modern fashion.

Another important element is the fact that addressing these issues can be done in collaboration with other colleagues, students from teaching colleges who can participate in the development of electronic teaching tools during their practical training at a school where they can co-operate with the relevant teacher or they can develop these topics in their graduation thesis.

4. Conclusions

The biggest issue in terms of the development of electronic teaching tools is the conservative mind-set of people-teachers and their personal belief in potential usefulness and advantages of multimedia and ICT as established by many years of research. Their frequent argument is: "Kids understand these things, but it is not for me, ... ", "I do not understand this technical stuff, ... ", "Computers steal time and are harmful to health, ... ", ".. chalk and blackboard will do for me... " etc. These arguments are natural as they strive to maintain the existing situation without any significant changes, to keep the existing status quo.

If an adult is systematically adopting computer skills and becoming familiar with ICT it is not a childish entertainment of clicking through, but it is a serious learning process aimed at mastering this kind of technology. The targeted and accurate communication of the project makes the difference that distinguishes them from an intuitive, memory based approach of children to this technology.

In the same way, communication materials where there is no mouse clicking can be developed as part of the teaching project in order to boost the pupils' creativity and thinking. The active involvement of pupils helps them gain a better understanding of mathematical principles, they can try everything, they are not only passively receiving the information but they experience the joy of "exploration", which is the biggest motivation. Inspiring teaching based on a constructive approach aims to develop the mathematical thinking of children through a wide range of mathematical activities. The teacher concentrates on the pupil, his/her understanding of relations and how the acquired knowledge is applied and used by the pupil. The teacher seeks for, and incorporates into teaching, such ideas that help the pupil with a better visualisation of the concepts and phenomena taught and understanding of the process and relationship.

It is obvious that this method develops pupils' creativity but it is also demanding in terms of teachers' personal involvement and creativity.

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THE USE OF TRIAL EXAMS RESULTS FOR COMPARISON OF CHANGES OVER 2005 AND 2006

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Abstract. A few years ago preparing and coneching cesed to be the responsibility of individual schools. The range of knowledge and skills written in the "Basis Program" and "Standards" have not changed since then. For obvious reasons exam papers are different every year. Hence, there is no way of obtaining answers for the changes that have been happening over a period of time. International PISA research showed that in Poland there was a major increase in knowledge and skills of fifteen year olds between 2000 and 2003 [5]. I have been conducting trial exams for six years in Łódź. For the first few years The College of Computer Science was the organizer, currently it is Academy of Humanities and Economics in Łódź [3,4]. Every year more than 2000 students attend these trial exams. The 2005 and 2006 exam papers were redone with some small changes. In the work I am going to analyse the results of these tests and apply to changes over 2005 and 2006.

1. Results comparison

The PISA research provides multiple results, long-term tendencies, measurement being one of them. Two surveys have were taken in Poland in 2000 and 2003. Thus, one can speak of the first recorded change.

The aim of this work is to ascertain if the tendency maintains in the succeeding PISA research.

The research was conducted in 2005 and 2006 in Łódź. Each year, over 2000 pupils from the third grade of junior high school took part in the research. The majority of schools partook in the research each year.

The following changes have been made in the tests: exercise 11 – utterly modified, exercise 14 – content corrected, exercise 32 – grading pattern changed.

The modified exercises have been excluded from further research.

Basic parameters:

Year	Average	Standard divergence
2005	26,04	7,43
2006	26,00	7,48

Fig. 1.

Fig. 2.

Figure 1 presents solvability of given exercises, whereas Figure 2 shows coefficient of correlation between the solution of given the exercises and the exam results. As the pictures reveal it, the difference between 2005 and 2006 is slight.

Fig. 3.

Figure 3 presents the exam grading pattern. The patterns are close to the general pattern and are not diverse.

13 middle schools (circa 70% of population) took the exam in both 2005 and 2006. Figure 4 presents middle schools' average scores in selected years. It is interesting, that each year the schools scored similar amount of points.

The coefficient of correlation of the middle schools' scores between 2005 and 2006 equals 0,83.

Fig. 4.

2. Sex and the exam results

The PISA research shows that in various countries there are differences in abilities to solve exercises between girls and boys. In Norway, Sweden, Finland, in the entire test girls got higher scores than boys. No such differences have been observed in Poland. Nevertheless, some differences are visible in the exercise level.

Figures 5 and 6 show coefficients of correlation between solving given exercises and the sex. (positive coefficient of correlation inclines toward boys, negative toward girls). In 2005 and 2006, exercise 3 was, statistically vital, solved better by boys and exercise 10 by girls.

The exercises' content was as follows:

Exercise 3 (0-1)

Two horses draw sledges with the speed of 10 m/s. How far will the sledge be drawn in the time of 5 minutes?

A. 50 km. B. 30 km. C. 3 km. D. 0,5 km.

Exercise 10 (0-1)

(The yogurt characteristic chart)

Which yogurt should Kate eat in order not to get on weight?

A. Yogurt A. B. Yogurt B. C. Yogurt C. D. Yogurt D.

Exercise 3 confirms that boys are much better in operating such concepts as speed and energy. The question why exercise 10 is better solved by girls does not require any comments.

Fig. 5.

Fig. 6.

3. Conclusion

The results presented in the work allow to state the hypothesis that the level of knowledge and abilities between 2005 and 2006 in Poland did not change. Therefore, the inhibition of growing knowledge level tendency's in the PISA research may be expected.

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GEOMETRICAL ACTIVITIES AS A TOOL FOR STIMULATING MATHEMATICAL THINKING OF 4-7 YEARS OLD CHILDREN

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Rationale

In the constructivist approach to teaching mathematics, great emphasis is put on the way how children use the language. Talking during lesson is perceived from two different perspectives: as the tool for communication (social function) and - as the tool for shaping and determining the thinking process. Talk is not a result of a fully developed thought - although is created through a course of word statement.

When leading my own research on geometrical intuition of 4 - 7 years old children I put a hypothesis that geometrical activities can be used as the tool for stimulating mathematical thinking in that group of children. Theoretical analyses and survey of literature supporting this hypothesis are as following:

[1] Following quotation from the main Vygotskian book Thinking and Speaking (1989, p. 333) summarizing one of important aspects of his theory about relation between thoughts and words: The child's thought, precisely because it is born as a dim, amorphous whole, must find expression in a single word. The child chooses a word as an eligible dress for his thought. As his thought becomes more differentiated, the child is less apt to express it in single words but constructs a composite whole. Conversely, progress in speech to the differentiated whole of a sentence helps the child's thoughts to progress from a homogeneous whole to well-defined parts.

[2] Teaching the mathematical thinking means - among others - teaching how to analyse some phenomena, how to perceive relations and properties. Those relations should construct the whole in the domain of

analysed situation, and if go beyond of its scope, they should build a structure, the net of general properties. In general, mathematical thinking consists of finding rules, perceiving a general in the particular.

[3] Theories about teaching geometry underlines the idea, that at the first level of geometrical knowledge a child perceives geometrical object as a whole. That idea is a base for van Hiele theory, in a similar way M. Hejny describes the "pre-concept level". So, the first geometrical thoughts are global, amorphous whole. On the next levels, attendants and relations start to exist as independent phenomena: the amorphous whole becomes more differentiated, more and more separated elements constitute whole.

[4] Small children' learning mainly consists of acquiring experiences by own activities. The special importances have for them all information, gathered by perception. This is a main reason why geometrical concepts, built by perception, are closer to child' abilities than arithmetical ones.

Methodology

The aim of my research was as following: to study children's language while they are solving geometrical tasks packed with relations and properties.

The research was lead in 2004 - 2007, among pre-school children at different ages. Teachers were told not to interrupt children's work. From our previous experience we know that very often a teacher wants to give children some "hints", but in fact - his/her remark only disturbs children's way of thinking.

Example 1.

Ania, (6 years old) draws the geometrical pattern. Before the work she prepares all felt-tip pens she needs and while drawing she chooses colors very carefully.
A teacher starts to talk with the girl during the work, bur she only answers on questions without any symptoms of willingness to say something.
Girl starts to work without any word. She is very concentrated on drawing.

[1] Teacher : what is this, this figure?
[2] Ania: (silence)
[3] Teacher: so, how many sides does it have? (she shows sides by her finger)
[4] Ania: (counts loudly): 1,2,3, - three - angles?

Examples of children' work

Analyzing children' work we can put hypothesis that their competence grows with age, both in drawing and in argumentation. I will illustrate this statement by examples.

4 years old children

Example 2. (Krystian, Michal)

T: I have prepared a pattern for you. Do you know, how to continue the drawing?

Krystian: A little person.

Michal: House.

T: How do you know it?

K: Because I know everything.

In this situation children take a space on a paper as a place for their own free creativity. They did not treat this task as a continuity pattern with regularities and relations between figures. They ignored information given by a teacher that they have "to continue"what is already began. Those children are not ready to perceive (and look for) regularities yet.
But the next dialog shows that the patterns can be used as a good tool for discovering regularities.

Example 3. (Krystian, Pawel, Grzes, 4 years old)

T: I prepared this pattern for you. Look carefully on it - do you know how to continue this drawing?

Krystian: In my book for painting I had an airplane.

Pawel: I know!! I can do it too!! I can do it too!

Grzes: How do you know it?

P: Because here it is shown (he points his finger at dotted figures).

The first boy - Krystian - is still thinking about "free drawing". But the other two boys are already in the world of regularities. For Pawel, the doors to this world have just opened: he discovered the relations, he understood the task, but he was not able to play in this game. The only what he could do was to repeat the motif (he coated the dotted line in the second motif). But his reaction (a big enthusiasm: I know!, I can do it too!!) shows that he was delighted by understanding the task. The member of the same group - Grzes, created the pattern where - in spite of many manual obstacles - a lot of relations between figures are retained.

5 years old children

In this group of children, the ability of talking about perceiving relation is on the higher level than in the group of 4 year old children. This is the group that more often utter some statements about patterns and the way how to continue it. They support their own utterances by gestures - this means that they used gestures as symbols for coding some relations between figures.

Example 4. Ola, 5 years old, Natalka 4 years old

T: I have prepared you a pattern. Do you know how to continue this drawing?

Ola: Big - small - big - small.

T: And how will you draw it?
O: [draws in the air by her finger the way how she will draw it]
T: And what does Natalka think about Ola's statement? Do you agree with Ola?
Natalka: [she nods that she agrees and she shows by the finger how she will draw]
T: So, how will you continue the drawing?
Na: Big - small - big - small

We can observe very interesting phenomena during this dialog. Natalka, younger than Ola, accepts arguments, vocabulary and gestures given by Ola and repeats them as her own. But in her work she is independent. She tries to depict the general structure of the pattern. She is critical towards her own work - she makes some corrections during drawing. Repeated triangles put in an "upside down"position show that this way of putting them was especially difficult for her. This phenomenon can be treated as an argument that although geometrical properties are within her reach, the verbal utterances about them are difficult.
For children, building their own, independent argumentation is difficult at this stage. It is shown in the net dialog

Examples 5. (Kuba, Wojtek, 5 yeasr old)
T: Do you know, how to continue?
Kuba: yes [Wojtek - silence]
T: So, how?
K: Triangles.
T: Could you tell me, how will you draw those triangles, Kuba?
K: Hey! I have an idea!! Triangle - triangle - circle - circle, can I do it like this?
T: And what do you think, Wojtek, about Kuba's idea?
Wojtek [disorientated] Do I have to continue by drawing circles?
K: No, the next is triangle and circle.
T: Do you agree with Kuba's opinion?
W: I agree.
T: But look carefully and say what do you think. What will you draw next?
W: Squares.

Kuba is a very creative boy. He discovered, that in this task some regularities exist, but he did not want to continue patter. He decided to make his own pattern and suggests to his colleague to work accordingly to his idea. This situation throws Wojtek off balance - he did not see any connection between this what he sees and this what Kuba says.
But this is not true that at this age children are not able to think independently and critically. These abilities are presented by the boy in the next dialogue.

Example 6. (Patryk, Bartek 5 year old)
T:I prepared a pattern for you. Do you know how to continue this pattern?
Bartek: I will draw like this [he creates a shape of triangle by the finger in the air]
T: And next?
B: [he "drawsą small triangle in the opposite position]
T: And next?
B: [he shows a big one]
T: And next?
B: [he shows the small one, in the opposite position]
T: And what is Patryk's idea?
Patryk: I will draw one like this - one like that- one like this - one like that [he shows by fingers elements from the motif]
T: And how do you know Patryk how to draw?
P: Because it is shown.
. . . .
B: [he looks at the Patryk's work] You have to make the same angles, not so big ones [he shows a big triangle, because Partyk draws big triangles first]
B: I will join those triangles [he draws big triangles first, after that he puts the small ones]

6 years old children

Utterances in this group of children are more accurate. They express more details and relationships. The argumentation is also more exact and independent. This is an example:

Example 7. (Iza i Ola - 6 years old)
T: Do you know how to continue the drawing?
Iza: yes
Ola: yes
T: Could you say Iza, what will you draw and how?
I: I will draw it like this [she shows the pattern prepared by the teacher]
N: And Olga?
O: Here one square and upwards, and here the next square and downwards . . .
I: Following the pattern . . .
O: Because here are these dots so follow the pattern......

Summary

The process of creation of the argumentation by 4-7 years old children in the geometrical environment consists of a few levels:

- Perceiving regularities

- Realizing regularities
- Verbalizing regularities.

Words are representations of concepts and ideas. Using words requires an intellectual effort as well as while using symbols. Geometrical symbol is "in the middle"between an abstractive symbol and a real thing. Geometrical figures, at the early stage of its understanding, can be treated as a not-finished picture of a real object. A square, even when drawn very correctly, can be interpreted by child as a ńot-finished window"(Hejny, 1993) but as a symbol it can be used very correctly in communication. Parallelity, as a relation between two objects, is perceived both by a child and by a mathematician, although a mathematician would connect the eligible sign with the abstract idea and a child would say: "two sticks ale lying equally". Gruszczyk-Kolczyńska writes: "the process of coding and en-coding in teaching mathematics starts from a very high level of abstraction and requires skills in operational reasoning on the real level (p. 19). In my opinion, this statement relates to arithmetical formulas. Geometry as rhythms and patterns gives a chance to code and encode rules and formulas at a very low level. Of course, it is not an easy way. The passage from perception, through geometrical symbolic representation to verbal mathematical description is very long, and often strange to children. Sometimes, the moment a child sees the pattern, it comments on it in a spontaneous way. The child says: it is easy, I can do it without any problem. After correct drawing, during a talk with a teacher he or she reacts: why do you ask me: how do you know how to draw it? - it is shown, it is obvious. But their effort they put in forming the utterances that describe the rule of drawing, has a very import didactical meaning. Children start to transform these relations and connections into words. Previously, they realized them without words, often in non-conscious way. By the talk, these relations gain a status of existence. They emerge gradually from experience and start to be the facts related to the mathematical world.

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USING NON-GRAPHICAL PROGRAMS FOR TEACHING MATHEMATICS (SPECIALIZED FOR GEOMETRY)

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Abstract. The paper deals with the possibilities of drawing the figures with program Microsoft Office Word 2003. Some advantages and disadvantages of using graphical and non-graphical programs for educational goals are mentioned. The built-in automatic shapes in program MS Office Word 2003 are described and all presented figures of geometric objects are created in it.

1. Introduction

There are many ways and possibilities for drawing pictures and figures today. Some specialized graphical programs allow drawing anything what user needs and wants. But it is often very difficult to master their tools and they are not accessible to everyone.

On the other side, there are non-graphical programs, which are accessible for most of users. They have often graphical environment with tools and shapes, which allow drawing figures suitable for teachers not only of mathematics but also of descriptive geometry. From the variety of programs for teachers, Microsoft Office Word has been chosen.

2. Drawing with program Microsoft Office Word 2003

At the beginning it is necessary to mention, that the program Word was not designed for precise graphic work, but many of its tools allow to draw quite accurate and complicated figures too. In Word, one can draw only with help of built-in automatic shapes and with using graphical tools. The advantage of such created figures is the step by step construction, which is for many users very intuitive and resembles using classic drawing instruments – ruler and pair of compasses.

Another advantage becomes apparent by changing the figure size – the thickness of lines remains unchanged. This does not happen with regular figures inserted into the document from other graphical programs. This advantage has sense for user, who wants to print the document. Another one is that these created figures can be easily used also in presentation where the construction is presented step by step.

Some disadvantages of using Word to create figures are: time-consu-ming work, precision complications (offered 500% zoom is not sufficient), division of simple constructions into more details (point in exact distance, midpoint, circle with exact radius and midpoint ...) and other.

Of course, combination of graphical programs and Word can be also used. The figure can be drawn at first in graphical program and then redrawn in Word. Sometimes the conversion of the figures from graphical programs to Word can be used for creating figures too, if there is such possibility (some graphical programs don't support this way).

3. Drawing tools in MS Word 2003

All the tools for creating figures can be found in tool bar *Drawing* (Fig. 1).

Fig. 5: Tool bar *Drawing*

There are submenus *Draw* with graphic tools (Fig. 2), *automatic shapes* (Fig. 3), *other objects*(text box, WordArt, diagram or organization chart, Clip Art, picture) and also their characteristics (fill color, line color, text color, thickness of line, type of line, shape and size of arrows, shadow and space effects of object).

In the submenu *Draw*, there are functions for grouping, ungrouping or regrouping of objects, setting the order of objects, setting the grid, aligning and distributing of objects, rotating and flipping of objects, setting the wrapping of text and other.

In the submenu *Automatic shapes*, there are functions for drawing many types of lines, connectors, basic shapes, arrows, diagrams, stars, callouts and possibility of choosing other automatic shapes too.

The *Shortcut menu* is often used for drawing automatic shapes. It can be displayed after right clicking on the object, or on the set of objects. After clicking on the item *Format AutoShape* (Fig. 4) other properties of objects can be set up. For example it is their size, layout and so on.

Figure 3

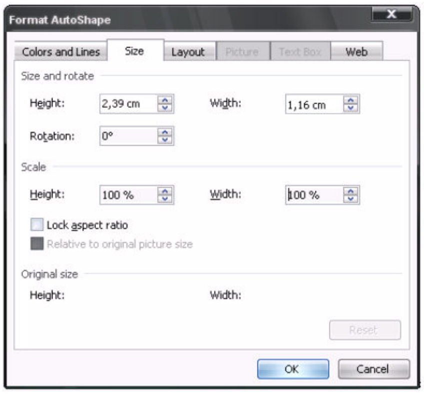


Figure 4

4. Examples and sequences of drawing figures

First of all, some recommendations:

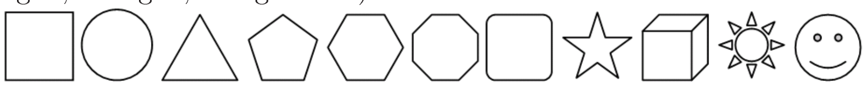
- If the precision is important, it is good to draw figures with lines with thinnest thickness (0,25 pt, or it can be chosen also thickness 0 pt which is printable) with maximal zoom 500% (zoom can be modified with mouse wheel with key Ctrl).
- Closed objects and shapes are drawn without fill.
- If some parts of construction are repeated, then it is useful to change colors of objects.
- For creating new objects with the same setting the tool *Set AutoShape Defaults* can be used.

For creating and modifying objects the keys "*Shift*", "*Ctrl*"and "*Alt*"are used very often. They make the work easier. The demonstration of using these keys follows. For visualization of the changes, the original object (shape) is drawn with continuous line and the new one, which will be created by pulling at the handle of object with use of the mentioned keys, is drawn with dashed line:

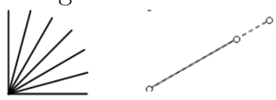
KeyŚhift":

This key can be used for preserve proportion. Some concrete examples:

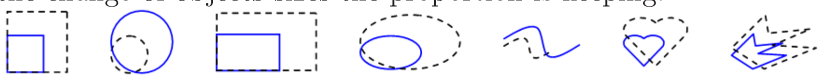
- Drawing square, circle and regular figures (equilateral triangle, regular pentagon, hexagon, octagon …).



- The new line is drawing with multiples of 15° angle and already drawn line is lengthening without change of direction.



- Objects are rotating with multiples of 15° angle around the midpoint.
- With the change of objects sizes the proportion is keeping.

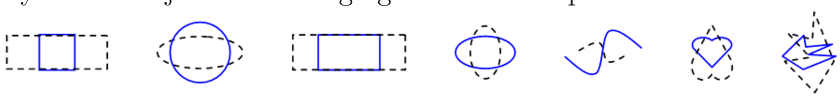


- Objects are moving only in vertical or horizontal direction.
- More objects can be marked at once.

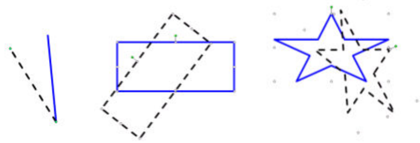
KeyĆtrl":

Most frequently this key is used for copying objects, but there are other possibilities for using it. Some examples:

- New objects are created from the midpoints.
- Already drawn objects are changing from the midpoints.



- Copying the objects only with moving.
- Objects are rotating around the handle opposite to rotate handle.



KeyĄlt":

It is used for work with grid. It causes snapping to grid even when it is turned off.

Of course, the keys can be combined and other shapes with special properties can be created.

Some examples of creating geometric figures with characteristic properties with using mentioned keys and graphical tools:

Parallel lines can be simply drawn by copying (Ctrl).

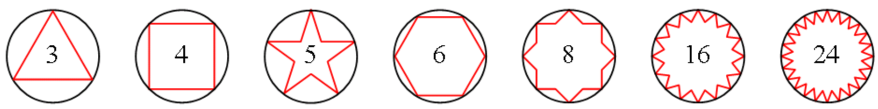
Perpendicular lines can be drawn with the tool *Rotate right/left* 90° (*Draw* → *Rotate or flip* ...).

Circle with the midpoint can be drawn only with the automatic shape *Oval* starting from the midpoint (Ctrl+Shift).

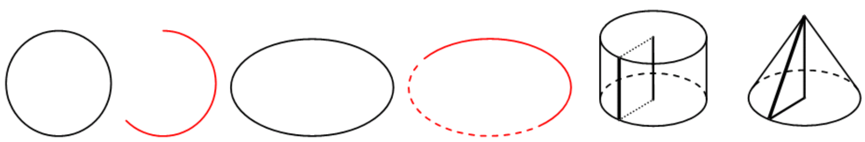
Axis of the segment can be drawn also in this way: At first, the original line is copied to the clipboard (Ctrl+C). Then it is rotated by 90 degrees (Shift) and then the original line is pasted (Ctrl+V). **The midpoint of the segment** can be created in the similar way.

Centrally and axially symmetrical shapes can be drawn with tools *Flip vertically/horizontally* (*Draw* → *Rotate or flip* ...).

Circle can be **divided into equal parts** using regular figures or stars.



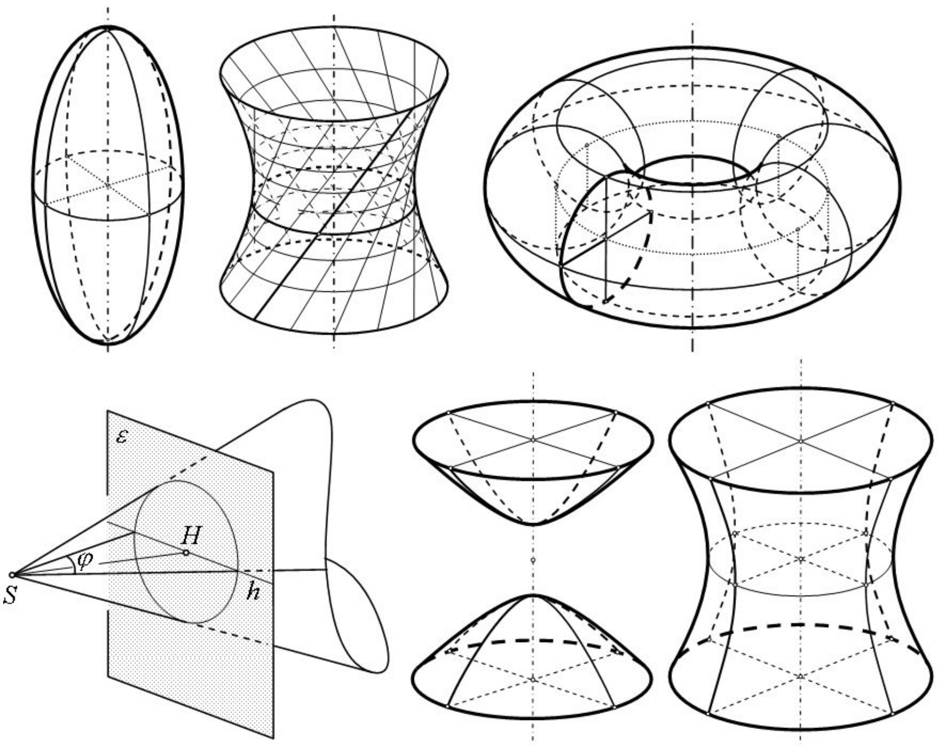
Parts of the circle or **ellipse** can be drawn using the shape *Arc*.



Conic sections (except ellipse) and other **curves** can be approximately drawn with automatic shape *Curve* with various modifications of its main points. The modification can be called from *Shortcut menu* after clicking on *Edit points* and then successive right clicking on single points of the curve. There are several available point characteristics: *Automatic*, *Smooth* (Fig. 5 – tangent vectors have the same direction and length), *Straight* (Fig. 6 – tangent vectors have the same direction but different length) and *Corner point* (Fig. 7 – tangent vectors have different direction and different length) and the shape of the curve can be modified by tangent vectors on these points.

Figure 5:	Figure 6:	Figure 7:
Smooth point	*Straight point*	Corner point

Some other pictures drawn in Word:



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WHY LOGARITHMS?

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Abstract. In 16th and 17th century, the need for speed in complex calculation spurred the invention of a powerful mathematical tool known as LOGARITHM. The reduction of multiplication and division to addition and subtraction (likewise the reduction of a complex mathematical structure to more simple ones) is in the spirit of "prosthaphaeretic rulesóf ancient Greeks. We discuss some mathematical ideas related to logarithms and present some historical notes.

Rarely in the history of science has an abstract mathematical idea been received more enthusiastically by the entire scientific community than the invention of logarithms. The sixteenth and early seventeenth centuries saw an enormous expansion of scientific knowledge in every field. Discoveries in geography, physics and astronomy, rapidly changed man's perception of the universe: Copernicus's heliocentric system, Magellan's circumnavigation of the globe in 1521, the new world map published in 1569 by Gerhard Mercator, inventions and needed new knowledge in numerical computation, in formulated new physics laws, for example mechanics (Galileo Galilei) and astronomy (Johannes Kepler, his three laws of planetary motion).

These developments involved an ever increasing amount of numerical data, forcing scientists to spend much of their time doing numerical computation. The times called for an invention that would free scientists once and for all from this burden. Napier took up the challenge. We have no account of how Napier first stumbled upon the idea that would ultimately result in his invention. He was well versed in trigonometry, e.g. he was familiar with

$$\sin A \sin B = \frac{1}{2}\left[\cos\left(A - B\right) - \cos\left(A + B\right)\right]$$

This formula and similar ones for $\cos A \cos B$ and $\sin A \sin B$ are known as the *prosthaphaeretic rules*, from the Greek word meaning ąddition and subtraction". Their importance lays in the fact that the product of two trigonometric expressions such as $\sin A \sin B$ could be computed by finding the sum or difference of other trigonometric expressions, in this case $\cos(A-B)$ and $\cos(A+B)$. Since it is easier add and subtract than to multiply and divide, these formulas provide primitive system of reduction from one arithmetic operation to another, simpler. Roughly, this was originally the idea of John Napier.

1. Invention of logarithms

This idea is better illustrated for the terms of a geometric progression, i.e. a sequence of numbers with a fixed ratio between successive terms. For example, the sequence $1, 2, 4, 8, 16, \ldots$ is a geometric progression with the common ratio 2. If we denote the common ratio by q, starting with 1, the terms of progression are $1, q, q^2, q^3, \ldots$ (note that the n-th term is q^{n-1}).

Long before Napier's time, it had been noticed that there exists a simple relation between the terms of a geometric progression and the corresponding exponents. Nicola Oresme in his book *De proportionibus proportionum* generalized some rules for combining proportions in year 1360. Nowadays expressed as $q^m q^n = q^{m+n}$ and $(q^m)^n = q^{mn}$. These relations are exactly formulated by Michael Stifel in his book *Arithmetica integra* as follow : "*If we multiply any two terms of the progression* $1, q, q^2, \ldots,$ *the result would be the same as if we had added the corresponding exponents... dividing one term by another term is equivalent to subtracting their exponent.*"

The problem arises, if the exponent of the denominator is greater than that of the numerator. To get around this difficulty, we simply define $q^{-n} = \frac{1}{q^n}$ and $q^0 = 1$, so that $\frac{q^3}{q^5} = q^{-2} = \frac{1}{q^2}$ and $\frac{q^3}{q^3} = q^0 = 1$. With this definition, we can extend a geometric progression indefinitely in both directions, $\ldots, q^{-3}, q^{-2}, q^{-1}, q^0, q^1, q^2, \ldots$. Each term is a power of the common ratio q, and that the exponents $\ldots, -3, -2, -1, 0, 1, 2, \ldots$ form an arithmetic progression. This relation is the key idea behind logarithms. Stiffel had in mind only integer values of the exponent, but Napier's idea in his book *Mirifici Logarithmorum Canonis descriptio* was to extend it to a continuous range of value's .

His thoughts proceeded as follows: If we could write any positive number as a power of some given fixed number then multiplication and division of positive numbers would be equivalent to addition and subtraction of their exponents. We illustrate the idea with number 2 as the base.

Suppose we need to multiply 16 by 64. We look in the table for the exponents corresponding to 16 and 64 and find them as 4 and 6. Adding these exponents gives us 10. We now reverse the process, looking for the number

n	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10
2^n	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	256	512	1024

whose corresponding exponent is 10. This number is 1024 and we have the desired answer. As the next example, suppose we want to find 4^4. We find the exponent corresponding to 4, namely 2 and this time multiply it by 4 to get 8, then look for the number whose exponent is 8 and find it to be 256. And indeed $4^4 = \left(2^2\right)^4 = 2^8 = 256$.

Of course, such an elaborate scheme is unnecessary for computing strictly with integers. But, for this to happen, we must first fill in the large gaps between the entries of our table. We can do this in one of two ways:

- using fractional exponents
- choosing for the base a number small enough so that its powers will grow reasonably slowly.

2. Napier logarithms

Napier choose the second option. A question arose, how to choose the base so that a change of exponent causes a small change of powers of the base. It seems that it should be a number close to 1. Napier decided $0,9999999$ or $1-10^{-7}$. Napier spent twenty years of his life to complete the task, that it will do the job. His initial table contained just 101 entries, starting with 10^7 and followed $10^7\left(1-10^{-7}\right) = 9999999$, then $10^7\left(1-10^{-7}\right)^2 \doteq 9999998$ and so on up to $10^7\left(1-10^{-7}\right)^{100} \doteq 9999900$. The difference between two sides is only $0,000495$, which we neglect. Each term being obtained by subtracting from the preceding term its $0,9999999 \doteq 10^7\left(1-10^{-7}\right)^{i+1} - 10^7\left(1-10^{-7}\right)^i = \left(1-10^{-7}\right)^i$. He then repeated the process again, starting once with 10^7, but this time taking as his proportion the ratio of the last number to the first in the original table, that is $\frac{9999900}{10000000} = 0,99999$ or $1-10^{-5}$. This second table contained 51 entries. The first was $1-10^{-7}$, followed $10^7\left(1-10^{-5}\right) = 9999900$, the last being $10^7\left(1-10^{-5}\right)^{50} \doteq 9995001$. A third table with 21 entries using the ratio $\frac{9995001}{10000000}$, the last entry in this table was $10^7\left(0,9995\right)^{20} = 9900473$. Finally, from each entry in this table Napier created 68 additional entries using the ratio $\frac{9900473}{10000000} \approx 0,99$ and the last entry then was $10^7 0,99^{68} \approx 4998609$, it is roughly half the original number.

In modern notation, this amounts to says that if $N = 10^7\left(1-10^{-7}\right)^L$, then the exponent L is the Napier logarithm of N. Napier's definition of logarithms was different in several respects from the modern definition (introduced in 1728 by Leonhard Euler). If $N = b^L$, where b is a fixed positive number other than 1, then L is the logarithm (with the base b) of N. Hence

$L = 0,\qquad$ Nap log $10^7 = 0$

$L = 1,\qquad$ Nap log $9999999 = 1$, etc.

Notice that the computations using Napier logarithm are more complicated then those using the modern logarithms. for example :

If $L_1=$ Nap log N_1, hence $10^7 N_1 = 10^7\left(1-10^{-7}\right)^{L_1}$ and

$L_2=$ Nap log N_2, hence $10^7 N_2 = 10^7\left(1-10^{-7}\right)^{L_2}$,

then $\frac{N_1N_2}{10^7} = 10^7\left(1-10^{-7}\right)^{L_1+L_2}$, $nL_1=$ Nap log $\frac{N_1^n}{10^{7(n-1)}}$.

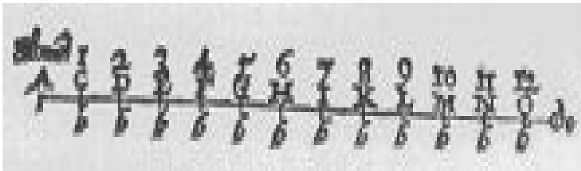
Relationship between the modern logarithm and the Napier logarithm :

Nap log $N = \frac{\ln \frac{N}{10^7}}{\ln\left(1-10^{-7}\right)}$,

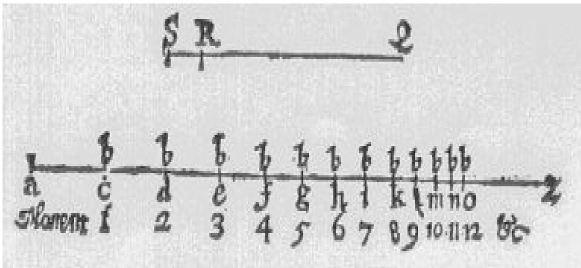
Nap log $N_1N_2 =$ Nap log N_1+ Nap log $N_2+\frac{\ln 10^7}{\ln\left(1-10^{-7}\right)}$.

3. Geometric definition of the Napier logarithm

The principles of his work explained in geometric terms have been presented in first in the article about logarithms: *Mirifici Logarithmorum Canonis descriptio* Assume that a point P moves along ACZ (starting from A) with the



uniform speed B. Now, in the first moment, let P move from A to C, in the second moment from C to D, etc. Let SQ be a line segment and let AZ be a halfline, see picture.



Let a point R start from S and move along SQ with variable speed decreasing in proportion to its distance from Q. During the same time let a point P start from A and move along ACZ with uniform speed. Napier called this variable distance AP the logarithm of the distance RQ.

"The Logarithme therfore of any sine is a number very neerely expressing the line, which increased equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed and the beginning equally swift."

Napier's geometric definition is, in agreement with the numerical description given above. Let |RQ|$=x$ and |AP|$=y$. If |SQ| is taken 10^7 and if the initial speed R is also taken 10^7, then in modern calculus notations we have $\frac{dx}{dt} = -x$ and $\frac{dy}{dt} = 10^7$. The initial boundary condition are $x_0 = 10^7$ and $y_0 = 0$. Then $\frac{dy}{dx} = -\frac{10^7}{x}$ or $y = -10^7 \ln cx$, where c is found from the initial condition $c = 10^{-7}$. Hence $y = -10^7 \ln \frac{x}{10^7} = 10^7 \log_{\frac{1}{e}} \frac{x}{10^7}$

4. Conclusion

The creation of the idea of logarithm by Napier (connection greek's words $\lambda o\gamma o\varsigma$ - ratio, $\alpha\rho\iota\tau\mu o\zeta$ - number) is a revolutionary milestone in history. His invention was quickly adopted by scientists all across Europe and even in faraway China. Henry Briggs (professor of geometry at Gresham College in London) was impressed by the new invention and has said to Napier : *"My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz the logarithm ..."*

At that meeting, Briggs proposed two modifications that would make Napier's table convenient

to have the logarithm of 1, rather than of 10^7, equal to 0 and

to have the logarithm of 10 equal an appropriate power of 10.

The beauty and the power of logarithm can be presented for example by computing $x = \sqrt[3]{\frac{493{,}823{,}67^2}{5{,}104}}$. To perform the computation it suffices to use "the table of logarithms from 10 to 52".

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STOCHASTIC GRAPHS AND THEIR APPLICATIONS

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Abstract. The article deals with one example of so called stochastic graph. The paper demonstrates some of possible applications of stochastic graphs in practice using the well known example about seven brigdes of the town of Königsberg.

In Figure 1 is a schematic city plan of Königsberg with seven bridges over the river of Pregel. In 1736 L. Euler showed that it is not possible to cross all seven bridges without crossing one of the bridges twice. Since then has been this task used as a motivation and introductory problem to graph theory. We use this problem differently.

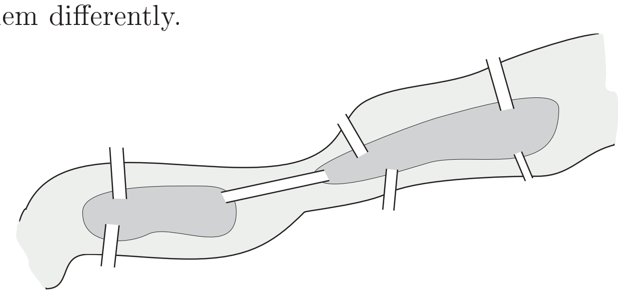


Fig. 1.

Let's suppose that spring flooding can destroy each of the bridges (independent on the others) with probability $0 < p < 1$. What is the probability that it is possible to cross the river of Pregel during the spring flooding?

*The paper was financially supported by MSM 6007 665 806 and GA CR 402/07/P209

We can approach the problem in various ways.

1. In total, there are seven bridges. Each of them will either śurviveór will be destroyed. (i.e. there are only two possibilities). Taking into account the independence, we obtain $2^7 = 128$ situations in total. We can determine for each of the situation whether it is possible to cross the river and also the probability of its happening. The resulting probability is equal to the sum of partial probabilities. This method is more time-consuming with the increasing number of bridges.

2. The situation in the city can be illustrated by the following schema:

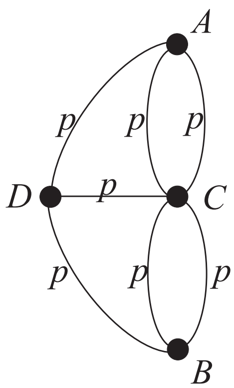


Fig. 2.

Let's denote
AB - event of crossing the river (i.e. from point A to point B).
$\overline{AB}$ - event when it is not possible to cross the river.

We will take advantage of the following characteristics of dependent probability

$$P(AB) = P(AB \mid Y) \cdot P(Y) + P(AB \mid \overline{Y}) \cdot P(\overline{Y}). \qquad (0.1.0.1)$$

As event Y we choose the event of destruction of bridge 7.
We obtain two disjunct cases illustrated by Figures 3 and 4:

destruction of bridge 7

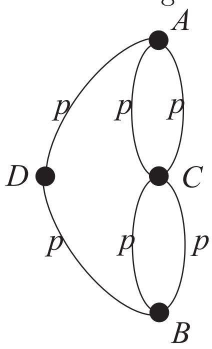


Fig. 3.

survival of bridge 7

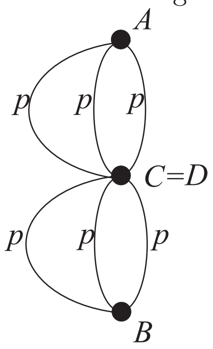


Fig. 4.

Set
ACB - event of going from point A through point C to point B
ADB - event of going from point A to point B through D .

In case of Figure 3 for $P(AB \mid Y)$ it is true that:

$$P(AB \mid Y) = P(ACB) + P(ADB) - P(ACB \cap ADB).$$

As $P(ACB) = P(AC) \cdot P(CB)$ using 1a), b) we obtain $P(AC) = P(CB) = 1 - p^2$ and $P(ADB) = (1-p)^2$. from which flows that

$$P(AB \mid Y) = (1-p^2)^2 + (1-p)^2 - (1-p^2)^2 \cdot (1-p)^2 =$$

$$= 1 - 4p^3 + 2p^4 + 2p^5 - p^6.$$

Considering situation in Figure 4, it is true for $P(AB \mid \overline{Y})$ that:

$$P(AB \mid \overline{Y}) = P(AC) \cdot P(CB).$$

Using 1a) we obtain $P(AC) = P(CB) = 1 - p^3$. This results in

$$P(AB \mid \overline{Y}) = (1-p^3)^2.$$

As $P(Y) = p,\ P(\overline{Y}) = (1-p)$ we can substitute into (1) and then we obtain

$$P(AB) = \left(1 - 4p^3 + 2p^4 + 2p^5 - p^6\right) \cdot p + (1-p^3)^2 \cdot (1-p),$$

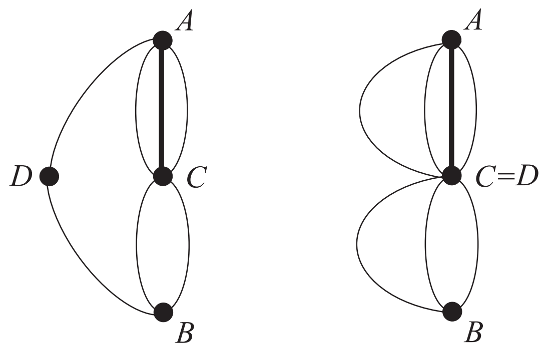
resulting in polynom

$$P(AB) = -2p^7 + 3p^6 + 2p^5 - 2p^4 - 2p^3 + 1.$$

For concrete values of parameter p, it is easy to determine the probability of the possibility to crass the river.

Let's talk about a new situation when we can built a new bridge connecting either points AC, AD, or CD. It is obvious that if we build the new bridge at any of the places, it will be possible to cross all the bridges just once. Now, the task is where to build the bridge to increase the possibility of crossing the river during the flooding to maximum.

1. The new bridge connects points A and C
   Like in previous case, we use (1) to obtain two disjunct cases (first one with probability p, the second one with probability $(1-p)$)



$$P_1(AB \mid Y) = (1-p^3)\cdot(1-p^2) + (1-p)^2 - (1-p^3)\cdot(1-p^2)\cdot(1-p)^2,$$

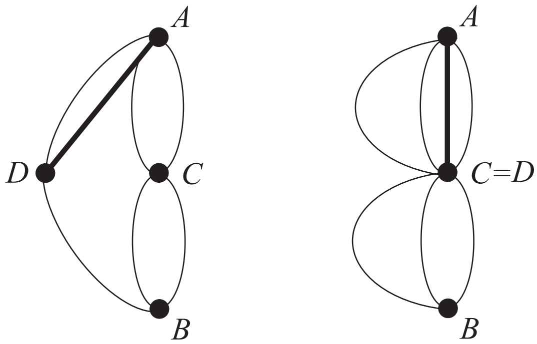
$$P_1(AB \mid \overline{Y}) = (1-p^4)\cdot(1-p^3)$$

resulting in

$$P_1(AB) = \Big((1-p^3)\cdot(1-p^2) + (1-p)^2 - (1-p^3)\cdot(1-p^2)\cdot(1-p)^2\Big)\cdot p +$$

$$+\Big((1-p^4)\cdot(1-p^3)\Big)\cdot(1-p) = -2p^8 + 3p^7 + p^6 - 2p^4 - p^3 + 1.$$

2. The new bridge connects points A and D
   Like in previous case, we use (1) to obtain two disjunct cases (first one with probability p, the second one with probability $(1-p)$)



$$P_2(AB \mid Y) = (1-p^2)^2 + (1-p^2)\cdot(1-p) - (1-p^2)^2\cdot(1-p^2)\cdot(1-p),$$

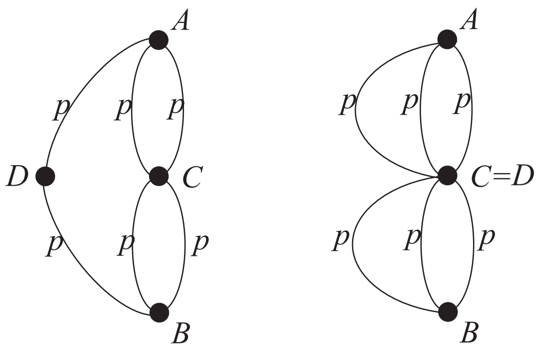
$$P_2(AB \mid \overline{Y}) = (1-p^4)\cdot(1-p^3)$$

resulting in

$$P_2(AB) = \Big((1-p^2)^2 + (1-p^2)\cdot(1-p) - (1-p^2)^2\cdot(1-p^2)\cdot(1-p)\Big)\cdot p +$$

$$+\Big((1-p^4)\cdot(1-p^3)\Big)\cdot(1-p) = -2p^8 + 2p^7 + 3p^6 - p^5 - 2p^4 - p^3 + 1.$$

3. The new bridge connects points C and D
   In this case, the first situation happens with probability of p^2, the second with probability of $1-p^2$



$$P_3(AB \mid Y) = (1-p^2)^2 + (1-p)^2 - (1-p^2)^2\cdot(1-p)^2,$$

$$P_3(AB \mid \overline{Y}) = (1-p^3)^2$$

resulting in

$$P_3(AB) = \Big((1-p^2)^2 + (1-p)^2 - (1-p^2)^2 \cdot (1-p)^2\Big) \cdot p^2 +$$
$$+\Big((1-p^3)^2\Big) \cdot (1-p^2) = -2p^8 + 2p^7 + 3p^6 - 2p^5 - 2p^3 + 1.$$

Let's compare first $P_1(AB)$ and $P_2(AB)$:

$$P_1(AB) - P_2(AB) = p^5 \cdot (1-p)^2 \geq 0.$$

Now, we compare $P_2(AB)$ and $P_3(AB)$:

$$P_2(AB) - P_3(AB) = p^3 \cdot (1-p)^2 \geq 0.$$

Now (for completeness), we compare $P_1(AB)$ and $P_3(AB)$:

$$P_1(AB) - P_3(AB) = p^3 \cdot (p^2+1) \cdot (1-p)^2 \geq 0.$$

It is obvious that it is most effective to build the bridge between points A and C.

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CLASSIFICATION OF PRIMARY SCHOOL TEACHERS ACCORDING TO THEIR ATTITUDES TO ICT EDUCATIONAL IMPLEMENTATION

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Abstract. The contribution deals with preliminary results of *Attitudes towards Computer Assistant Teaching* (abbr. as ĄTCAT") research into the issue of educational implementation of ICT tools in the primary school context. In this respect, I focus on primary mathematics. The model of teacher development with respect to the level of ICT implementation (the ACOT research) is given as a theoretical base of the presented classification of teachers according to their attitudes to utilising computers when teaching primary mathematics.

Computers have become a common part of our everyday life. It has become obvious that individual success in the global information society will depend on the ability to obtain, analyze and utilise information rather then on the amount of actual factual encyclopaedic knowledge. School as an integral part of the society should reflect current requirements of the society and prepare individuals in the framework of the information society which is being formed. New economic conditions should initiate the transformation of "traditional school"into a modern teaching environment based on the new humanism philosophy. In order to fulfil all the above-mentioned requirements, the school has to implement ICT tools not only as a natural component of its educational environment but also as a tool of transition from a transmisive concept to constructivist concept. ICT implementation into educational reality is not an isolated process equivalent to providing schools with computers. Technology should not be the aim but a didactic tool and educational environment encouraging change. We focus on the personality of the teacher, who is – according to V. Spilková (*Spilková, 1997*) the initiator of changes. Educational implementation of ICT tools is a long-term innovative process.

The changes as well as implementation of new technologies can be successful only on condition that they are willingly accepted by the majority of teachers. Only then, as Žilková points out (*Žilková, 2006*) can successful ICT integration into teaching mathematics be accompanied by a relevant and efficient level of use.

The general features of the innovative process were described by E. Rogers (*Rogers, 1995, Brdička, 2004*) in the beginning of 1960s. Rodgers was a prominent diffusionist. His theory of the progress of innovation acquisition in society was created in order to describe cases of implementing radical inventions such as the ancient invention of the process of iron production or the use of mobile telephones. He suggested that the spread of new inventions or solutions is made possible by spreading information via various channels. The innovative process can be successful only when suitable conditions are present. Even then the process is a jump one not a gradual one. At first, the innovation is adopted by a handful of enthusiast, whose risk in case of failure is great. They face general scepticism, indifference and lack of understanding. Should the innovation be successful, the number of those who adopt it must rise until a critical amount is surpassed. The process then becomes irreversible. In theory, every single adopter experiences five phases: discovery, interest, trial, decision, and adoption.

Rogers (*Brdička, 2004*) categorises the "innovative process adopters"into five categories depending in which phase the adoption occurred: innovators – enthusiasts (2,5%), early adopters – visionaries (13,5%), early majority – pragmatists (34%), later majority – conservatives (34%) and laggards - skeptics (16%). The innovation process (i.e. a number of teachers – adopters as a function of time) has been graphed in the Cartesian system (see Fig. 1) using the S-curve.

In the early 1990s Rogers' theory was applied on the issue of educational implementation of computers in the framework of the *American Apple Classrooms of Tomorrow* research (abbr. as ĄCOT") (*Haymore, Ringstaff, 1990*). A diffusion model of the teacher development with respect to the level of ICT implementation was suggested. Four phases are recognised: *survival*, *mastery*, *impact* and *innovation*.

The first phase – *survival* – is connected to implementation of new standards of qualification, which are usually very explicit about the necessity of information literacy of teachers. Facing existential threats, teachers not qualified in this respect try to overcome the obstacle themselves. They usually fight mastering basic operations, often using the trial – error method. As knowledge of technology grows, the phase of *mastery* connected with mastering the technical aspects of work with computer comes. The use of computers spreads and the efficiency of use rises. This is the phase of new strategies acquisition and better teaching models implementation. The dependence on IT specialists decreases. Teachers at this level use technologies in an instructive way. In the *impact* phase the orientation of teachers shifts to the pupil. Technologies are not an aim but a tool, which is commonly used in many educational activities of teachers. The teacher seeks the most effective way of using the technologies in his teaching methods. Instructive attitudes are supplemented by constructive ones. Finally, (some) teachers reach the diffusion phase, i.e. the phase of full *innovation*. Such teachers are able to adjust the curriculum (subject syllabi) as well as teaching methods and are able to reach beyond the generally given teaching aims.

Fig. 1. S-curve of innovation phases.

The process of the teacher's personality development is a complex one. It is an interrelated total which is closely connected to the level of teacher's competence reached in the ICT area. The process of the teachers' development includes development of their knowledge, skills, attitudes and understanding so that they could handle a number of complex decision making situations in education.

The *Attitudes towards Computer Assistant Teaching* research (abbr. as ĄTCAT") was carried out in the winter term of 2005/6 as a part of the FRVŠ grant *Multimedia in teaching mathematics at the primary school level*. Preparing the primary school teachers typology with respect to their attitudes towards using computers in teaching mathematics was one of the research aims. Total of 148 respondents – 3rd, 4th and 5th class teachers – were questioned. The research had a form of a questionnaire research with a semantic differential method.

We used cluster analysis (*Chráska, 2004*) in order to identify the basic types of teachers according to their attitudes to using computers in teaching primary mathematics. We built on the evaluation of average assessment factors and energy of individual respondents with respect to the concept indicator B: *The computer and I* in teaching mathematics. The cluster analysis resulted in assigning a given number of clusters to individual teachers. Each cluster was characterized by an average assessment factor and an average energy factor. The cluster analysis was carried out for 2, 3 and 4 clusters consecutively. When doing so we observed the way students are differentiated by the given number of clusters. The 3–cluster analysis seemed the most suitable. The obtained results are included as Tab. 1 and Fig. 2.

Description statistics – 3 clusters									
	cluster 1 – A 32 cases			cluster 2 – B 80 cases			cluster 3 – C 36 cases		
fact.	*avg*	*SD*	*D*	*avg*	*SD*	*D*	*avg*	*SD*	*D*
B_e	5.33	4.11	0.53	4.11	0.53	0.28		0.78	0.61
B_h	3.39	4.56	0.47	4.56	0.47	0.22	6.01	0.58	0.34

Tab. 1. Cluster analysis results.

Fig. 2. Cluster analysis results.

Type A (cluster 1, $n_1 = 32$) Type A teachers are characterized by a low assessment of benefit of computer use in teaching mathematics and an opinion that using computers in teaching implies great amount of energy, i.e. it is demanding and implies overcoming obstacles. Most of Type A teachers are probably early users, for which even their own work as users is difficult.

Type B (cluster 2, $n_2 = 80$) Type B teachers see certain sense and benefits in incorporating IT into teaching primary mathematics. They give an ąveragełevel of assessment and an ąveragełevel of energy. Most likely, these are teachers who ądoptedćomputers as a useful tool. However, applying computers in teaching is still quite difficult for them.

Type C (cluster 3, $n_3 = 36$) Type C teachers are aware of possible benefits of computers in teaching primary mathematics. They give the best assessment. Type C teachers are probably advanced computer users as the educational implementation of computers does not imply any relevant rise of energy for them.

When interpreting the results we tried to find out whether and to what extent our typology corresponded to the above given "model of teacher development with respect to the level of ICT implementation". Having studied respondent reactions, we believe that phases typical of primary school teachers are survival and mastery while impact and innovation are only a perspective to come. The survival phase corresponds to Type A teachers. Such teachers fight mastering basic operations, often using the trial – error method. As Brdička (*Brdička, 2002*) points out, using computers is often ą necessary condition of survival at their current positions". The possibility of direct implementation of computers into teaching mathematics is not considered or regarded as relevant. The computer is an enemy rather than a partner or assistant.

The mastery phase brings technical and user knowledge. Techniques of work with computers are improved while the level of dependency on computer specialists decreases. This phase is typical for Type C teachers.

Type B teachers are a transition type between Type A and Type C teachers. They adopted computers as a useful tool but they do not know how to incorporate it in teaching in a sensible way. They are aware of the benefits of computer use but preparation of relevant computer aided activities and classes as well as their performance are still demanding for them.

The educational implementation of ICT tools is a long-term innovative process. The fact that a number of teachers already adopted computers as a useful and motivating tool is comforting. However, there are teachers who did not accept the challenge yet. Universities preparing primary school teachers can see this as a challenge too – they should adjust their pre- and post gradual information training in such a way that the number of "practising"teachers rises beyond the imaginary critical amount.

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MODULO ARITHMETIC AND MODULO DESIGN

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Abstract. This paper deals with one application of Modular Arithmetic intended to students of secondary schools. The method of creating so called modulo designs by means of modulo numbers is shown. The designs can be created by means of the Cabri Geometry. This access enables pupils to develop effectively their creative thinking within not only mathematics.

Modular arithmetic is a notation and set of mathematics that were introduced by Karl Friedrich Gauss in 1801. The main idea is that equations can be analyzed from the perspective of remainders. While órdinaryęquations use the "$='$śigns, modular arithmetic uses the "$\equiv'$śigns. Two values that are "$\equiv$"to each other are said to be congruent relative the modulus. More precisely, we say that the integers a and b are congruent modulo m (where m is an integer greater than 1) if the difference $a - b$ is divisible by m; we write $a \equiv b \pmod{m}$.

Modulo numbers can be use to create so called modulo designs.

The aim of this paper is to give a notice by means of the Cabri Geometry that there are some patterns among modulo designs. Their proof is a quite different matter; let us say even the second stage. We'd like to show here without a big number of modulo designs' pictures one could hardly even find these patterns. With this respect the Cabri Geometry plays very significant role in detecting of the patterns.

Let n, k are two positive integers such that k is less or equal than n. To construct the (n, k) modulo design proceed as follows:

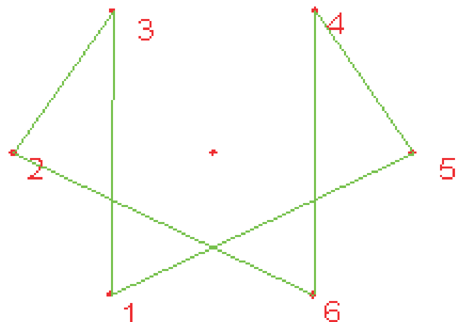
Draw a regular $n-1$ polygon and label its vertices as 1, 2, $\ldots n-1$. Now we connect the pair of vertices according this rule:

If $\mathbf{i}.k \equiv \mathbf{j} \pmod{n}$, where $i = 1, 2, \ldots, n-1$,

then we connect vertices labeled by numbers **i** and **j**; otherwise said we draw the **segment** with endpoints **i** and **j**.

So for example the modulo design (7, 3) we get by connecting these vertices of a regular hexagon:

Because $\mathbf{1}.3 \equiv \mathbf{3} \pmod 7$, we connect vertices **1** and **3**, because $\mathbf{2}.3 \equiv \mathbf{6}$ (mod 7), we connect vertices **2** and **6**, because $\mathbf{3}.3 \equiv \mathbf{2} \pmod 7$, we connect vertices **3** and **2**, etc. till because $\mathbf{6}.3 \equiv \mathbf{4} \pmod 7$, we connect vertices **6** and **4**.



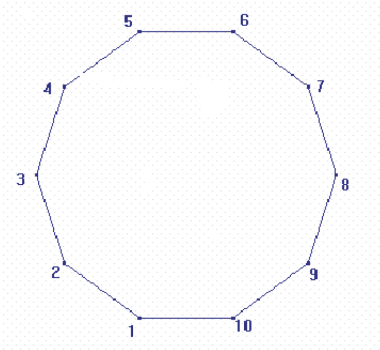
(7, 3)

Using of the Cabri Geometry appears to be very fruitful. It enables students to discover some patterns concerning in the symmetry of modulo designs and their mutual "pair"identity. The Cabri Geometry namely facilitates drawing of big number of figures for different n even big one. And only by means of these figures can students at the beginning first detect and then intuitively formulate above mentioned patterns. Let us state the advice, that can students find the patterns: **To shade in the regions you have found!** Let us formulate some interesting results that students cam detect by means of sufficient number of figures drawn by the Cabri Geometry: Let n be a prime number. Then

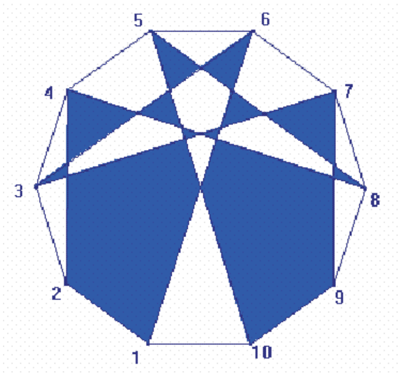
[1] **All modulo designs (i.e. all figures) are symmetric by the segment with endpoints 1 and $n-1$.**

[2] **Modulo designs (n, k) and (n, l) are identical (i.e. figures are same) if $k.l \equiv 1 \pmod n$.**

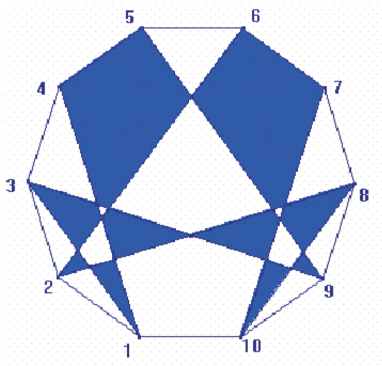
So, let n be e.g. 11, then we can obtain by means of the Cabri Geometry the figures – **modulo designs** $(\mathbf{11}, \mathbf{k})$, for $k = 1, 2, \ldots, 10$ as follows



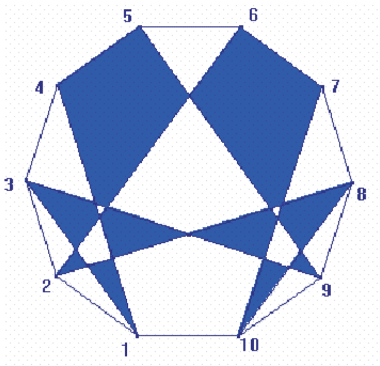
Design (11, 1)



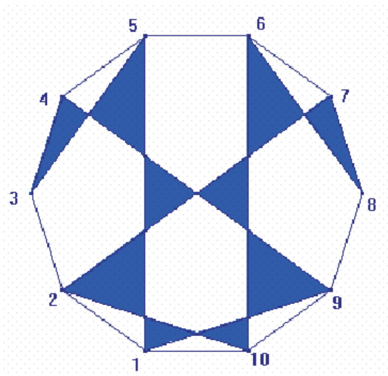
Design (11, 2)= Design(11, 6)



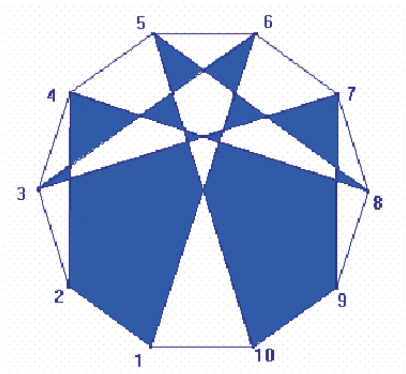
Design (11, 3) = Design(11, 4)



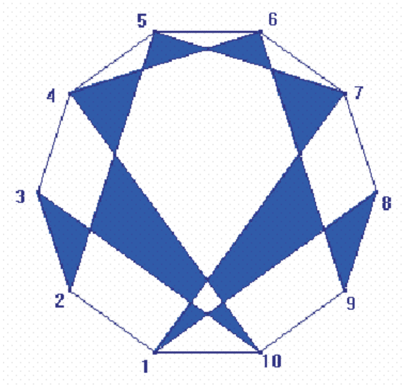
Design (11, 4)= Design(11, 3)



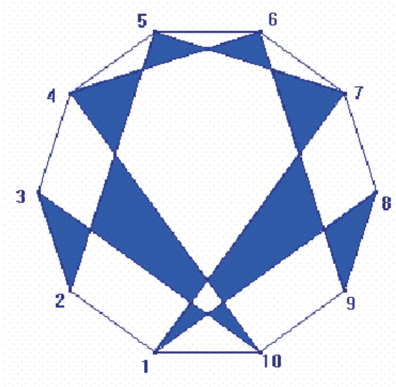
Design (11, 5) = Design(11, 9)



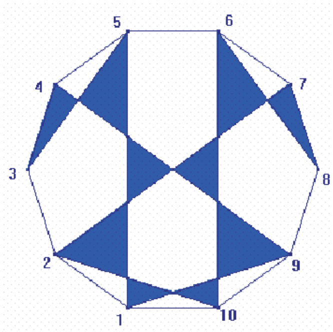
Design (11, 6)= Design(11, 2)



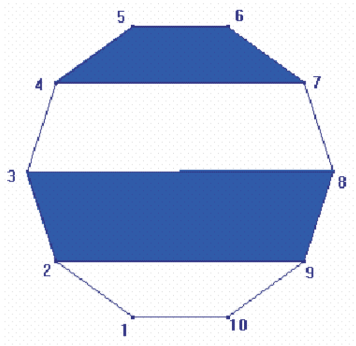
Design (11, 7) = Design(11, 8)



Design (11, 8)= Design(11, 7)



Design (11, 9) = Design(11, 5)



Design (11, 10)

In conclusion we summarize the meaning of this paper was also to show to students there are two mathematical ways of thinking:

- Inductive reasoning – the method of making observations, noticing patterns and forming conclusions (the students detected patterns by means of big number of figures).
- Deductive reasoning – the method of drawing conclusions from other ideas we accept as true by using logic (students should prove "theoremsłabeled by **1.** and **2.** respectively).

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