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PART I

MATHEMATICS

AND ITS APPLICATIONS

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NOTION OF DISTANCE FOR EUCLIDEAN PLANE

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Abstract. One motivation for developing axiomatic systems is to determine precisely which properties of certain objects can be deduced from which other properties. The purpose is to choose a certain fundamental set of properties from which the other properties of the object can be deduced. Some of axioms of Euclidean plane based on the notion of distance are considered. The notions of linear and planar sets are introduced in terms of distance. Thus Euclidean plane is regarded as a distance space with a metric satisfying the corresponding properties.

1. Introduction

Axiomatic geometry made its debut with the Greeks in the sixth century BC, who insisted that statements be derived by logic and reasoning rather than trial and error. This systematization exteriorized in the thirteen volume *Elements* by Euclid (300 BC). Euclid's geometry prevailed until the 19th century when the discovery of non-Euclidean geometry. Several great mathematicians including Pasch, Peano, Pieri, Veblen, Levi and Hilbert have made splendid improvements in Euclidean geometry as a mathematical (axiomatic) system. Hilbert partitioned his axioms for Euclidean geometry into five groups: connection, order, congruence, parallels, continuity.

We describe some properties of various sets for the usual Euclidean 2–space in terms of distance (or metric) as the only initial notion.

2. Metric space

Definition 2.1. Let S be an arbitrary nonempty set. A function $d : S \times S \to \mathbb{R}$ is a *distance function* or *metric* on S if, and only if, for each $p, q, r \in S$

- $d(p,q) \geq 0$, and $d(p,q) = 0$ iff $p = q$ (POSITIVE PROPERTY);
- $d(p,q) = d(q,p)$ (SYMMETRIC PROPERTY);
- $d(p,q) \leq d(p,r) + d(r,q)$ (TRIANGLE INEQUALITY).

We call the set S endowed with this metric a *metric space* and, for all $p, q \in S$, we call the number $d(p,q)$ the *distance* between p and q (with respect to the metric d).

Example 2.1. Define d on $\mathbb{R}^2 \times \mathbb{R}^2$ as follows: $d(p,p) = 0$ for all $p \in \mathbb{R}^2$ and for $p, q \in \mathbb{R}^2$ with $p \neq q$, $d(p,q) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$ if neither p nor q is the origin $(0,0)$ of $\mathbb{R}^2$ and $d(p,q) = 1 + \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$ otherwise. Then d has the positive property and is symmetric. Moreover, d *coincident* with the usual Euclidean metric on $\mathbb{R}^2$ except when exactly one of p and q is the origin. For all $p, q, r \in \mathbb{R}^2$, we have $d(p,(0,0)) \leq d(p,r) + d(r,(0,0))$ and $d(p,q) \leq d(p,(0,0)) + d((0,0,q))$, by the triangle inequality for the Euclidean metric on $\mathbb{R}^2$, so that d also satisfies the triangle inequality and it is consequently a metric on $\mathbb{R}^2$.

Definition 2.2. Suppose (S,d) and (T,e) are metric spaces and $f : S \longrightarrow T$. Then f is called an *isometry* or an *isometric map* if, and only if, $e(f(p), f(q)) = d(p,q)$ for all $p, q \in S$. If f is an isometry, we say that the metric subspace $(f(S), e)$ of (T,e) is an *isometric copy* of the space (S,d).

3. Sets in metric space

Metrics are designed to measure distance between points. Now, we consider the notion of linear set in an arbitrary metric space (see [2]).

Definition 3.1. Let A, B, and C be three points in a metric space (S,d). We say that $\{A, B, C\}$ is a *linear set* in S if

$$d(A,B) = \pm d(A,C) \pm d(C,B). \tag{LS}$$

If both summands on the right-hand side are not zero and have sign plus, then we say that C lies *between* A and B, and we write $A - C - B$.

Definition 3.2. The set X in a metric space (S,d) is called *linear set* if each three different points in X form a linear set.

Definition 3.3. The *segment* AB *with endpoints* A *and* B is the set $\overline{AB} = \{A, B\} \cup \{C : A - C - B\}$. The *distance* $d(A,B)$ from A to B is

the length $|AB|$ of $\overline{AB}$. The union $\overleftrightarrow{AB}$ of all segments containing $\overline{AB}$ is the (straight) *line* through A and B.

Definition 3.4. Suppose A and B are points in a metric space (S, d). The *ray* AB outgoing from A towards B is the set $\overrightarrow{AB}$ consisting of $\overline{AB}$ and all points C with $|AC| = |AB| + |BC|$. The ray of line AB emanating from A in the direction opposite to B is the set $\overleftarrow{AB}$ of all points C such that $|CB| = |CA| + |AB|$.

The concept of linear sets in a metric space without additional conditions is not interesting. For example, a line is not necessarily a linear set, as the next example shows.

Example 3.1. Let $S = \{A, B, C, D\}$, $d(A, B) = 6$, $d(A, C) = 4$, $d(A, D) = d(B, D) = d(C, D) = 3$ and $d(B, C) = 2$. It is easy to see that $S = \overline{AB}$ complies with the axioms of metric space. On the other hand, the set $\{A, C, D\}$ is not linear, hence the segment $\overline{AB}$ is not linear. We also see that three points can lie on a line which is not a linear set, whenever a line is not the union of two rays (half-lines) emanating from a point in opposite directions. To show this, consider the line $BC = \overline{AB}$. We have $\overrightarrow{BC} = \{A, B, C\}$ and $\overleftarrow{BC} = \{B, C\}$. It follows that $BC \neq \overrightarrow{BC} \cup \overleftarrow{BC}$.

Remark 3.1. If a finite set S is linear, then there exists a segment containing S with endpoints from S. Really, among all pairs belonging to S, there exists a pair A, B with maximal distance. Let C be another point of S. Then $\{A, B, C\}$ is a linear set. From the definition of the betweenness follows that $|AB| = |AC| + |CB|$, and thus $C \in \overline{AB}$.

Note, the linearity of a metric space (S, d) does not imply the uniqueness of point C (with respect to the points $A, B \in S$) satisfying the equation (LS). Indeed, suppose (S, d) is metric space, $S = \{K, L, M, N\}$ and $d(K, L) = d(L, M) = d(M, N) = d(N, K) = 1$, $d(K, M) = d(L, N) = 2$. Each three points of S form a linear set in which one of the distances is 2 and the others are 1. In this case, K and M lie between L and N, and we have $d(K, L) = d(L, M)$ and $d(M, N) = d(N, K)$.

Next, we consider the notion of linearly coupled sets and the notion of a triangle in an arbitrary metric space (see [2]).

Definition 3.5. Two sets are *linearly coupled* if their intersection contains at least two points.

Definition 3.6. A finite system of sets in (S, d) is *linearly coupled* if they can be numbered so that every two neighboring sets are linearly coupled.

Definition 3.7. Suppose A, B, C are points of the metric space (S, d). We say that they define a *triangle* ABC. The segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ are the *sides* of the triangle, and A, B, C are the *vertices*. The number $|ABC| = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the lengths of the sides and $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle, is the *area* of $\triangle ABC$. A triangle with zero area is said to be *degenerated*.

Definition 3.8. We say that a point M of the space (S, d) *belongs to a triangle* ABC if

$$|ABC| = |MAB| + |MBC| + |MCA|. \tag{$\triangle$}$$

In this sense, any triangle is regarded as the set of points belonging to it.

From the definition of a triangle follows that if $\triangle ABC$ is degenerated, then $\{A, B, C\}$ is a linear set. In this case, one of the factors in $s(s-a)(s-b)(s-c)$ vanishes. For example, let $s - c = 0$, i.e., $c = a + b$ and we can assume (without loss of generality) $a = |CB|$, $b = |AC|$, $c = |AB|$. We obtain $|AB| = |AC| + |CB|$, i.e., $A - C - B$.

4. The metric axioms

Next, we introduce one of possible modifications of metric axioms, which hold true in Euclidean 2-space (cf. [2]).

M1. (UNIQUENESS AXIOM) For each two points A and B, a point M satisfying the condition $|AB| = \pm|AM| \pm |MB|$, where $|AM|$, $|MB|$ and the sign are fixed, is uniquely determined.

M2. (LINEARITY AXIOM) The union of sets of any linearly coupled system of linear sets is a linear set.

M3. (AXIOM OF FILLING A LINE) For each two points A and B and any relation $d(A, B) = \pm d_1 \pm d_2$, $d_1 \geq 0$, $d_2 \geq 0$, there exists a point C such that $d(A, C) = d_1$ and $d(C, B) = d_2$.

M4. There are three points not forming a linear set.

M5. For an arbitrary line l and any point C, we have $|ABC| = \lambda|AB|$, where λ does not depend on the choice of the segment $\overline{AB} \subset l$.

M6. (AXIOM OF SIMILARITY) Let $d(A, A') + d(A', O) = d(A, O)$ and $d(B, B') + d(B', O) = d(B, O)$, where all distances differ from zero, and let

$$\frac{d(A', O)}{d(A, O)} = \frac{d(B', O)}{d(B, O)}.$$

Then

$$\frac{d(A', O)}{d(A, O)} = \frac{d(B', O)}{d(B, O)} = \frac{d(A', B')}{d(A, B)}.$$

M7. (AXIOM OF CONGRUENCE) Let $d(A, B) = d(A', B')$. Then there exists an isometry $f : (S, d) \longrightarrow (S, d)$ such that $f(A) = A'$ and $f(B) = B'$.

Now, let us observe some connections between this system of axioms and the notions of planar set presented in the previous sections. First of all we can see that if a metric space satisfies axiom **M2**, then its segments are the linear sets. Indeed, let $\overline{AB}$ be a segment in (S, d). Then $A \neq B$ since all one-element sets are linear. Suppose that K, L, M are points of $\overline{AB}$. By the definition of a segment, we have $\{A, B, K\}$, $\{A, B, L\}$ and $\{A, B, M\}$ are linearly coupled linear sets, and by **M2**, their union $\{A, B, K, L, M\}$ is a linear set. This means, that its subset $\{K, L, M\}$ is linear, too.

Looking at definitions 3.7 and 3.8 above, we see that each vertex of the triangle belongs to it. The situation with the sides of triangle is not trivial. To prove this, we need to consider the axiom **M5**. Suppose $\triangle ABC$ is a triangle in (S, d). If $M \in \overline{AB}$ then $|AB| = |AM| + |MB|$. If we multiply both sides of this equation by real number λ (λ is the *altitude* of the triangle $\triangle ABC$ or the distance from C to AB) we have $|ABC| = |AMC| + |MBC|$. Then $|ABC| = |AMB| + |AMC| + |MBC|$ since $|AMB| = 0$. From this and by the definition of a triangle, we conclude that M belongs to $\triangle ABC$.

We recall here only one important theorem related to lines of a metric space.

Theorem (A. A. Ivanov [2]). If a metric space (S, d) satisfies metric axioms **M1-M3**, then any line of this space is isometric to the real line with the usual metric.

The system of metric axioms for Euclidean 2-space presented above may be viewed as a system equivalent to the Hilbert's axiomatic. The connections between them seems to be an interesting subject for investigation.

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THE COROLLARY OF GREEN'S THEOREM FOR CURVILINEAR INTEGRALS

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1. Introduction

One of the very important applications of the multidimensional real integrals in the technical practice is a calculation of areas and volumes of solids in generally n-dimensional space $\mathbf{R}_n$. In the paper [1], that problem is investigated as a topological problem and the formula for the calculation of the volume of the n-dimensional solid in the space $\mathbf{E}_n$ is proved there. For the calculation of these volumes, parametric descriptions of the surface areas of solids are necessary. Then the surface areas are smooth (respective by parts smooth) areas in Euclidean space of the corresponding dimension. Using that theory we need multidimensional real integrals of the dimension $n-1$ for calculation of volumes of solids in $\mathbf{E}_n$. The calculations of these integrals are easier. That method shows a new theoretical and practical approach to the solving of the known problems. In this paper, the correspondence between that new theory in $\mathbf{E}_2$ and the known result of the curvilinear integral theory (i.e. a calculation of an area of a closed bounded figure by a curvilinear integral) is presented.

2. Volumes of bounded closed solids by using parametric descriptions

To understand relationship (continuity) between the new theory presented in [1] and the calculation of an area of a closed bounded figure by a curvilinear integral, we must repeat the basic result of that theory.

Let x^α, $\alpha = 1, \dots, n$, be Cartesian co-ordinates of a point $\mathbf{x} \in \mathbf{E}_n$, u^a, $a = 1, \dots, n-1$, be Cartesian co-ordinates of a point $\mathbf{u} \in \mathbf{E}_{n-1}$, where $\mathbf{E}_n$ is Euclidean n-dimensional space, $n \geq 2$. Let Ω be the bounded closed domain in $\mathbf{E}_{n-1}$ and $x^\alpha\left(u^1, \dots, u^{n-1}\right)$, where $\alpha = 1, \dots, n$, given functions defined

on some domain $\mathbf{O} \subset \mathbf{E}_{n-1}$, $\Omega \subset \mathbf{O}$. Let us suppose that the vector function $\mathbf{x}(\mathbf{u}) = \{x^\alpha(u^a)\}$ has almost everywhere in Ω the continuous partial derivatives

$$B_a^\alpha := \frac{\partial x^\alpha}{\partial u^a} \text{ for } \alpha = 1, \dots, n,\ a = 1, \dots, n-1,$$

the rank of the matrix $(B_a^\alpha)_{n\times(n-1)} = \begin{pmatrix} B_1^1 & B_1^2 & \dots & B_1^n \\ B_2^1 & B_2^2 & \dots & B_2^n \\ & & \dots & \\ B_{n-1}^1 & B_{n-1}^2 & \dots & B_{n-1}^n \end{pmatrix}$ is maximal in Ω, i.e. it is equal to $n-1$ almost everywhere in Ω, the subset

$$P^0 := \{\mathbf{x} \in \mathbf{E}_n |\ \mathbf{x} = \mathbf{x}(\mathbf{u}),\ \mathbf{u} \in \operatorname{int}\Omega\}$$

of the matrix

$$P := \{\mathbf{x} \in \mathbf{E}_n |\ \mathbf{x} = \mathbf{x}(\mathbf{u}), \mathbf{u} \in \Omega\} \tag{1}$$

is a homeomorphic range of the set int Ω (of all interior points of the set Ω) in $\mathbf{E}_n$.

It follows from the given assumptions that the set P is bounded and it is a smooth hypersurface by parts in the space $\mathbf{E}_n$. This manifold does not intersect by itself and that divides the space $\mathbf{E}_n$ to two disjoint regions in $\mathbf{E}_n$, in which one is bounded and the second one is unbounded. The closure of relevant bounded region is called **the n-dimensional solid** in space $\mathbf{E}_n$. Let us denote it by W. Especially, for $n = 3$, let the closure be **the solid** in $\mathbf{E}_3$, for $n = 2$ **the closed area** in $\mathbf{E}_2$.

If we denote by $\operatorname{int} W$ the set of all internal points of the set W and by ∂W the boundary of it, it is obvious that

$$W = P \cup \operatorname{int} W, \quad \text{where } \partial W = P.$$

Let us consider the hypersurface P as the set of points immersed to Euclidean space $\mathbf{E}_{n+1} = \mathbf{E}_n \times \mathbf{E}_1$. Then this hypersurface P can be expressed P in a form

$$P := \{X = (\mathbf{x}; x^{n+1}) \in \mathbf{E}_{n+1} |\ \mathbf{x} = \mathbf{x}(\mathbf{u}), x^{n+1} = 0,\ \mathbf{u} \in \Omega\}. \tag{2}$$

Let us choose the point $V = (\mathbf{x}_0; x_0^{n+1})$ in the space $\mathbf{E}_{n+1}$, where $x_0^{n+1} > 0$, and the set of all half-lines

$$\begin{aligned} &\mathbf{x} = \mathbf{x}_0 + [\mathbf{x}(\mathbf{u}) - \mathbf{x}_0]t, \\ &x^{n+1} = x_0^{n+1}(1-t),\ \mathbf{u} \in \Omega,\ t \in \langle 0; +\infty). \end{aligned} \tag{3}$$

This set represents **the conical hypersurface** in $\mathbf{E}_{n+1}$ with the vertex V. With respect to the condition $x^{n+1} \geq 0$, we obtain the smooth by parts and closed bounded

(compacted) hypersurface in $\mathbf{E}_{n+1}$ from it. This hypersurface divides the space $\mathbf{E}_{n+1}$ into two disjoint regions, where one is bounded and the second one is unbounded. Let us denote the closure of the relevant bounded region by K. Then the set K be called **the hypercone** in the space $\mathbf{E}_{n+1}$ with the vertex $V = \left(\mathbf{x}_0; x_0^{n+1}\right)$, the set W be called **the base** of this hypercone, and the number $x^{n+1} > 0$ **the hight** of it. (Under the term "the closed hypersurface in $\mathbf{E}_n$", we understand the manifold in $\mathbf{E}_n$ in the topological sense which is a homeomorphic range of the hypersphere in $\mathbf{E}_n$, if this mapping exists.)

We would like to determine the volume of the hypercone K, i.e. to determine the measure of the set K. The measure of the set K is describes by the relation

$$I = \overbrace{\int\int\ldots\int}^{n+1}_{K} dx^1 dx^2 \cdot \ldots \cdot dx^{n+1}. \tag{4}$$

To calculate this integral, we introduce the generalized coordinates u^a, t, v in the space $\mathbf{E}_{n+1}$, where $a = 1, \ldots, n-1$. The relations

$$\begin{aligned} &\mathbf{x} = \mathbf{x}_0 + [\mathbf{x}(\mathbf{u}) - \mathbf{x}_0]t, \\ &x^{n+1} = v(1-t),\ \mathbf{u} \in \Omega,\ t \in \langle 0; 1\rangle, v \in \left\langle 0; x_0^{n+1}\right\rangle \end{aligned} \tag{5}$$

define these coordinates.

With respect to the given goal, we can suppose that $V = (0; x_0^{n+1})$, i.e. $x_0^\alpha = 0, \alpha = 1, \ldots, n$, without loss of generality. Then the relation (5) can be simplified to the form

$$\begin{aligned} &x^\alpha = x^\alpha(u^a)t,\ \alpha = 1, \ldots, n, \\ &x^{n+1} = v(1-t). \end{aligned} \tag{6}$$

If we use the coordinates determined by the relations (6) we obtain the integral of the form

$$I = \int\int\ldots\int_{K'} J du^1 du^2 \cdot \ldots \cdot du^{n-1} dt dv, \tag{7}$$

where the symbol J is the Jacobian associated to the transformation (6) and the set $K' = \Omega \times \langle 0; 1\rangle \times \left\langle 0; x_0^{n+1}\right\rangle$. It follows

$$J = \begin{vmatrix} \frac{\partial x^1}{\partial t} & \frac{\partial x^2}{\partial t} & \ldots & \frac{\partial x^n}{\partial t} & \frac{\partial x^{n+1}}{\partial t} \\ t\frac{\partial x^1}{\partial u^1} & t\frac{\partial x^2}{\partial u^2} & \ldots & t\frac{\partial x^n}{\partial u^1} & t\frac{\partial x^{n+1}}{\partial u^1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ t\frac{\partial x^1}{\partial u^{n-1}} & t\frac{\partial x^2}{\partial u^{n-1}} & \ldots & t\frac{\partial x^n}{\partial u^{n-1}} & t\frac{\partial x^{n+1}}{\partial u^{n-1}} \\ t\frac{\partial x^1}{\partial v} & t\frac{\partial x^2}{\partial v} & \ldots & t\frac{\partial x^n}{\partial v} & t\frac{\partial x^{n+1}}{\partial v} \end{vmatrix} =$$

$$= \begin{vmatrix} x^1(\mathbf{u}) & x^2(\mathbf{u}) & \dots & x^n(\mathbf{u}) & -v \\ B_1^1 & B_1^2 & \dots & B_1^n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ B_{n-1}^1 & B_{n-1}^2 & \dots & B_{n-1}^n & 0 \\ 0 & 0 & \dots & 0 & 1-t \end{vmatrix} \cdot t^{n-1}$$

for the Jacobian from (6).

Let us denote
$$\Delta(\mathbf{u}) := \begin{vmatrix} x^1(\mathbf{u}) & x^2(\mathbf{u}) & \dots & x^n(\mathbf{u}) \\ B_1^1 & B_1^2 & \dots & B_1^n \\ \dots & \dots & \dots & \dots \\ B_{n-1}^1 & B_{n-1}^2 & \dots & B_{n-1}^n \end{vmatrix} \tag{8}$$

Then the absolute value of the Jacobian can be expressed shortly in the form

$$|J| = \left(t^{n-1} - t^n\right) \cdot |\Delta(\mathbf{u})|.$$

We obtain for the integral (7)

$$I = \int\int_{\Omega} \dots \int |\Delta(\mathbf{u})| du^2 \cdot \dots \cdot du^{n-1} \cdot \int_0^1 \left(t^{n-1} - t^n\right) dt \cdot \int_0^{x^{n+1}} dv =$$
$$= \frac{1}{n(n+1)} x_0^{n+1} \int\int \dots \int |\Delta(\mathbf{u})| du^1 du^2 \cdot \dots \cdot du^{n-1}.$$

Thus

$$I = \frac{x_0^{n+1}}{n+1} \int\int \dots \int \frac{1}{n} |\Delta(\mathbf{u})| du^1 du^2 \dots du^{n-1}. \tag{9}$$

The number

$$\mu_W := \frac{1}{n} \int\int \dots \int |\Delta(\mathbf{u})| du^1 du^2 \dots du^{n-1} \tag{10}$$

is called **the volume of the n-dimensional solid W in the space $\mathbf{E}_n$.**

Especially, for $n = 3$ this number is called **the volume of the solid W in $\mathbf{E}_3$**, for $n = 2$, **the area of the bounded figure W in $\mathbf{E}_2$.**

3. The corollary of Green's theorem for curvilinear integrals

Starting to find areas of closed figures in $\mathbf{E}_2$ by the theory above, we must find the suitable parameterizations of the projection of these figures to $\mathbf{E}_1$ and calculate the determinant $\Delta(\mathbf{u})$. In this case, the projection of the area P into the x-coordinate (into $\mathbf{E}_1$) is the closure of the interior of a curve K. The area P is of the form of a sector (Fig. 1) and its projection into $\mathbf{E}_1$ is described by the parametric equations $x = \varphi(t), y = \psi(t)$ for each $t \in \langle \alpha; \beta \rangle$.
Then $\Delta(\mathbf{u}) = \Delta(t)$. Thus

$$\Delta(t) = \begin{vmatrix} \varphi(t) & \psi(t) \\ \varphi'(t) & \psi'(t) \end{vmatrix} = \varphi(t)\psi'(t) - \psi(t)\varphi'(t),$$

$$\mu_W = \frac{1}{2}\int_\alpha^\beta |\Delta(t)|dt = \frac{1}{2}\int_\alpha^\beta |\varphi(t)\psi'(t) - \psi(t)\varphi'(t)|dt. \tag{11}$$

It is **the known corollary of Green's theorem** of the curvilinear integral theory when we calculate the area of the closed figure so that we find the parameterization of the curve AB only because the integrals along the segment lines OA and OB are equal to zero.

Fig. 1. The sector P.

Examples

Let us present some examples of these parameterizations for known curves and solve relevant areas. The following examples demonstrate that the theory above can be also use to the known cases in a plane and that we obtain corresponding solutions.

Example 1: Calculate the area of the figure bounded by the ellipse defined by the set P where

$$P := \left\{ \mathbf{x} \in \mathbf{E}_2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \ a > 0, \ b > 0.$$

Solution:

The parametric equations of the ellipse:

$$x = a\cos t, \quad y = b\sin t, \quad t \in \langle 0; 2\pi \rangle.$$

Then with respect to the new theory above:

$$B_1^1 = \frac{\partial x}{\partial t} = -a\sin t, \ B_1^2 = \frac{\partial y}{\partial t} = b\cos t.$$

The determinant $\Delta(t)$ defined in (8) is equal to

$$\Delta(t) = \begin{vmatrix} a\cos t & b\sin t \\ -a\sin t & b\cos t \end{vmatrix} = ab.$$

The area:

$$\mu_W = \frac{1}{2}\int_0^{2\pi} abdt = \underline{ab\pi}$$

Example 2: Calculate the area of the figure bounded by the lemniscate defined by the set P, where

$$P := \left\{ \mathbf{x} \in \mathbf{E}_2 |\ \left(x^2 + y^2\right)^2 = a^2 \left(x^2 - y^2\right) \right\},\ a > 0.$$

Solution:

The first we must find the parametric equations of the lemniscate. We will use the relation $y = x \operatorname{tg} t$. If we put this expression to the equation of the lemniscate we obtain the known parametric equations of the form

$$x = a \cos t \sqrt{\cos 2t}, \quad y = a \sin t \sqrt{\cos 2t}, \quad t \in \left\langle -\frac{\pi}{4}; \frac{\pi}{4} \right\rangle.$$

Both loops of the lemniscate bound figures of the same areas. Then

$$\Delta(t) = \begin{vmatrix} a \cos t \sqrt{\cos 2t}; & a \sin t \sqrt{\cos 2t} \\ -a \sin t \sqrt{\cos 2t} - \dfrac{2a \cos t \sin 2t}{2\sqrt{\cos 2t}}; & a \cos t \sqrt{\cos 2t} - \dfrac{2a \sin t \sin 2t}{2\sqrt{\cos 2t}} \end{vmatrix} =$$

$$\ldots = a^2 \cos 2t.$$

The area μ_W is equal to $2S_1$ where S_1 is area of one loop, thus $\mu_W = 2 \cdot \frac{1}{2} \int\limits_{-\pi/4}^{\pi/4} a^2 \cos 2t \, dt = 2 \cdot \frac{a^2}{2} \left[\frac{\sin 2t}{2} \right]_{-\pi/4}^{\pi/4} = \underline{a^2}$.

4. Conclusions

We see the new theory corresponds to the known theory of curvilinear integrals. There exist lots of examples and technical applications where the presented method may be used in a real practice. If we are able to find the suitable parameterizations of the smooth or smooth by parts areas, the calculations of areas (and of course volumes, generally in the n-dimensional space) can be easier because we calculate with integrals of less dimensions. However, the process of the finding the parameterization proves to be a rather complicated problem in specific cases.

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TRIANGULAR STRUCTURES AND DUALITY

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Abstract. We introduce and study the category AFD the objects of which are generalized convergence D-posets (with more than just one greatest element) of maps into a triangle object T and the morphisms of which are sequentially continuous D-homomorphisms. The category AFD can serve as a base category for antagonistic fuzzy probability theory. AFD-measurable maps can be considered as generalized random variables and ADF-morphisms, as their dual maps, can be considered as generalized observables.

1. Introduction

In generalized probability theory (cf. [6,11,13,19,20]) basic notions are events, states (generalized probability measures), and observables (notions dual to generalized random variables). Difference posets (abbr. to D-posets) of fuzzy sets have been introduced by F. Kôpka in 1992 (see [15]). More general D-posets (cf. [16]) form a category in which classical, fuzzy, and quantum phenomena can be modeled. D-posets are equivalent to effect algebras (cf. [7]). Recall that ID (cf. [10,18]) is the category the objects of which are suitable convergence D-posets of maps into the closed unit interval $I = [0, 1]$ and the

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morphisms of which are sequentially continuous D-homomorphisms. Note that ID is a suitable base category for generalized probability theory and (unlike in the classical Kolmogorov probability theory) in ID both observables and states are morphisms.

The category theory (cf. [14]) and in particular the language of mathematical structures (cf. [1]) is natural and suitable to carry out and describe various constructions used in the foundations of generalized probability, e.g. the duality between observables and random variables. The sequential convergence and sequential continuity of the morphisms play a key role (cf. [9]).

We are motivated by the intuitionistic fuzzy probability theory (see [2,21]). Our goal is to generalize the theory of measurable spaces and measurable maps developed in [18] replacing the cogenerator $[0,1]$ by a suitable triangular object T, to prove a duality theorem, and to indicate some applications to generalized probability theory.

Recall that in intuitionistic logic the law of excluded middle does not hold. An *intuitionistic fuzzy event* $A \subseteq X$ is a pair (μ_A, ν_A) of membership functions $\mu_A, \nu_A \in I^X$ such that $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. The intuitionistic fuzzy events are partially ordered ($(\mu_B, \nu_B) \leq (\mu_A, \nu_A)$ whenever $\mu_B \leq \mu_A$ and $\nu_A \leq \nu_B$) and carry suitable operations. *Intuitionistic fuzzy probability* sends intuitionistic fuzzy events to closed subintervals of I.

2. D-posets of fuzzy sets

Let X be a set, let $\mathcal{X} \subseteq I^X$ be a family of functions of X into I, for each $c \in I$ let $c_{\mathcal{X}}$ be the corresponding constant function, let "$\leq$" be the pointwise partial order on $\mathcal{X}$, and let "$\ominus$" be the pointwise partial difference defined for $v \leq u$ by $(u \ominus v)(x) = u(x) - v(x),\ x \in X$. The quintuple $(\mathcal{X}, \leq, 0_{\mathcal{X}}, 1_{\mathcal{X}}, \ominus)$, abbreviated to $\mathcal{X}$, is a *D-poset of fuzzy sets* called an *ID-poset*. The pair $(X, \mathcal{X})$ is called an *ID-measurable space*. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be ID measurable spaces and let $f : X \longrightarrow Y$ be a map such that $v \circ f \in \mathcal{X}$ for each $v \in \mathcal{Y}$. Then f is said to be a $(\mathcal{Y}, \mathcal{X})$*-measurable map*. Let h be a map of an ID-poset $\mathcal{Y}$ into an ID-poset $\mathcal{X}$ which preserves "the ID-structure". Then h is said to be a *D-homomorphism*.

Denote MID is the category of ID-measurable spaces and measurable maps. It is known that the category ID and a distinguished subcategory of MID (consisting of sober objects) are dually naturally equivalent (cf. [10,18]).

Example 2.1. Let $(\Omega, \mathbf{A})$ be a classical measurable space. Then $\mathbf{A}$ can be considered as an ID-poset via identifying $A \in \mathbf{A}$ and its characteristic function and defining $A \ominus B = A \setminus B$ whenever $B \subseteq A$.

Example 2.2. Let $(\Omega, \mathbf{A})$ be a classical measurable space. Let $\mathcal{P}(\mathbf{A})$ be the set of all probability measures on $\mathbf{A}$. Let $\{a\}$ be a singleton. Then each $p \in \mathcal{P}(\mathbf{A})$

is an ID-morphism of $\mathbf{A}$ into $I = I^{\{a\}}$. Denote $ev(A) = \{p(A);\ p \in \mathcal{P}(\mathbf{A})\}$ and $ev(\mathbf{A}) = \{ev(A);\ A \in \mathbf{A}\}$. For $X = \mathcal{P}(\mathbf{A})$ and $\mathcal{X} = ev(\mathbf{A})$, $(X, \mathcal{X})$ is a typical ID-measurable space.

3. AFD-system

Denote $T = \{(a, b) \in I \times I;\ a + b \leq 1\}$. Then T carries the pointwise partial order defined by $(a, b) \preceq (c, d)$ whenever $a \leq c$ and $b \leq d$, a partial difference operation defined by $(c, d) \oslash (a, b) = (c - a, d - b)$ whenever $(a, b) \preceq (c, d)$, and the pointwise sequential convergence defined by $(a, b) = \mathrm{Lim}_{n\to\infty}(a_n, b_n)$ whenever $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

Denote $(T, \preceq, \oslash, \mathrm{Lim})$ the resulting structure; it will be abbreviated to T and called the *triangle* T.

Let X be a set and let T^X be the set of all maps of X into T. If X is a singleton $\{a\}$, then $T^{\{a\}}$ will be condensed to T. Let $u \in T^X$. Then there are two maps u_l and u_r of X into I such that for each $x \in X$ we have $u(x) = (u_l(x), u_r(x))$; we shall write $u = (u_l, u_r)$. In T^X there are three distinguished constants defined as follows:

$b_{\mathcal{X}} = (0_{\mathcal{X}}, 0_{\mathcal{X}}),\ b_{\mathcal{X}}(x) = (0, 0)$ for all $x \in X$;

$l_{\mathcal{X}} = (0_{\mathcal{X}}, 1_{\mathcal{X}}),\ l_{\mathcal{X}}(x) = (0, 1)$ for all $x \in X$;

$r_{\mathcal{X}} = (1_{\mathcal{X}}, 0_{\mathcal{X}}),\ r_{\mathcal{X}}(x) = (1, 0)$ for all $x \in X$.

The system T^X carries the pointwise partial order, the pointwise partial difference (we shall use the same symbols for the pointwise partial order and the pointwise partial difference on T and T^X), and the pointwise sequential convergence induced by the triangle T.

Definition 3.1. *Let X be a set and let $\mathcal{X}$ be a set of maps of X into the triangle T such that $b_{\mathcal{X}}, l_{\mathcal{X}}, r_{\mathcal{X}} \in \mathcal{X}$ and $\mathcal{X}$ is closed with respect to the pointwise partial difference. Then $\mathcal{X}$ carrying the pointwise order, the pointwise partial difference, and the pointwise sequential convergence is said to be an* AFD-system* *and $(X, \mathcal{X})$ is said to be an* AFD-measurable space. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be AFD-measurable spaces and let $f:\ X \longrightarrow Y$ be a map such that $v \circ f \in \mathcal{X}$ for each $v \in \mathcal{Y}$. Then f is said to be a $(\mathcal{Y}, \mathcal{X})$-*measurable map.

In what follows, all AFD-systems will be *reduced*, i.e., for each $x,\ y \in X$, $x \neq y$, there exists $u \in \mathcal{X}$ such that $u(x) \neq u(y)$.

Denote MAFD the category of AFD-measurable spaces and measurable maps.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be AFD-measurable spaces (remember $\mathcal{X}$ and $\mathcal{Y}$ are reduced) and let $f:\ X \to\ Y$ be a $(\mathcal{Y}, \mathcal{X})$-measurable map. Define the dual map $f^{\lhd}$ of $\mathcal{Y}$ into $\mathcal{X}$ as follows: $f^{\lhd}(v) = v \circ f,\ v \in \mathcal{Y}$.

*The notion is derived from "antagonistic".

Lemma 3.2.

(i) The dual map $f^{\triangleleft}$ is sequentially continuous and preserves the structure of AFD-systems.

(ii) Let f and g be measurable maps of X into Y. If $f \neq g$, then $f^{\triangleleft} \neq g^{\triangleleft}$.

Proof. (i) First, let $\langle v_n \rangle$ be a sequence converging pointwise in $\mathcal{Y}$ to v. Since $(f^{\triangleleft}(v))(x) = v(f(x))$ and $(f^{\triangleleft}(v_n))(x) = v_n(f(x))$, $x \in X, n \in N$, the sequence $\langle f^{\triangleleft}(v_n) \rangle$ converges pointwise in $(X, \mathcal{X})$ to $f^{\triangleleft}(v)$. Thus $f^{\triangleleft}$ is a sequentially continuous map of $(X, \mathcal{Y})$ to $(X, \mathcal{X})$. Second, we have to verify that

a) $f^{\triangleleft}$ sends each distinguished constant of $\mathcal{Y}$ into the corresponding distinguished constant of $\mathcal{X}$: $f^{\triangleleft}(b_{\mathcal{Y}}) = b_{\mathcal{X}}$, $f^{\triangleleft}(l_{\mathcal{Y}}) = l_{\mathcal{X}}$, and $f^{\triangleleft}(r_{\mathcal{Y}}) = r_{\mathcal{X}}$;

b) $f^{\triangleleft}$ preserves the partial order: if $u, v \in \mathcal{Y}$ and $u \leq v$, then $f^{\triangleleft}(u) \leq f^{\triangleleft}(v)$ in $\mathcal{X}$;

c) $f^{\triangleleft}$ preserves the partial operation: if $u, v \in \mathcal{Y}$ and $u \leq v$, then $f^{\triangleleft}(v \oslash u) = f^{\triangleleft}(v) \oslash f^{\triangleleft}(u)$ in $\mathcal{X}$.

Once again, all three conditions follow from the fact that for each $w \in \mathcal{Y}$ and for each $x \in \mathcal{X}$ we have $(f^{\triangleleft}(w))(x) = w(f(x))$. For example, if $u \leq v$ in $\mathcal{Y}$, i.e., $u(y) \leq v(y)$ for all $y \in Y$, then also $(f^{\triangleleft}(u))(x) = u(f(x)) \leq v(f(x)) = (f^{\triangleleft}(v))(x)$ for all $x \in X$, and hence $f^{\triangleleft}(u) \leq f^{\triangleleft}(v)$. Other conditions can be verified analogously.

(ii) Assume that there exist $x \in X$ such that $f(x) \neq g(x)$. Since $\mathcal{Y}$ is reduced, there exists $u \in \mathcal{Y}$ such that $u(f(x)) \neq u(g(x))$. Consequently $(f^{\triangleleft}(u))(x) = u(f(x)) \neq u(g(x)) = (g^{\triangleleft}(u))(x)$ and hence $f^{\triangleleft} \neq g^{\triangleleft}$. □

Definition 3.3. *Let h be a map of an AFD-system $\mathcal{Y}$ into an AFD-system $\mathcal{X}$ preserving the structure of AFD-systems. Then h is said to be an* AFD-homomorphism.

Let $\mathcal{X} \subseteq T^X$ be an AFD-system. Then each $x \in X$ can be considered as a sequentially continuous AFD-homomorphism ev_x of $\mathcal{X}$ into T defined by $ev_x(u) = u(x), u \in \mathcal{X}$. Denote X^* the set of all sequentially continuous AFD-homomorphisms of $\mathcal{X}$ into T. For $u \in \mathcal{X}$ put $u^*=\{ev_x(u);\ x \in X^*\}$ and $\mathcal{X}^* = \{v^*;\ v \in \mathcal{X}\}$. It is easy to see that $\mathcal{X}^*$ is an AFD-system and X^* is the set of all AFD-homomorphisms of $\mathcal{X}^*$ into T. Observe that if $a, b \in \mathcal{X}, a \neq b$, then $ev_a \neq ev_b$. Indeed, $\mathcal{X}$ is reduced and hence $u(a) \neq u(b)$ for some $u \in \mathcal{X}$.

Definition 3.4. *Let $\mathcal{X} \subseteq T^X$ be an AFD-system. Then $\mathcal{X}^*$ is said to be the* sobrification *of $\mathcal{X}$. If $X = X^*$, then $\mathcal{X}$ and $(X, \mathcal{X})$ are said to be* sober.

Theorem 3.5. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be sober AFD-measurable spaces and let h be a sequentially continuous AFD-homomorphism of $\mathcal{Y}$ into $\mathcal{X}$. Then there exists a unique AFD-measurable map f of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$ such that $f^{\triangleleft} = h$.*

Proof. For each $x \in X$, the composition $ev_x \circ h$ is a sequentially continuous AFD-homomorphism of $\mathcal{Y}$ into T. Since $\mathcal{Y}$ is reduced and sober, there exists a unique $y \in Y$ such that $ev_y = ev_x \circ h$. Put $y = f(x)$. This defines a map f of X into Y. Let $u \in \mathcal{Y}$. Then for each $x \in X$ we have $(h(u))(x) = ev_x(h(u)) = (ev_x \circ h)(u) = ev_{f(x)}(u)$. Hence $h = u \circ f = f^{\triangleleft}$. It follows from the preceding lemma that if g is a measurable map of X into Y such that $g^{\triangleleft} = h$, then $f = g$. □

Corollary 3.6. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be sober AFD-measurable spaces. Then $f \mapsto f^{\triangleleft}$ yields a one-to-one correspondence between $(\mathcal{Y}, \mathcal{X})$-measurable maps and AFD-homomorphisms of $\mathcal{Y}$ to $\mathcal{X}$.*

4. Duality and applications

Denote AFD the category of AFD-systems and sequentially continuous AFD-homomorphisms. Denote SMAFD the subcategory of MAFD consisting of sober AFD-measurable spaces and denote SAFD the corresponding subcategory of AFD consisting of sober objects.

Theorem 4.1. *The categories SMAFD and SAFD are dually isomorphic.*

Proof. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be sober AFD-measurable spaces. Denote $F((X, \mathcal{X})) = \mathcal{X}$ and $F((Y, \mathcal{Y})) = \mathcal{Y}$. Let f be an AFD-measurable map of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$. Denote $F(f) = f^{\triangleleft}$. A straightforward calculation shows that F yields a contravariant functor of SMAFD into SAFD and that F is a dual isomorphism. □

Observe that the categories AFD and SAFD are isomorphic (indeed, analogously as in [18] it can be proved that $\mathcal{X} \mapsto \mathcal{X}^*$ yields an isomorphism of AFD and SAFD) and, consequently, AFD and SMAFD are dually naturally equivalent. It can be shown that AFD can serve as a base category for antagonistic fuzzy probability theory, AFD-measurable maps can be considered as generalized random variables and their duals can be considered as generalized observables.

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STABILITY OF THE EQUATION OF RING HOMOMORPHISMS

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Abstract. Let $\mathcal{R}$ be a unitary ring and $(\mathcal{A}, \|\cdot\|)$ stand for a Banach algebra with a unit. In connection with some stability results of R. Badora [1] and D.G. Bourgin [2] concerning the system of two Cauchy functional equations

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = f(x)f(y) \end{cases} \tag{$*$}$$

for mappings $f : \mathcal{R} \longrightarrow \mathcal{A}$, we deal with Hyers-Ulam stability problem for a *single* equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y)\,. \tag{$**$}$$

The basic question whether or not equation $(**)$ is equivalent to the system $(*)$ has widely been examined by J. Dhombres [3] and the present author in [4] and [5].

1. Introduction

D. G. Bourgin has shown in [2] that given a surjective map f from a ring into a Banach algebra such that both additivity and multiplicativity of f are assumed merely with some (ε, δ)-exactness, i.e.

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

and

$$\|f(xy) - f(x)f(y)\| \le \delta\,,$$

then f has to be a ring homomorphism, i.e. f has to satisfy the system of two Cauchy functional equations

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = f(x)f(y) \end{cases} \tag{$*$}$$

exactly. This stability result has been then generalized by R. Badora in [1] who was applying different methods to get rid of, among others, the surjectivity assumption upon the map in question.

The functional equation

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y)\,, \tag{$**$}$$

resulting from summing up side by side the two Cauchy equations occurring in the system $(*)$, has been studied by J. Dhombres in [3] with the chief concern of finding possibly mild conditions quaranteeing that this *single* equation establishes a ring homomorphism. Under various alternative and less restrictive assumptions this problem was later examined by the present author in [4] and [5].

Bearing these two ideas in mind a natural question arises whether or not the functional equation $(**)$ is stable in the sense just described. In what follows, we are answering that question in affirmative.

2. The result

It should be emphasized that the so called *hyperstability* result obtained by D. G. Bourgin in [2] ((ε,δ)-exactness and the exact validity of the system are equivalent) can hardly be expected when dealing with equation $(*)$. Actually, given a positive ε a straightforward verification proves that an *arbitrary* map f from a ring into a normed algebra, enjoying the property that

$$\|f(x)\|\le\eta \qquad \text{where} \quad 4\eta+\eta^2\le\varepsilon\,,$$

satisies equation $(**)$ with ε-exactness. Observe also that taking arbitrary elements a and r from the domain and the range of the solution f of equation $(**)$, respectively, we can easily check that the map

$$x\longmapsto af(rx)$$

yields a solution to $(**)$ as well, provided that $a^2=a$ and $r^2=r$. Therefore, the maps for which such shifts are bounded are, in a sense, uninteresting in the context spoken of. What about the others? The following result provides an answer to that question.

Theorem. *Let $\mathcal{R}$ be a ring with a unit 1 and let $(\mathcal{A},\|\cdot\|)$ stand for a commutative Banach algebra with a unit e. Given an $\varepsilon\ge 0$ assume that a map $f:\mathcal{R}\longrightarrow\mathcal{A}$ is such that $f(0)=0$, $f(1)=e$, $f(2)=2e$, and*

$$\|f(x+y)+f(xy)-f(x)-f(y)-f(x)f(y)\|\le\varepsilon \quad \textit{for all} \quad x,y\in\mathcal{R}\,. \tag{1}$$

Then either there exist an $a \in \mathcal{A} \setminus \{0\}$ *and an* $r \in \mathcal{R} \setminus \{0\}$ *such that the map*

$$\mathcal{R} \ni x \longmapsto af(rx) \in \mathcal{A} \quad \textit{is bounded} \tag{b}$$

or

$$f \textit{ establishes a ring homomorphism between } \mathcal{R} \textit{ and } \mathcal{A}. \tag{h}$$

Proof. By setting $y = 1$ in (1) we get

$$\|f(x+1) - f(x) - e\| \leq \varepsilon, \quad x \in \mathcal{R}\,,$$

whence

$$\|f(x+2)-f(x)-2e\| \leq \|f((x+1)+1)-f(x+1)-e\|+\|f(x+1)-f(x)-e\| \leq 2\varepsilon,$$

holds true for all $x \in \mathcal{R}$. Now, putting $y = 2$ in (1), we infer that

$$\begin{aligned}\varepsilon &\geq \|f(x+2) + f(2x) - f(x) - 2e - 2f(x)\| \\ &= \|(f(2x) - 2f(x)) - (2e + f(x) - f(x+2))\| \\ &\geq \|f(2x) - 2f(x)\| - \|f(x+2) - f(x) - 2e\| \geq \|f(2x) - 2f(x)\| - 2\varepsilon\end{aligned}$$

and, therefore,

$$\|f(2x) - 2f(x)\| \leq 3\varepsilon\,, \quad x \in \mathcal{R}\,. \tag{2}$$

A standard procedure applied already by D. H. Hyers in [6] gives now, by virtue of the completness of the algebra $(\mathcal{A}, \|\cdot\|)$, the convergence of the Hyers function sequence $(g_n)_n \in \mathbb{N}$ given by the formula

$$g_n(x) := \frac{1}{2^n} f\left(2^n x\right)\,, \quad x \in \mathcal{R},\ n \in \mathbb{N}\,,$$

along with the estimation

$$\|g_n(x) - f(x)\| \leq 3\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right)\varepsilon\,, \quad x \in \mathcal{R},\ n \in \mathbb{N}\,, \tag{3}$$

Consequently, by setting

$$g(x) := \lim_{n\to\infty} g_n(x)\,, \quad x \in \mathcal{R}\,,$$

and applying (1) with x replaced by $2^n x$ we arrive at

$$\left\|\frac{1}{2^n} f\left(2^n x + y\right) + g_n(xy) - g_n(x) - \frac{1}{2^n} f(y) - g_n(x) f(y)\right\| \leq \frac{1}{2^n}\varepsilon,$$

whence, passing to the limit as $n \longrightarrow \infty$, we deduce that

$$\lim_{n\to\infty} \frac{f\left(2^n x + y\right)}{2^n} = g(x)f(y) + g(x) - g(xy)\,, \quad x, y \in \mathcal{R}\,. \tag{4}$$

On the other hand, with x and y replaced by $2^n x$ and $2^n y$, respectively, inequality (1) implies that

$$\left\| \frac{1}{2^n} \left[g_n(x+y) - g_n(x) - g_n(y) \right] + g_{2n}(xy) - g_n(x) \cdot g_n(y) \right\| \leq \frac{1}{2^{2n}} \varepsilon \,,$$

which, after passing to the limit as $n \longrightarrow \infty$, gives the estimation

$$\| 0 \cdot [g(x+y) - g(x) - g(y)] + g(xy) - g(x) \cdot g(y) \| \leq 0 \,,$$

valid for all $x, y \in \mathcal{R}$. This states that g is multiplicative, i.e.

$$g(xy) = g(x)g(y) \,, \quad x, y \in \mathcal{R} \,. \tag{5}$$

Now, equalities (4) and (5) imply that

$$\lim_{n \to \infty} \frac{f\left(2^n x + y\right)}{2^n} = g(x)[f(y) - g(y) + e] \,, \quad x, y \in \mathcal{R} \,,$$

whereas (3) leads obviously to

$$\| f(x) - g(x) \| \leq 3\varepsilon \,, \quad x \in \mathcal{R} \,. \tag{6}$$

On setting $h := f - g + e$ we get finally that

$$\lim_{n \to \infty} \frac{f\left(2^n x + y\right)}{2^n} = g(x)h(y) \,, \quad x, y \in \mathcal{R} \,. \tag{7}$$

On account of the associativity of the addition in the ring $\mathcal{R}$ we derive from (1) the following two inequalities

$$\| f(x+y+z) + f((x+y)z) - f(x+y) - f(z) - f(x+y)f(z) \| \leq \varepsilon$$

and

$$\| - f(x+y+z) - f(x(y+z)) + f(x) + f(y+z) + f(x)f(y+z) \| \leq \varepsilon$$

valid for all triples (x, y, z) from $\mathcal{R}^3$. Summing them side by side and applying the triangle inequality we obtain the estimation

$$\begin{aligned} &\| f((x+y)z) - f(x+y) - f(z) - f(x+y)f(z) \\ &\quad - f(x(y+z)) + f(x) + f(y+z) + f(x)f(y+z) \| \leq \varepsilon \,, \end{aligned}$$

for any $(x, y, z) \in \mathcal{R}^3$.

Replacing here the variable z by $2^n z$ and dividing both sides by 2^n we arrive at

$$\|g_n((x+y)z) - \frac{1}{2^n} f(x+y) - g_n(z) - f(x+y)g_n(z)$$
$$-\frac{1}{2^n} f(2^n xz + xy) + \frac{1}{2^n} f(x) + \frac{1}{2^n} f(2^n z + y) + \frac{1}{2^n} f(x) f(2^n z + y)\| \le \frac{1}{2^n}\varepsilon\,,$$
$$(x,y,z) \in \mathcal{R}^3\,,\ n \in \mathbb{N}.$$

Passing to the limit as $n \longrightarrow \infty$ and applying (7) we deduce that

$$g((x+y)z) - g(z) - f(x+y)g(z) - g(xz)h(xy) + g(z)h(y) + f(x)g(z)h(y) = 0\,,$$

which in view of the multiplicativity of g (see (5)) and the commutativity of $\mathcal{A}$ states that

$$[g(x+y) - e - f(x+y) - g(x)h(xy) + h(y) + f(x)h(y)]\, g(z) = 0\,, \qquad (8)$$

for every triple $(x,y,z) \in \mathcal{R}^3$.

Let $c := g(1) - e$. If $c \neq 0$, then for every $x \in \mathcal{R}$ one has $g(x) = g(1 \cdot x) = g(1)g(x)$, i.e. $c\, g(x) \equiv 0$ whence, by means of (6),

$$\|cf(x)\| = \|cf(x) - c\, g(x)\| \le \|c\| \cdot \|f(x) - g(x)\| \le 3\,\varepsilon\, \|c\|$$

for all $x \in \mathcal{R}$, i.e. we have (b) with $a := c$ and $r := 1$.

If $c = 0$, i.e. $g(1) = e$, then setting $z = 1$ in (8) we obtain an equation

$$h(y) - g(x)h(xy) + f(x)h(y) = f(x+y) - g(x+y) + e = h(x+y)\,, \quad x,y \in \mathcal{R}\,.$$

Since $f = h + g - e$ the latter equation may equivalently be written in the form

$$h(x+y) - h(x)h(y) = g(x)[h(y) - h(xy)]\,, \quad x,y \in \mathcal{R}\,.$$

In particular, on account of the symmetry of the left hand side, one has

$$g(x)[h(y) - h(xy)] = g(y)[h(x) - h(yx)]\,, \quad x,y \in \mathcal{R}\,,$$

whence, by setting here $y = 1$ we conclude that

$$g(x)[e - h(x)] = 0\,, \quad x \in \mathcal{R}\,,$$

because of the equality $h(1) = f(1) - g(1) + e = e$. Consequently, for all $x,y \in \mathcal{R}$ we obtain

$$g(xy)[e - h(x)] = g(y)g(x)[e - h(x)] = 0\,.$$

If we had $b := h(x_0) - e \neq 0$ for some $x_0 \in \mathcal{R} \setminus \{0\}$ (note that $h(0) = e$), we would get $bg(x_0y) \equiv 0$ whence, by means of (6),

$$\|bf(x_0y)\| = \|bf(x_0y) - b\,g(x_0y)\| \leq \|b\| \cdot \|f(x_0y) - g(x_0y)\| \leq 3\,\varepsilon\,\|b\|$$

for all $y \in \mathcal{R}$, i.e. we have (b) with $a := b$ and $r := x_0$.

Thus, the final possibility is: $h(x) \equiv e$ which says nothing but the equality $f = g$. Since g is multiplicative inequality (1) states that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad \text{for all} \quad x, y \in \mathcal{R}\,.$$

The celebrated D. H. Hyers theorem from [6] gives now the existence of an additive map $A : \mathcal{R} \longrightarrow \mathcal{A}$ such that

$$\|f(x) - A(x)\| \leq \varepsilon \quad \text{for every } x \in \mathcal{R}\,. \tag{9}$$

Observe now that for any $x \in \mathcal{R}$ one has

$$f(2x) = g(2x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^{n+1}x\right) = 2 \lim_{n\to\infty} \frac{1}{2^{n+1}} f\left(2^{n+1}x\right) = 2g(x) = 2f(x)\,,$$

whence $f(2^n x) = 2^n f(x)$ for all $x \in \mathcal{R}$ and $n \in \mathbb{N}$. This jointly with (9) implies that

$$2^n \|f(x) - A(x)\| = \|f(2^n x) - A(2^n x)\| \leq \varepsilon\,, \quad x \in \mathcal{R},\ n \in \mathbb{N}\,,$$

which forces f to coincide with A. Consequently, f is both additive and multiplicative, i.e. f establishes a ring homomorphism between $\mathcal{R}$ and $\mathcal{A}$. Thus the proof has been completed.

3. Concluding remarks

The assumptions $f(0) = 0$ and $f(1) = e$ seem to be natural while dealing with homomorphisms. Note that none of them results from inequality (1). The same applies to $f(2) = 2e$; inequality (1) forces only the distance $\|f(2) - 2e\|$ to be majorized by ε. The question whether the commutativity of the target algebra is essential remains open.

The assertion of the Theorem would certainly be more readable if we had simply the alternative: either f is bounded or f is a homomorphism (classical *superstability* effect). Plainly, that is actually the case whenever both the domain ring $\mathcal{R}$ and the Banach algebra $\mathcal{A}$ in question are fields. If $\mathcal{A}$ is a field then f yields a homomorphism provided that no function of the form $x \longmapsto f(rx)$, $r \in \mathcal{R} \setminus \{0\}$, is bounded. If $\mathcal{R}$ is a field then f yields a homomorphism provided that no function af, $a \in \mathcal{A} \setminus \{0\}$, is bounded.

If either $\mathcal{R}$ or $\mathcal{A}$ has no unit the situation becomes sophisticated even while examining equation $(**)$ itself (see [4] and [5]). Thereby, the study of its stability behaviour seems to be even more difficult.

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SPECIAL PARTIAL ORDERINGS IN SIMPLE GRAPHS

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Abstract. We show an algorithm checking whether in a given simple graph G it is possible to introduce a partial ordering whose covering relation agrees with the adjacency relation in G.

We start with recalling some basic definitions of the graph theory.

An *(undirected) graph* $G = (V, E)$ consists of a non-empty set V, called the *vertex-set* and a set E of two element subset $\{u, v\}$ of the set V, called the *edge-set*.

Vertices u and v of a graph $G = (V, E)$ are *adjacent vertices* if $\{u, v\} \in E$. Then, it is said that they are joined by the edge $e = \{u, v\}$. We say that the vertices u and v are *incident* with the edge e, and the edge e is *incident* with the vertices u and v.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exists a bijection $\phi : V \to V'$ such that for every $u, v \in V$

$$\{u, v\} \in E \iff \{\phi(u), \phi(v)\} \in E'.$$

A subgraph of a graph $G = (V, E)$ is a graph $G_1 = (V_1, E_1)$ such that $V_1 \subseteq V$ and $E_1 \subseteq E$.

Assume that $\mathcal{U} = \{A_i; 1 \leq i \leq n\}$ is a family of finite sets.
A sequence $(a_1, a_2, \dots, a_n)$ such that $a_i \in A_i$ for $i = 1, 2, \dots, n$ and $a_i \neq a_j$ for $i \neq j$ is called a transversal or a system of distinct representatives of the family $\mathcal{U}$.

Every transversal of a partition of the vertex-set V of the graph G determines a subgraph of G.

Now, we introduce the notion of an acyclic partition of the vertex-set V.

Definition 1. Let $G = (V, E)$ be a graph. The family $\mathcal{P}$ of non-empty, pairwise disjoint sets V_i, $i = 1, \dots, n$, such that

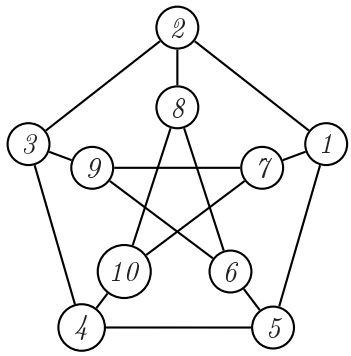
[1] $\bigcup_{i=1}^{n} V_i = V$;

[2] if $\{u, v\} \in E$, then $u \in V_i$ and $v \in V_j$ for $i \neq j; i, j \in \{1, \dots, n\}$;

[3] none of transversals of the family $\mathcal{P}$ contains a cycle

is called an acyclic partition of the vertex-set V.

Example 1. *Consider the following graph G.*



$\mathcal{P}_1 = \{\{1, 3, 6, 10\}, \{2, 4, 7\}, \{5, 8, 9\}\}$;
$\mathcal{P}_2 = \{\{1, 6\}, \{2, 5\}, \{3, 7\}, \{4, 8\}, \{9, 10\}\}$
are acyclic partitions of the vertex-set of G.

Theorem 1. *A graph $G = (V, E)$, in which there exists a cycle of length 3, does not possess an acyclic partition of the vertex-set V.*

Proof. If $\mathcal{P} = \{V_i\}_{1 \leq i \leq n}$ is any acyclic partition of V then, according to the definition, v_1, v_2 and v_3 have to belong to different elements, let say V_1, V_2, V_3 of the partition. However, choosing v_1, v_2 and v_3 as the representatives of V_1, V_2 and V_3 we violate the third condition of the definition. □

By an ordered partition of the set V we mean a sequence $(V_i)_{1 \leq i \leq n}$ of disjoint subsets of V such that $\bigcup_{i=1}^{n} V_i = V$.
Let $G = (V, E)$ be a graph, for which there exists an acyclic ordered partition $\mathcal{P} = (V_i)_{1 \leq i \leq n}$ of the vertex-set V. Let $f_{\mathcal{P}}(v) = i$ for $v \in V_i$.

Theorem 2. *Let $G = (V, E)$ be a graph, for which there exists an acyclic ordered partition $\mathcal{P} = (V_i)_{1 \leq i \leq n}$ of the vertex-set V. Then a binary relation $<_{\mathcal{P}}$ on V defined by*

$u <_{\mathcal{P}} v$ iff there exists a path d between u and v such that $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$ and for every $w \in V \setminus \{u, v\}$ if $w \in d$, then $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(w) <_{\mathcal{P}} f_{\mathcal{P}}(v)$

is a strong partial ordering on V.

Proof. It is clear that the relation $<_{\mathcal{P}}$ is antireflexive and antisymmetric. We will show that it is transitive. Let $u, v, w \in V$ and $u <_{\mathcal{P}} v, v <_{\mathcal{P}} w$. Then there exist the paths d_1 and d_2 between u and v and between v and w respectively, such that $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$ and for every $a \in V \setminus \{u, v\}$ if $a \in d_1$ then $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{<_{\mathcal{P}}}(a) <_{\mathcal{P}} f_{\mathcal{P}}(v)$ and $f_{\mathcal{P}}(v) <_{\mathcal{P}} f_{\mathcal{P}}(w)$ and for every $b \in V \setminus \{v, w\}$ if $b \in d_2$ then $f_{\mathcal{P}}(v) <_{\mathcal{P}} f_{\mathcal{P}}(b) <_{\mathcal{P}} f_{\mathcal{P}}(w)$. Thus, there exists the path d between u and w such that $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(w)$ and for every $c \in V \setminus \{u, w\}$ if $c \in d$, then $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(c) <_{\mathcal{P}} f_{\mathcal{P}}(w)$. We obtain $u <_{\mathcal{P}} w$. Therefore the relation $<_{\mathcal{P}}$ is transitive. □

Let $G = (V, E)$ be a graph with an acyclic ordered partition $\mathcal{P}$ and let the relation $\prec_{\mathcal{P}}$ be the covering relation associated to the relation $<_{\mathcal{P}}$ from Theorem 2.

Theorem 3. *Let $G = (V, E)$ be a graph with the acyclic ordered partition $\mathcal{P}$. For every $u, v \in V$ we have*

$$u \prec_{\mathcal{P}} v \Leftrightarrow (\{u, v\} \in E \wedge f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)).$$

Proof. Let's assume that v covers u. Thus $u <_{\mathcal{P}} v$ and there is no vertex w of V with $u <_{\mathcal{P}} w <_{\mathcal{P}} v$. Then there exists a path d between u and v and $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$. Since none of paths between u and v goes through the vertex w different from u and v, thus the path d is of length 1. Therefore $\{u, v\} \in E$ and $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$.

Now, we assume that $\{u, v\} \in E$ and $f_{\mathcal{P}}(u) <_{\mathcal{P}} f_{\mathcal{P}}(v)$. Since none of transversals of the partition $\mathcal{P}$ contains a cycle, thus the edge $\{u, v\}$ is the only one path between u and v. Therefore v covers u. □

Corollary 1. *If in a graph G there is a cycle of length 3, then it is not possible to introduce a partial ordering whose covering relation agrees with the adjacency relation in G.*

We show an algorithm for finding an acyclic partition of the vertex-set V in the graph $G = (V, E)$, $|V| = n$.

[1] Let V be a vertex-set in graph $G = (V, E)$ and let $(V_i)_{1 \le i \le n}$ be a sequence of sets. We assume that $V_i = \emptyset$ for every $1 \le i \le n$.

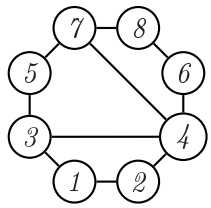
[2] While $V \neq \emptyset$ do:
choose any vertex $v \in V$. Look for i such that none of vertices of the set V_i is joined by an edge with the vertex v and none of transversals of the family of sets $V_1, \ldots, V_{i-1}, V_i \cup \{v\}, V_{i+1}, \ldots, V_n$ contains a cycle.

- if there exists such a set V_i, then we remove the vertex v from the set V and add it into the set V_i.
- else there is not an acyclic partition of the vertex-set V.

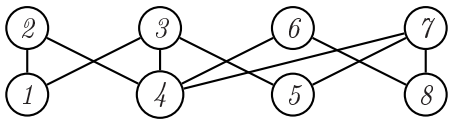
[3] The family $\mathcal{P}$ of non-empty, pairwise disjoint sets V_i of the sequence $(V_i)_{1 \le i \le n}$ is an acyclic partition of the vertex-set V.

It is clear that using this algorithm we can obtain many different acyclic partitions of the vertex-set V, depending on the order of chosen elements.

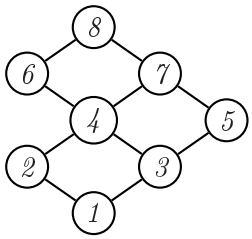
Example 2. *We use this algorithm to introduce a partial ordering relation in the following graph.*



Using this algorithm, we can obtain the following acyclic partition of the vertex-set V: $V_1 = \{1, 4, 5, 8\}$, $V_2 = \{2, 3, 6, 7\}$. In this case, we have the following Hasse diagram



We can also obtain the following acyclic partition of the vertex-set V : $V_1 = \{1\}$, $V_2 = \{2, 3\}$, $V_3 = \{4, 5\}$, $V_4 = \{6, 7\}$, $V_5 = \{8\}$ and now, the Hasse diagram looks as follow



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RANDOMLY kC_n GRAPHS

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Abstract. A graph G is said to be a randomly H graph if and only if any subgraph of G without isolated vertices, which is isomorphic to a subgraph of H, can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H. The problem of characterization of randomly H graphs, where H is $r-$regular graph on p vertices, was given by Tomasta and Tomová. In general, the characterization of such graphs seems to be difficult. However, there exist several results for the case $r = 2$. Chartrand, Oellermann, and Ruiz characterized randomly C_n graphs. Híc and Pokorný characterized randomly $2C_n$ graphs, as well as randomly $C_n \cup C_m$ graphs, where $n \neq m$. In this paper, the problem of randomly H graphs, where $H = kC_n, k > 2$ is discussed.

1. Introduction

In 1986 Chartrand, Oellermann, and Ruiz [2] introduced the term 'randomly H graph' as follows: Let G be a graph containing a subgraph H without isolated vertices. Then G is called a randomly H graph if whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H.

Note that a subgraph H_1 does not have to maintain the partial isomorphism between F and the subgraph of H.

In [1] and [2] authors pointed out the problems concerning isolated vertices in the definition of randomly H graphs. That is why we consider that both H and F are free of isolated vertices.

Every nonempty graph is randomly K_2, while every graph G without isolated vertices is a randomly G graph. K_n is randomly H for every

$H \subseteq K_n$. The graph $K_{3,3}$ is randomly H for every subgraph H of $K_{3,3}$ (see [2, Theorem 1]).

2. Preliminaries

The general question is 'For what classes of graphs H is it possible to characterize all those graphs G that are randomly H?'.

In [5] the characterization of randomly $K_{r,s}$ graphs was given, but in terms of H-closed graphs. In [1] Alavi, Lick, and Tian studied randomly complete n-partite graphs and characterized them.

The problem of characterization of randomly H graphs, where H is r-regular graph on p vertices, was given by Tomasta and Tomová (see [9]). In general the characterization of such graphs seems to be difficult. However, there exist several results for $r = 2$. The list of the most important results about randomly $2-$regular graphs follows.

Theorem A (see Chartrand, Oellermann, and Ruiz [2]) A graph G is randomly C_3 if and only if each component of G is a complete graph of order at least 3.

Theorem B (see Chartrand, Oellermann and Ruiz [2]) A graph G is randomly C_4 if and only if

1. $G = K_p$, where $p \geq 4$, or
2. $G = K_{r,s}$, where $2 \leq r \leq s$.

Theorem C (see Chartrand, Oellermann, and Ruiz [2]) A graph G is randomly C_n, $n \geq 5$, if and only if

1. $G = K_p$, where $p \geq n$, or
2. $G = C_n$, or
3. $G = K_{n/2,n/2}$ and n is even.

Theorem D (see Tomasta and Tomová [9]) Let H be connected n-vertex regular graph of degree $r \geq 2$ different from K_3 and C_4. Then a p-vertex graph G where $p > n$ is randomly H if and only if $G = K_p$.

Theorem E (see Híc and Pokorný [7]) A graph G is randomly $2C_3$ if and only if

1. $G = K_p,\ p \geq 6$, or
2. $G = K_{p_1} \cup K_{p_2} \cup ... \cup K_{p_n}$, where $n \geq 2$, $p_i = 3$ or $p_i \geq 6$.

Theorem F (see Híc and Pokorný [7]) A graph G is randomly $2C_{2n+1}$, where $n \geq 2$, if and only if

1. $G = 2C_{2n+1}$, or
2. $G = 2K_{2n+1}$, or

3. $G = C_{2n+1} \cup K_{2n+1}$, or
4. $G = K_p$, $p \geq 2(2n+1)$.

Theorem G (see Híc and Pokorný [7]) A graph G is randomly $2C_4$ if and only if
1. $G = K_{r,s}$, where $4 \leq r \leq s$, or
2. $G = 2C_4$, or
3. $G = 2K_4$, or
4. $G = C_4 \cup K_4$, or
5. $G = K_p$, where $p \geq 8$.

Theorem H (see Híc and Pokorný [7]) A graph G is randomly $2C_{2n}$, where $n \geq 3$, if and only if
1. $G = 2K_{2n}$, or
2. $G = 2C_{2n}$, or
3. $G = 2K_{n,n}$, or
4. $G = C_{2n} \cup K_{n,n}$, or
5. $G = C_{2n} \cup K_{2n}$, or
6. $G = K_{n,n} \cup K_{2n}$, or
7. $G = K_{2n,2n}$, or
8. $G = K_p$, $p \geq 4n$.

Theorem I (see Híc and Pokorný [8]) A graph G is randomly $C_n \cup C_m$, where $3 \leq n < m$ if and only if
1. $G = C_n \cup C_m$, or
2. $G = K_n \cup C_m$, or
3. $G = K_{\frac{n}{2},\frac{n}{2}} \cup C_m$ where n is even, or
4. $G = K_{\frac{m+n}{2},\frac{m+n}{2}}$ where both m and n are even, or
5. $G = K_p$, where $p \geq m+n$.

This paper deals with randomly $2-$ regular graphs H, where $H = kC_n$ for $k \geq 3$. All the terms used in this paper can be found in [4]. Especially, if H is a subgraph of G, we will use $G - H = \langle V(G) - V(H) \rangle$ to denote the induced subgraph of the graph G with the vertex set $V(G) - V(H)$.

3. Results

Theorem 1 A graph G is randomly kC_3 if and only if
(i) $G = K_p,\ p \geq 3k$, or
(ii) $G = K_{p_1} \cup K_{p_2} \cup ... \cup K_{p_n}$, where $n \geq 2$, $p_i = 3$ or $p_i \geq 3k$, $\sum p_i \geq 3k$.

Proof: The sufficiency is obvious, as both $G = K_p$, where $p \geq 3k$, and $G = K_{p_1} \cup K_{p_2} \cup ... \cup K_{p_n}$, where $n \geq 2$, $p_i = 3$ or $p_i \geq 3k$, are randomly kC_3 graphs.

For necessity, let G be a randomly kC_3 graph. Let G be connected. We will prove that no edge is missing.

Assume the contrary. Let the edge $\{u, v\}$ be missing. Since G is connected, there exists a $u-v$ path. Let us choose the shortest $u-v$ path P. Obviously, the length of P is at least 2. Then no subpath of P of the length 2 cannot be expanded into C_3, a contradiction with the assumption that G is a randomly kC_3 graph.

Now, let G not to be connected. Then G cannot contain components K_{p_i}, where $p_i \neq 3, p_i < 3k$. The case $p_i = 2$ is trivial. If $p_i \in 4, 5, 6, ..., k-1$ those components contain a subgraph F isomorphic to $\lfloor \frac{p_i}{2} \rfloor K_2$. Although F is isomorphic to a subgraph of kC_3, F cannot be extended to kC_3, a contradiction. If $p_i \in k, k+1, k+2, ..., 3k-1$ those components contain a subgraph F isomorphic to kK_2. Although F is isomorphic to a subgraph of kC_3, F cannot be extended to kC_3, a contradiction. If some component of G has exactly three vertices, then the component is isomorphic to K_3, as G is a randomly kC_3 graph. If some component of G has more than $3k-1$ vertices, then the component is a complete subgraph of G. Otherwise, we can produce a contradiction similarly to the connected case.

Lemma 1: Let G be a disconnected randomly kC_n graph, where $n > 3$. Then G has k components and each of the components has exactly n vertices.

Proof: a) Let G have more than k components. Let a subgraph F of G consist of $k+1$ edges which belong to $k+1$ different components of G. Subgraph F is isomorphic to some subgraph of kC_n. However, the subgraph F cannot be extended to kC_n, a contradiction.

b) Let G have less than k components. Then there exists a component that has more than n vertices. Let C be a component of G which has at least $n+1$ vertices. Then there exists a subgraph F_1 of C, which consists of a path on $n-1$ vertices and of one edge. Let $\{u, v\}$ be an edge of the other component of G. Let $F = F_1 \cup \{u, v\}$. Obviously F is a subgraph of G which is isomorphic to a subgraph of kC_n. However, F cannot be extended to kC_n, a contradiction.

c) Let G have k components. Obviously none of the components has less than n vertices. Similarly to case b we can prove that none of the components has more than n vertices.

According to a), b) and c), G has k components and each component has n vertices.

Theorem 2: A graph G is randomly kC_{2n+1}, where $n \geq 2$, if and only if
(i) $G = H_1 \cup H_2 \cup ... \cup H_k$, where H_i is either C_{2n+1} or K_{2n+1}, or
(ii) $G = K_p$, $p \geq k(2n+1)$.

Proof: It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse we assume that G is a randomly kC_{2n+1} graph.

1. Assume first that G is disconnected. We will prove that G fits part (i) in the statement of the theorem.

Lemma 1 implies that G has k components and each component has exactly $2n+1$ vertices. Since G is randomly kC_{2n+1}, each component is randomly C_{2n+1}. According to Theorem C, each component of G is either $G = C_{2n+1}$ or $G = K_{2n+1}$, so G fits part (i).

2. We henceforth assume that G is connected. We will prove that $G = K_p$, $p \geq k(2n+1)$. Let us use the mathematical induction. If $k = 2$, then by Theorem F, part 4 holds $G = K_p$. Assume that the statement holds for every $r < k$. Let H be a subgraph of G which is isomorphic to C_{2n+1}.

a) Let $G' = G - H$. Let us notice that G' is a randomly $(k-1)C_{2n+1}$ graph. By the inductive hypothesis, G' is a complete graph. Now we will prove that $G'' = \langle V(H) \rangle$ is complete, too. Assume the contrary.

Let $G'' = C_{2n+1}$. Then $V(G'') = V(H) = \{v_1, v_2, ..., v_{2n+1}\}$ and $E(G'') = \{\{v_i, v_{i+1}\}; i = 1, 2, ...2n\} \cup \{\{v_{2n+1}, v_1\}\}$. Since G is connected, there exists an edge $\{u, v\}$, where $u \in V(H)$, $v \in V(G')$. Without loss of generality we may assume that $v = v_1$. Let us construct the path $u, v_1, v_2, ..., v_{2n-1}, v_{2n}$. This path can be extended to C_{2n+1} only by adding the edge $\{v_{2n}, u\}$. Now let us construct the path $v_{2n+1}, v_{2n}, u, v_1, v_2, ..., v_{2n-3}, v_{2n-2}$. This path can be extended to C_{2n+1} only by adding $\{v_{2n-2}, v_{2n+1}\}$. So G'' is not isomorphic to C_{2n+1}, a contradiction. Then, by Theorem C, $G'' = K_{p'}$, $p' = 2n+1$.

b) Now it is necessary to prove that for every $u \in V(G')$, $v \in V(G'')$ the graph G contains the edge $\{u, v\}$. It is sufficient to choose $u - v$ path on $2n+1$ vertices. Since both G' and G'' are complete and G is connected, the path always exists and can be extended to C_{2n+1} only if $\{u, v\}$ edge exists. Since both u and v are arbitrary vertices, G is complete.

The formula $p \geq k(2n+1)$ follows from the fact that G is randomly kC_{2n+1}.

Lemma 2: Let G be a randomly kC_4 graph, $K_{r,s} \subset G$, $K_{r,s} \neq G$, where $2k \leq r \leq s$. Let $V(G) = V(K_{r,s})$. Then G is a complete graph.

Proof: Let $G \supset K_{r,s}$ and $V(G) = V(K_{r,s}) = \{u_1, ..., u_r\} \cup \{v_1, ..., v_s\}$. Let $\{u_i, u_j\} \in E(G)$, $\{u_i, u_j\} \notin E(K_{r,s})$. Let v_q and v_l be arbitrary vertices which belong to the different partite set than u_i and u_j. Let us construct the path v_q, u_i, u_j, v_l. Because G is randomly kC_4, the path can be extended to C_4 only by adding the edge $\{v_q, v_l\}$. Since v_q and v_l are arbitrary vertices, $\{v_q, v_l\} \in E(G)$ for every q, l. If we use a similar method with the edge $\{v_i, v_j\} \in E(G)$, we prove that G is a complete graph.

Theorem 3: A graph G is randomly kC_4 if and only if
(i) $G = K_{r,s}$, where $2k \leq r \leq s$, or
(ii) $G = H_1 \cup H_2 \cup ... \cup H_k$, where H_i is either C_4 or K_4 , or
(iii) $G = K_p$, where $p \geq 4k$.

Proof: It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse then, we assume that G is randomly kC_4.

1. Assume first that G is disconnected. We will prove that G fits part (ii) in the statement of the theorem.

Lemma 1 implies that G has k components and each component has exactly four vertices. Since G is randomly kC_4, each of its component has to be randomly C_4. According to Theorem B the graph G fits the part (ii) in the statement of the theorem.

2. We henceforth assume that G is a connected randomly kC_4 graph. We will prove that either $G = K_{r,s}$, where $2k \leq r \leq s$ or $G = K_p$, where $p \geq 4k$. We will use the mathematical induction. If $k = 2$, then it is true by Theorem G, parts 1 and 5.

Assume that the statement holds for every $r < k$. Let H be a subgraph of G isomorphic to C_4. Let $G' = G - H$. Since G is randomly kC_4, G' is randomly $(k-1)C_4$. By the inductive hypothesis, either $G' = K_t$, $t \geq 4(k-1)$ or $G' = K_{a,b}$, $2(k-1) \leq a \leq b$.

a) Let $G' = K_{a,b}$, $2(k-1) \leq a \leq b$. Let $\{w_1, w_2, ..., w_b\}$ and $\{s_1, s_2, ..., s_a\}$ be a partition of $V(G')$ into two sets. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be a partition of $V(H)$ into two sets. Since G is connected, it contains an edge which starts in $V(H)$ and ends in $V(G')$. Without loss of generality we may assume that the edge is $\{v_1, w_1\}$. Since the graph G is randomly kC_4, every path u_i, v_1, w_1, s_j can be extended to C_4 by adding the edge $\{s_j, u_i\}$ for every $i = 1, 2$ and $j = 1, 2, ..., a$. Using the same method with the edge $\{u_1, s_1\}$ we prove that the graph G contains all edges $\{v_i, w_j\}$ for every $i = 1, 2$ and $j = 1, 2, ..., b$. So $G \supset K_{a+2,b+2}$ and $V(G) = V(K_{a+2,b+2})$.

If $G \neq K_{a+2,b+2}$, by Lemma 2 $G = K_p$, $p \geq 4k$. Otherwise $G = K_{r,s}$, where $2k \leq r \leq s$.

b) Let $G' = K_t$, $t \geq 4$. Using the same method as in part a) it is not difficult to prove that G is a complete graph, which means that $G = K_p$, $p \geq 4k$.

Lemma 3: Let G be a randomly kC_{2n} graph, $K_{2n,2n} \subset G$, where $n > 2$. Let $V(G) = V(K_{2n,2n})$. Then G is a complete graph.

Proof: The proof is similar to that of Lemma 2. A major difference is that we have to consider a path on $2n$ vertices instead of a path on 4 vertices.

Theorem 4: A graph G is randomly kC_{2n}, where $n \geq 3$, if and only if
(i) $G = H_1 \cup H_2 \cup ... \cup H_k$, where H_i is C_{2n} or K_{2n} or $G = K_{n,n}$, or
(ii) $G = K_{kn,kn}$, or
(iii) $G = K_p$, $p \geq 2kn$.

Proof: It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse then, we assume that G is randomly kC_{2n}.

1. Assume first that G is disconnected. We will prove that G fits part (i) in the statement of the theorem.

Lemma 1 implies that G has k components and each component has exactly $2n$ vertices. Since G is randomly kC_{2n}, each of the components is randomly C_{2n}.

According to Theorem H, parts 1-6, G fits the part (i).

2. We henceforth assume that G is connected.

a) Let $|V(G)| > 2kn$. We will prove that G is complete. We will use the mathematical induction. If $k = 2$, it is true by Theorem H, part 8.

Assume that the statement holds for every $r < k$. Let H be a subgraph of G isomorphic to C_{2n}. Let $G' = G - H$. Since G is randomly kC_{2n}, G' is a randomly $(k-1)C_{2n}$ graph of order $p' > 2(k-1)n$. Therefore, by the inductive hypothesis, G' is complete.

Similarly, if we choose a subgraph of G' isomorphic to C_{2n}, we will obtain that $\langle V(H) \rangle$ is complete.

Now it is sufficient to prove that for every $v \in V(H)$ and for every $w \in V(G')$ there exists the edge $\{v, w\} \in E(G)$. As G is connected and both $\langle V(H) \rangle$ and G' are complete, there exists a $v - w$ path on $2n$ vertices. Let us choose this path. Because G is randomly kC_{2n}, the path can be extended to C_{2n} by adding the edge $\{v, w\}$. That is why G is complete.

b) Let $|V(G)| = 2kn$. Using mathematical induction we will prove that either $G = K_{kn,kn}$ or $G = K_p, p \geq 2kn$. If $k = 2$, it is true by Theorem H, cases 7 and 8.

Assume that the statement holds for every $r < k$. Let us construct graphs H and G' similarly to case a). Since the order of G' is $2(k-1)n$ and G is randomly kC_{2n}, by the inductive hypothesis, G' is isomorphic to $K_{(k-1)n,(k-1)n}$ or $K_{p'}, p' = 2(k-1)n$.

b$_1$) If G' is isomorphic to $K_{p'}, p' = 2(k-1)n$ or $K_{(k-1)n,(k-1)n}$. Similarly to Theorem 2, part a we can prove that $\langle V(H) \rangle$ is not isomorphic to C_{2n}.

b$_2$) Let G' be isomorphic to $K_{(k-1)n,(k-1)n}$ and $\langle V(H) \rangle$ be isomorphic to $K_{n,n}$. Let $\{w_1, w_2, ..., w_{(k-1)n}\}$ and $\{s_1, s_2, ..., s_{(k-1)n}\}$ be a partition of $V(G')$ into two sets. Let $\{u_1, u_2, ..., u_n\}$ and $\{v_1, v_2, ..., v_n\}$ be a partition of $V(H)$ into two sets. We will prove that for every $i, j \in \{1, 2, ..., n\}$ $E(G)$ contains the

edge $\{s_i, v_j\}$. Since G is connected, it contains an edge which starts in $V(H)$ and ends in $V(G')$. Without loss of generality we may assume that the edge is $\{v_1, w_1\}$. Let us construct the path $v_i, u_1, w_1, s_1, w_2, s_2, ..., w_{j-1}, s_{j-1}, w_{j+1}, s_{j+1}, ..., s_{n-1}, w_n, s_j$. Since G is randomly kC_{2n}, the path can be extended to C_{2n} by adding the edge $\{s_j, v_i\}$, so $\{s_j, v_i\} \in E(G)$.

Using the similar method we can prove that $\{w_i, u_j\} \in E(G)$ for every $i, j \in \{1, 2, ..., n\}$. So $G \supset K_{kn,kn}$. If $G \neq K_{kn,kn}$, then by Lemma 3 $G = K_{2kn}$.

b$_3$) If G' is isomorphic to $K_{2(k-1)n}$ and $\langle V(H)\rangle$ is isomorphic to K_{2n} or $K_{n,n}$, then similarly to b$_2$ case we can prove that $G \supset K_{kn,kn}$ and $G \neq K_{kn,kn}$. Then, by Lemma 3, $G = K_{2kn}$.

4. Conclusion

In the paper a complete characterization of randomly H graphs where $H = kC_n$ is given. The case of 2−regular randomly H graphs, which contain more than two circuits of which at least two are different, remains open.

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REMARKS ON CONNECTIVITY AND I-CONNECTIVITY

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The following definition has been introduced by Jadwiga Knop and Małgorzata Wróbel in 2006 (see [3]).

Definition 1. (J. Knop & M. Wróbel – 2006) A subset A of a topological space X is called to be i-connected if it is connected and $\mathrm{Int}\,(A)$ is nonempty and connected.

Of course this definition requires much more from a set as for usual connectedness. However, in the space of real numbers endowed by natural topology each connected set fulfils the condition from definition 1. Some of properties of i-connected sets were described in that article. We want to discuss the problem, in what kinds of spaces each connected set is also i-connected.

If $\mathscr{L}$ is any ideal of sets which does not contain any nonempty open set, then Hashimoto topology generated by this ideal fulfils our requirement.

It is not difficult to observe that any space which is homeomorphic to the space of real numbers endowed by the described Hashimoto topology fulfils our requirements i.e. each connected set with nonempty interior is i-connected as well.

Another example of such spaces is the so called "long line" ([4]) and each space which is homeomorphic to that space.

One can observe that a circle on a plane with topology generated by Euclidean space $\mathbb{R}^2$ also fulfils the considered condition.

Now we will consider some of necessary conditions for a topological space to fulfill the considered condition. From now on $\mathscr{J}$ will denote the class of topological spaces in which every connected set with nonempty interior is i-connected. Before we formulate next theorem we will remind the denotation. For a subset E of a topological space X by A^d, $\mathrm{Int}\,(E)$ and $\overline{E}$ we will denote

the set of all accumulation (adjoint) points of the set E, the interior of the set E and closure of the set E, respectively.

We defined so called k-connected subsets of a topological space, which are a little different than i-connected sets, but have some interesting properties.

Definition 2. [1] A subset A of a topological space X is called to be k-connected if its interior is connected and $A \subset \overline{\mathrm{Int}\,(A)}$.

Of course, each k-connected set is i-connected as well.

It is quite easy to see that for euclidean space of real numbers $\mathbb{R}$, a subset A is k-connected if and only if it is connected.

By $\mathscr{K}$ we will denote the class of topological spaces in which every connected set with nonempty interior is k-connected. Hence $\mathscr{K} \subset \mathscr{J}$.

As we could observe, k-connected sets are similar to i-connected ones and we will make use of this notion, so it is worth to compare those kinds of sets.

Theorem 1. [1] A subset A of a topological space is i-connected if and only if it can be represented in the form

$$A = B \cup C,$$

such that B is k-connected and

$$B \cap C = \emptyset, \quad \mathrm{Int}\,(C) = \emptyset$$

and each component of C is not separated with B.

Theorem 2. Let X be a topological space in which each connected set with a nonempty interior is i-connected. Then:

(1) for each nonempty, open and disjoint and connected subsets A, B and C of the space X

$$\overline{A} \cap \overline{B} \cap \overline{C} = \emptyset.$$

Proof. Let $X \in \mathscr{J}$ and suppose that condition (1) is not fulfilled. There exist then nonempty disjoint open and connected sets A, B and C such that

$$\overline{A} \cap \overline{B} \cap \overline{C} \neq \emptyset.$$

Let $x \in \overline{A} \cap \overline{B} \cap \overline{C}$. Consider the set $A \cup B \cup \{x\}$ and denote it by E. Since the set A is not separated from $\{x\}$ and the set B is not separated from $\{x\}$ then E is a connected set.

From condition $x \in \overline{C}$ it follows that $U \cap C \neq \emptyset$ for each neighbourhood U of the point x. Hence

$$x \notin \mathrm{Int}\,(A \cup B)\,.$$

From here we infer that

$$\operatorname{Int}(E) = A \cup B.$$

The set $\operatorname{Int}(E)$ is not connected and in consequence E is connected has a nonempty interior and is not i-connected. Contradiction completes the proof. □

By a cut point of a connected space X we mean a point x such that $X \setminus \{x\}$ is not connected. A cut point is called strong cut point of a connected space X if the set $X \setminus \{x\}$ has 2 components.

One can see that not every point belonging to a space X from the class $\mathscr{J}$ is a cut point. It may happen that there is no cut point of such a space. However:

Corollary 1. If $X \in \mathscr{J}$ and x is a cut point, then it is a strong cut point of X.

Theorem 3. Let X be a topological space in which each connected set with a nonempty interior is i-connected. Then:

(2) for each nonempty, open and disjoint and connected subsets A and B of X

$$\left(\overline{A} \cap \overline{B}\right)^d = \emptyset.$$

Proof. Let $X \in \mathscr{J}$ and suppose that there exist two disjoint nonempty open and connected sets A and B such that

$$\left(\overline{A} \cap \overline{B}\right)^d \neq \emptyset.$$

Let $x \in \left(\overline{A} \cap \overline{B}\right)^d$ and $E = A \cup B \cup \{x\}$.

Since the sets A and B are connected and not separated from $\{x\}$, then the set E is connected. As before it is not difficult to notice that $x \notin \operatorname{Int}(E)$. It follows then that $\operatorname{Int}(E)$ is not connected, i.e. E is not i-connected, what completes the proof of condition (2). □

Theorem 4. If X is a locally connected Hausdorff space in which every i-connected set is k-connected, then

(3) there is no connected set A having at least two elements such that $\operatorname{Int}(A) = \emptyset$.

Proof. Suppose that there exists a connected set A and points x and y such that

$$\operatorname{Int}(A) = \emptyset, \quad x \in A, \quad y \in A, \quad x \neq y.$$

It follows from local connectedness of X that there exists a neighbourhood U of x such that $y \notin U$.

Let $B = U \cup A$. The set B is connected because of both set A and U are connected and non-disjoint.

Now we will show that

(4) $\operatorname{Int}(B) \subset \overline{U}$.

Suppose that $a \in \operatorname{Int}(B) \setminus \overline{U}$. There exists a neighbourhood V of the point a such that

$$a \in V \quad \text{and} \quad V \subset B \setminus \overline{U}.$$

On the other hand:

$$V = \operatorname{Int}(V) = \operatorname{Int}(V \cap B) = \operatorname{Int}(V \cap (U \cup A)) =$$

$$= \operatorname{Int}((V \cap U) \cup (V \cap A)) = \operatorname{Int}(V \cap A) = \emptyset.$$

Contradiction completes the proof of the required inclusion (4). Inclusions $U \subset B$ and (4) imply inclusions

$$U \subset \operatorname{Int}(B) \subset \overline{U},$$

which prove, that $\operatorname{Int}(B)$ is connected.

In that way we have proved that the set B is i-connected. Hence it is k-connected as well. Thus

$$B \subset \overline{\operatorname{Int}(B)} \subset \overline{U}$$

what is impossible in view of $y \in B$ and $y \notin \overline{U}$. □

We remind that a topological space is called totally disconnected if its every component is a singleton.

Theorem 5. If X is a topological space such that there is no connected set A having at least two elements such that $\operatorname{Int}(A) = \emptyset$, then any i-connected set E can be represented in the form $E = B \cup C$ such that B is k-connected, C is totally disconnected and $B \cap C = \emptyset$.

Proof.

Let E be i-connected set which is not k-connected. Then E and $\operatorname{Int}(E)$ are connected and

$$E \not\subset \overline{\operatorname{Int}(E)}.$$

Let A be any component of the set $E \setminus \overline{\operatorname{Int}(E)}$. The set A is nonempty. Moreover $\operatorname{Int}(A) = \emptyset$, since

$$\operatorname{Int}(A) \subset \operatorname{Int}\left(E \cap (X \setminus \overline{\operatorname{Int}(E)}\right) = \operatorname{Int}(E) \setminus \overline{\operatorname{Int}(E)} = \emptyset.$$

In view of assumptions the set A must not have more than one element, hence it is a singleton. In that way we proved that

$$E = \left(E \cap \overline{\operatorname{Int}(E)}\right) \cup \left(E \setminus \overline{\operatorname{Int}(E)}\right),$$

where $B = E \cap \overline{\operatorname{Int}(E)}$ is k-connected, C, where $C = E \setminus \overline{\operatorname{Int}(E)}$, is totally disconnected and $B \cap C = \emptyset$. □

Theorem 6. If X is a locally connected Hausdorff space and each component of X belongs to the class $\mathscr{J}$, then $X \in \mathscr{J}$.

Proof. If E is a connected set in X with a nonempty interior, then it is contained in one of the components of X, say C. Since $C \in \mathscr{J}$ then E is i-connected in C. The set C is open then

$$\operatorname{Int}_C(E) = C \cap \operatorname{Int}_X(E) = \operatorname{Int}_X(E).$$

Moreover $\operatorname{Int}_C(E)$ is connected, thus $\operatorname{Int}_X(E)$ is connected.

In this way we have proved that E is i-connected. □

It is quite obvious that if a locally connected Hausdorff space X is in the class $\mathscr{J}$, then each component of X also belongs to the class $\mathscr{J}$. Thus:

Corollary 2. If X is a locally connected Hausdorff space, then $X \in \mathscr{J}$ if and only if each component of X belongs to the class $\mathscr{J}$.

Local connectedness of the space X is necessary, since if

$$X = \bigcup_{n=0}^{\infty} C_n,$$

where

$$C_n = \left\{[0,1] \times \left\{\frac{1}{n}\right\} : n \in \mathbb{N}_+\right\}, \quad n \in \mathbb{N}_+$$

and

$$C_0 = [-1,2] \times \{0\},$$

then X is not locally connected and each of the components belongs to the class $\mathscr{J}$, but X does not, since C_0 is a connected subset of X and $\operatorname{Int}(C_0) = ([-1,0) \cup (1,2]) \times \{0\}$ is nonempty and is not connected.

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SOME PROPERTIES OF i–CONNECTED SETS (PART II)

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Abstract. A generalization theorem for i–connected sets in the Hashimoto topology is given. Moreover, i–connectivity in the topology of at most countable complements and in the order topology is presented.

1. Introduction

Let (X, T) be a topological space and let P stand for some property of the subsets of X. We denote by $\mathcal{P}$ the family of all subsets of X which satisfy P. We say that a subset A has the property P at the point $p \in X$, if there exists a neighbourhood V_p of p such that $V_p \cap A \in \mathcal{P}$. We introduce the symbol A^* to define the set of all points at which A does not have the property P, i.e.

$$A^* = \left\{ p \in X : \bigwedge_{V_p} (V_p \cap A \notin \mathcal{P}) \right\}. \tag{1}$$

Additionally, let us assume that the family $\mathcal{P}$ is ideal, i.e.

(2) the relations $A \in \mathcal{P}$ and $B \in \mathcal{P}$ imply $A \cup B \in \mathcal{P}$,

(3) the relations $A \in \mathcal{P}$ and $B \subset A$ imply $B \in \mathcal{P}$.

Moreover, let us assume that the property P satisfies the following

(4) $(A \in \mathcal{P}) \Leftrightarrow (A \cap A^* = \phi) \Leftrightarrow (A^* = \phi)$,

and that

(5) every single element subset of X belongs to $\mathcal{P}$.

Example 1. The family of the sets of the first category and the family of the sets of measure zero in the sense of Lebesgue satisfy conditions (2)-(5).

Let (X,T) be T_1 – space. Using an ideal $\mathcal{P}$ we can introduce the new topology on the set X, the so–called Hashimoto topology, defined by the formula

$$T^* = \{U \backslash F \subset X : U \in T \wedge F \in \mathcal{P}\}. \tag{6}$$

We see easily that $T \subset T^*$.

For simplicity of the notation we continue to write $\mathrm{int}A\,(\mathrm{cl}A)$ for the interior (the closure) of A in the topological space (X,T) and $\mathrm{int}_*A\,(\mathrm{cl}_*A)$ for the interior (the closure) of A in the space (X,T^*).

Because T^* is a stronger topology we have

(7) $\bigwedge\limits_{A\subset X} \mathrm{int}A \subset \mathrm{int}_*A,$

(8) $\bigwedge\limits_{A\subset X} \mathrm{cl}_*A \subset \mathrm{cl}A,$

(9) if $M \subset X$ is connected in the Hashimoto topology (X,T^*), then M is connected in (X,T).

Now let us recall the definition of an i–connected set (cf. [2]).

Definition 1. Let (X,T) be a topological space. A set $A \subset X$ is said to be *i–connected* if it has a nonempty interior and both A and $\mathrm{int}A$ are connected.

Example 2. In the natural topology on the straight line every connected set which has a nonempty interior is i–connected. Note that no similar fact holds for the Euclidean plane. For instance, a set consisting of two tangent discs is connected but it's interior is not.

2. The i–connected sets in the Hashimoto topology

We will need the following lemmas.

Lemma 1. (cf. [1], p. 6). Let (X,T) be a topological space. An open set G belonging to $\mathcal{P}$ is contained in $X \backslash X^*$.

Lemma 2. (cf. [1], p. 7). If the space (X,T) satisfies $X^* = X$, then $\mathrm{cl}G = \mathrm{cl}_*G$ for every $G \in T^*$.

Lemma 3. Let (X,T) be a topological space and let $X^* = X$. If $A \subset X$ is open and connected in (X,T), then A is connected in (X,T^*).

Proof. Let us suppose that A is disconnected in (X, T^*). Then we can represent the set A as the sum of two nonempty disjoint open subsets in the subspace (A, T^*_A). Since $A \in T$ and $T \subset T^*$, (6) shows that there exist two nonempty sets $U_1, U_2 \in T$ and two sets $F_1, F_2 \in \mathcal{P}$ such that $A = (U_1 \backslash F_1) \cup (U_2 \backslash F_2)$ and $(U_1 \backslash F_1) \cap (U_2 \backslash F_2) = \phi$. Hence $A \subset U_1 \cup U_2$ and

$$U_1 \cap U_2 \subset F_1 \cup F_2. \tag{10}$$

Because $U_1 \cap U_2 \in T, F_1 \cup F_2 \in \mathcal{P}$ and $X = X^*$, therefore, by Lemma 1 and (10), we obtain that $U_1 \cap U_2 = \phi$ which means that A is disconnected in (X, T), contrary to the assumption. □

Theorem 1. Let (X, T) be a topological space and let $X = X^*$. If a set $A \subset X$ has a nonempty and connected interior in the space (X, T) and $A \subset \mathrm{cl}\,\mathrm{int} A$, then A is i–connected in the space (X, T^*).

Proof. Since $\mathrm{int} A$ in open and connected in (X, T), by Lemma 3, it is connected in (X, T^*). By (7) we have

$$\mathrm{int} A \subset \mathrm{int}_* A \subset A \subset \mathrm{cl} A.$$

Moreover, by the assumption and Lemma 2, we get

$$\mathrm{cl} A = \mathrm{cl}\,\mathrm{int} A = \mathrm{cl}_*\,\mathrm{int} A$$

and finally,

$$\mathrm{int} A \subset \mathrm{int}_* A \subset A \subset \mathrm{cl}_*\,\mathrm{int} A.$$

Since $\mathrm{int} A$ and $\mathrm{cl}_*\,\mathrm{int} A$ (as the closure of a connected set) are connected in (X, T^*), the sets $\mathrm{int}_* A$ and A are connected in (X, T^*), and the proof is completed. □

Note that every set satisfying the assumptions of the above theorem is i–connected in (X, T). Therefore we obtain the following

Corollary 1. If a set $A \subset X$ is i–connected in the space (X, T) and $X = X^*$ and $\mathrm{cl} A = \mathrm{cl}\,\mathrm{int} A$, then A is i–connected in the space (X, T^*).

Remark 1. Taking in the above theorem $(X, T) = (\mathbb{R}^2, T_d)$ and $T^* = \{U \backslash F \subset \mathbb{R}^2 : U \in T_d \text{ and } \mu(F) = 0\}$, where μ denotes the Lebesgue measure in $\mathbb{R}^2$ and T_d is the family of open sets in the Euclidean plane, one gets Theorem 3 of [2].

3. The i–connectivity in the topology of at most countable complements

We start with the following lemma which gives the equivalent condition on connectivity of open sets.

Lemma 4. Let (X,T) be a topological space and let a nonempty and open set $A \subset X$ be fixed. Then a set A is connected in the space (X,T) if and only if the following condition is fulfilled

$$\bigwedge_{A_1 \subset A} [(A_1 \neq \phi \wedge A_1 \neq A \wedge A_1 \in T) \Rightarrow A \backslash A_1 \notin T]. \tag{11}$$

Proof.

$\Rightarrow$ Let us suppose, contrary to our claim, that there exists a nonempty and open set A_1 such that $A_1 \subset A$, $A_1 \neq A$ and $A \backslash A_1 \in T$. Then the set A can be represented as the union of two nonempty disjoint and open sets A_1 and $A \backslash A_1$, which is impossible.

$\Leftarrow$ Conversely, suppose that there exist two nonempty open sets $A_1, A_2 \subset X$ such that $A \cap A_1 \neq \emptyset, A \cap A_2 \neq \emptyset$, $(A \cap A_1) \cap (A \cap A_2) = \emptyset$ and $A = (A \cap A_1) \cup (A \cap A_2)$. Since A is open therefore $A \cap A_1 \in T$ and $A \cap A_2 \in T$. It follows that there exists a nonempty and open set $B = A \cap A_1$ such that $B \subset A, B \neq A$ and $A \backslash B \in T$, contrary to (11). $\square$

Let $\mathfrak{M}$ be a family of all subsets of X which satisfy (11), i.e.

$$\mathfrak{M} := \left\{ A \subset X : \bigwedge_{A_1 \subset A} [(A_1 \neq \phi \wedge A_1 \neq A \wedge A_1 \in T) \Rightarrow A \backslash A_1 \notin T] \right\}.$$

We present some properties of the family $\mathfrak{M}$:

(a) A nonempty and open set is connected if and only if it belongs to $\mathfrak{M}$.

(b) A nonempty and open set is i–connected if and only if it belongs to $\mathfrak{M}$.

(c) If a set is i–connected, then it's interior belongs to $\mathfrak{M}$.

(d) A topological space (X,T) is connected if and only if X belongs to $\mathfrak{M}$.

(e) A topological space (X,T) is connected if and only if every nonempty and closed subset of X belongs to $\mathfrak{M}$.

Remark 2. If we denote by $\mathcal{S}$ the family of all connected sets in the topological space (X,T), then

$$\mathcal{S} \cap (T \backslash \{\emptyset\}) = \mathfrak{M} \cap (T \backslash \{\emptyset\}).$$

Example 3. The base

$$\mathcal{B} = \{(a,b) : a,b \in R,\ a < b\}$$

of natural topology on the straight line is contained in the family $\mathfrak{M}$.

In the sequel the symbol card$F$ denotes the cardinality of the set $F \subset X$.

Theorem 2. Let X be an uncountable set and let T_S be a topology of at most countable complements, i.e.

$$T_s = \left\{ U \subset X : U = \phi \vee \bigvee_{F \subset X} (\mathrm{card} F \leq \chi_0 \wedge U = X \backslash F) \right\}.$$

Then every nonempty and open set belongs to $\mathfrak{M}$.

Proof. Let us choose arbitrary nonempty open sets U and A_1 such that $A_1 \subset U$ and $A_1 \neq U$. Then there exist two at most countable sets $F, F_1 \subset X$ such that $U = X \backslash F$, $A_1 = X \backslash F_1$, $F \subset F_1$ and $F \neq F_1$.
Since

$$U \backslash A_1 = U \backslash (X \backslash F_1) = U \cap (X \backslash (X \backslash F_1)) = U \cap F_1$$

therefore, the set $U \backslash A_1$ is nonempty and at most countable, and finally $U \backslash A_1 \notin T_S$ which completes the proof. $\square$

By the above theorem and Lemma 4 we have the following corollaries:

Corollary 2. Every nonempty and open set in (X, T_S) is connected.

Corollary 3. Every connected set in (X, T_S) which has a nonempty interior, is i–connected.

Therefore in the space (X, T_S) the connectivity of the sets with nonempty interiors is identical with the i–connectivity.

We end our study by the following observation.

Let (X,T) be a topological space such that X is an uncountable set and $T = \{\emptyset, X\}$. Moreover, let $\mathcal{P}$ be a family of subsets of X defined by the formula

$$\mathcal{P} = \{A \subset X : \mathrm{card} A \leq \chi_0\}.$$

It is easy to check that the family $\mathcal{P}$ fulfils conditions (2)-(5) and so we can introduce the Hashimoto topology T^* in the set X. It immediately follows that such defined topology is the same as the topology of at most countable complements.

4. The i–connectivity in the order topology

Let $(X, \leq)$ be linearly ordered set including at least two elements where the ordering relation $\leq$ is dense and does not have gaps (i.e. the relation $\leq$ is continuously ordered X).

Obviously in X we can introduce the so–called order topology $\mathcal{T}$ by defining a base consisting of all open intervals of the form

$$(a, b) = \{x \in X : a < x < b\}, \quad (\leftarrow, a) = \{x \in X : x < a\},$$

$(a, \rightarrow) = \{x \in X : a < x\}$, where $a, b \in X$ and $a < b$.

It follows immediately that in such defined topology every connected set is convex. Moreover, in [3] it has been shown that every convex set is connected. Hence for continuously ordered subsets (in particular on the straight line) the connectivity is equivalent to it's convexity.

Now, let us quote the following theorem.

Theorem 3 (cf. [3], p. 13). Let $(X, \leq)$ be linearly ordered set including at least two elements where the ordering relation $\leq$ deos not have gaps. Then every nonempty and convex subset is one of the form:

$$X, (a, b), (-\infty, a), (a, \rightarrow), [a, b], (\leftarrow, a], [a, \rightarrow), (a, b], [a, b) \tag{12}$$

where $[a, b] = \{x \in X : a \leq x \leq b\}, (\leftarrow, a] = \{x \in X : x \leq a\}, (a, \rightarrow) = \{x \in X : a \leq x\}, (a, b] = \{x \in X : a < x \leq b\}, [a, b) = \{x \in X : a < x \leq b\}$ for arbitrary $a, b \in X$ and $a < b$.

Corollary 4. Let $(X, \mathcal{T})$ be a topological space defined by the ordering relation $\leq$ which is dense and does not have gaps. Then the only nonempty connected sets are subsets of the form (12).

Corollary 5. Let $(X, \mathcal{T})$ be a topological space defined by the ordering relation $\leq$ which is dense and does not have gaps. Then every connected set which has a nonempty interior is i–connected.

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PROBLEM OF THE EXISTENCE OF $\omega^\star$-PRIMITIVES

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Abstract. If (X, ϱ) is a dense in itself metric space and $f\colon X \to \mathbb{R}$, then we define $\omega^*(f, x) = \inf_{r>0} \sup_{y,z\in\mathbf{B}(x,r)\setminus\{x\}} |f(y)-f(z)|$. We say that a function $F\colon X \to \mathbb{R}$ is an $\omega^\star$- primitive for $f\colon X \to \mathbb{R}$ if $\omega^\star(F, \cdot) = f$. We discuss problem of the existence of $\omega^\star$-primitives for an arbitrary upper semicontinuous function $f\colon X \to [0, \infty)$ defined on a dense in itself metric space. At the end we show that if an upper semicontinuous function $f\colon X \to [0, \infty)$ is defined on a nonmetrizable topological space, then $\omega^\star$-primitive may not exists.

Let (X, ϱ) be a metric space, $\mathbf{B}(x, r)$ is an open ball with center x and radius r and let $f\colon X \to \mathbb{R}$ be any function. Then we may define an oscillation of the function f as:

$$\omega(f, x) = \inf_{r>0} \sup_{y,z\in\mathbf{B}(x,r)} |f(y) - f(z)|.$$

It is well known that $\omega(f, \cdot) : X \to [0, +\infty]$ is an upper semicontinuous function vanishing at isolated points of X. There were investigate the following problem.

Problem 1. Let $f\colon X \to [0, +\infty]$ be any upper semicontinuous function vanishing at isolated points of X. Does there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$?

Positive answer was given by profesor J. Ewert and profesor S. Ponomarev:

Theorem 7 ([?]). Let (X, ϱ) be an arbitrary metric space. For every upper semicontinuous function $f\colon X \to [0, +\infty]$ vanishing at isolated points of X there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$.

In the paper we consider similar problem. Let (X, ϱ) be a dense in itself metric space and let $f\colon X \to \mathbb{R}$ be any function. Then we may define a function $\omega^*(f, \cdot)\colon X \to [0, +\infty]$,

$$\omega^*(f, x) = \inf_{r>0} \sup_{y,z \in \mathbf{B}(x,r)\setminus\{x\}} |f(y) - f(z)|.$$

Similarly, $\omega(f, \cdot)$ is an upper semicontinuous function. Although, definitions of $\omega(f, \cdot)$ and $\omega^*(f, \cdot)$ are similar, their properties may be quite different.

Example 1. Let $X = \{\frac{2k-1}{2^n} : k = 1, \dots, 2^{n-1}, n \geq 0\} \subset \mathbb{R}$ and $f\colon X \to \mathbb{R}$, $f(\frac{2k-1}{2^n}) = \frac{1}{2^n}$ for $\frac{2k-1}{2^n} \in X$.

It is easily seen that $\omega(f, \cdot) = f$ and $\omega^*(f, \cdot) = 0$. Hence, $\omega(f, x) \neq \omega^*(f, x)$ for $x \in X$.

So, we have the following question:

Problem 2. Let (X, ϱ) be a dense in itself metric space and let $f\colon X \to [0, +\infty]$ be an upper semicontinuous function. Does there exists a function $F\colon X \to \mathbb{R}$ such that $\omega^*(F, \cdot) = f$?

We say that a function $F\colon X \to \mathbb{R}$ is an ω^*- primitive for $f\colon X \to \mathbb{R}$ if $\omega^\star(F, \cdot) = f$.

First, we make some observations. For a function $F\colon X \to \mathbb{R}$ we may define upper and lower Baire functions:

$$M_f(x) = \inf_{r>0} \sup_{y \in \mathbf{B}(x,r)} f(y)$$

and

$$m_f(x) = \sup_{r>0} \inf_{y \in \mathbf{B}(x,r)} f(y).$$

Then $\omega(F, x) = M_f(x) - m_f(x)$ for $x \in X$.

Next, if (X, ϱ) is a dense in itself metric space then for a function $F\colon X \to \mathbb{R}$ we may define

$$\limsup_{t \to x} f(t) = \inf_{r>0} \sup_{y \in \mathbf{B}(x,r)\setminus\{x\}} f(y),$$

$$\liminf_{t \to x} f(t) = \sup_{r>0} \inf_{y \in \mathbf{B}(x,r)\setminus\{x\}} f(y)$$

and then

$$\omega^*(F, x) = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t)$$

for $x \in X$ (if we assume that $\infty - \infty = \infty = -\infty - (-\infty)$)

In the following we will need the following denotations. Let $\varrho(x, A) = \inf\{\varrho(x, a) : a \in A\}$ denotes the distance of the point x from the nonempty set A in a metric space (X, ϱ) and let

$$\mathbf{B}(A, \varepsilon) = \bigcup_{x \in A} \{t \in X : d(x, t) < \varepsilon\} = \bigcup_{x \in A} \mathbf{B}(x, \varepsilon).$$

for $\emptyset \neq A \subset X$ and $\varepsilon > 0$.

We will give the positive answer of Problem 2 in the case of upper semi-continuous functions with finite values $f\colon X \to [0, +\infty)$. We can prove even more. First, we start from the following technical lemma.

Lemma 1. Let (X, ϱ) be a metric space. For every subset M dense in X, nonempty set $A \subset X$ and $\varepsilon > 0$ there exists a set $T_{M,A,\varepsilon} \subset M$ such that

[1] $\varrho(z_1, z_2) \geq \varepsilon$ for every $z_1, z_2 \in T_{M,A,\varepsilon}$,

[2] $\varrho(z, A) < \varepsilon$ for every $z \in T_{M,A,\varepsilon}$,

[3] $\varrho(x, T_{M,A,\varepsilon}) < 2\varepsilon$ for every $x \in A$.

Proof. Observe that another way of stating (2) is to say $T_{M,A,\varepsilon} \subset \mathbf{B}(A, \varepsilon)$ and an equivalent formulation of (3) is $A \subset \mathbf{B}(T_{M,A,\varepsilon}, 2\varepsilon)$. Since M is a dense subset of X, $M \cap \mathbf{B}(A, \varepsilon) \neq \emptyset$.

Let $\mathfrak{B}$ be the set of all subsets B of X satisfying the following conditions

(a) $B \subset M \cap \mathbf{B}(A, \varepsilon)$,

(b) $\varrho(z_1, z_2) \geq \varepsilon$ for each $z_1, z_2 \in B$.

The family $\mathfrak{B}$ is nonempty because contains all singletons $\{x\}$ for $x \in M \cap \mathbf{B}(A, \varepsilon)$. Moreover, $\mathfrak{B}$ is partially ordered by inclusion. It is easily seen that if $\{B_s : s \in S\}$ is a chain in X then the set $B = \bigcup_{s \in S} B_s$ belongs to $\mathfrak{B}$ and B is above all elements from $\{B_s : s \in S\}$. Hence, by Zorn Lemma the family $\mathfrak{B}$ has a maximal element $T_{M,A,\varepsilon}$.

We will show that the set $T_{M,A,\varepsilon}$ fulfils all required properties. By (a) it is clear that $T_{M,A,\varepsilon} \subset M$ and $T_{M,A,\varepsilon} \subset \mathbf{B}(A, \varepsilon)$, so $\varrho(z, A) < \varepsilon$ for every $z \in T_{M,A,\varepsilon}$. Next $\varrho(z_1, z_2) \geq \varepsilon$ for $z_1, z_2 \in T_{M,A,\varepsilon}$ from (b).

Assume that $\varrho(x_0, T_{M,A,\varepsilon}) \geq 2\varepsilon$ for some $x_0 \in A$. Since M is a dense subset of X, there exists $z_0 \in M$ such that $\varrho(x_0, z_0) < \varepsilon$. Hence

$$\varrho(t, z_0) \geq \varrho(t, x_0) - \varrho(x_0, z_0) \geq \varrho(x_0, T_{M,A,\varepsilon}) - \varrho(x_0, z_0) > 2\varepsilon - \varepsilon = \varepsilon$$

for each $t \in T_{M,A,\varepsilon}$. It follows that $T_{M,A,\varepsilon} \cup \{z_0\} \in \mathfrak{B}$. Since $T_{M,A,\varepsilon}$ is a maximal element of $\mathfrak{B}$, this is a contradiction. Therefore $\varrho(x, T_{M,A,\varepsilon}) < 2\varepsilon$ for every $x \in A$ and the set $T_{M,A,\varepsilon}$ satisfies conditions $(1) - (3)$. $\square$

Remark 1. From condition (1) of the Lemma it follows that $T_{M,A,\varepsilon}$ is a closed and discrete set.

Now, we formulate the main theorem of the paper

Theorem 8. Let (X, ϱ) be a dense in itself metric space and let Y be dense subset of X. Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be a pair of functions such that f is upper semicontinuous, g is lower semicontinuous and $g \le f$. Then there exists **one** function $F\colon X \to \mathbb{R}$ for which

[1] $\limsup_{t\to x} F(t) = f(x)$ and $\liminf_{t\to x} F(t) = g(x)$ for $x \in X$,

[2] $F(x) = g(x)$ for $x \in X \setminus Y$.

Proof. Let

$$K = \{(n,k) \in \mathbb{Z} : -n^2 \le k < n^2\}.$$

Let $\preceq$ be a relation in K defined as follows

$$(n_1, k_1) \preceq (n_2, k_2) \Leftrightarrow n_1 < n_2 \vee (n_1 = n_2 \wedge k_1 \le k_2).$$

It is easily seen that K is well ordered by $\preceq$. Define

$$A_{n,k} = \left\{x \in X \,:\, \tfrac{k}{n} \le f(x) < \tfrac{k+1}{n}\right\}$$

and

$$B_{n,k} = \left\{x \in X \,:\, \tfrac{k}{n} \le g(x) < \tfrac{k+1}{n}\right\}$$

for $(n,k) \in K$. We shall construct two families $\{R_{n,k} : (n,k) \in K\}$ and $\{S_{n,k} : (n,k) \in K\}$ of closed and discrete subsets of X which satisfy the following conditions:

(a) $R_{n_1,k_1} \cap R_{n_2,k_2} = \emptyset = S_{n_1,k_1} \cap S_{n_2,k_2}$ for $(n_1,k_1), (n_2,k_2) \in K$, $(n_1,k_1) \ne (n_2,k_2)$ and $R_{n,k} \cap S_{i,j} = \emptyset$ for $(n,k), (i,j) \in K$,

(b) $\bigcup_{(n,k)\in K} (R_{n,k} \cup S_{n,k}) \subset Y$,

(c) $R_{n,k} \subset \mathbf{B}(A_{n,k}, \frac{1}{n})$, $S_{n,k} \subset \mathbf{B}(B_{n,k}, \frac{1}{n})$ for $(n,k) \in K$,

(d) $\varrho(x, R_{n,k}) < \frac{2}{n}$ for $x \in A_{n,k}$, $(n,k) \in K$ and $\varrho(x, S_{n,k}) < \frac{2}{n}$ for $x \in B_{n,k}$, $(n,k) \in K$.

If $(n,k) \in K$ and $A_{n,k} = \emptyset$ then we set $R_{n,k} = \emptyset$ and if $B_{n,k} = \emptyset$ then we set $S_{n,k} = \emptyset$. Thus we have to define $R_{n,k}$ if $A_{n,k} \neq \emptyset$ and $S_{n,k}$ if $B_{n,k} \neq \emptyset$. We will make it inductively. Let $R_{1,-1} = T_{Y,A_{1,-1},1}$ where $T_{Y,A_{1,-1},1}$ is the set from Lemma 1 for $M = Y$, $A = A_{1,-1}$ and $\varepsilon = 1$. Since $R_{1,-1}$ is a closed and discrete subset of X and X is dense in itself, the set $Y \setminus R_{1,-1}$ is dense in X. Thus we can set $S_{1,-1} = T_{Y\setminus R_{1,-1},B_{1,-1},1}$. Next, let

$\widetilde{Y}_{1,0} = Y \setminus (R_{1,-1} \cup S_{1,-1})$, $\quad R_{1,0} = T_{\widetilde{Y}_{1,0},A_{1,0},1}$ and $S_{1,0} = T_{\widetilde{Y}_{1,0}\setminus R_{1,0},B_{1,0},1}$.

Fix $(n,k) \in K$. Assume that the closed and discrete sets $R_{i,j}$ and $S_{i,j}$ satisfying conditions (a)-(d) are choosen for $(i,j) \prec (n,k)$ and let

$$\widetilde{Y}_{n,k} = Y \setminus \bigcup_{(i,j)\prec(n,k)} (R_{i,j} \cup S_{i,j}).$$

Define $\qquad R_{n,k} = T_{\widetilde{Y}_{n,k},A_{n,k},\frac{1}{n}}$ and $S_{n,k} = T_{\widetilde{Y}_{n,k}\setminus R_{n,k},B_{n,k},\frac{1}{n}}$.

It is obvious that the families

$$\{R_{n,k} : (n,k) \in K\} \quad \text{i} \quad \{S_{n,k} : (n,k) \in K\}$$

constructed inductively satisfy conditions $(a)-(d)$. Let us define a function $F\colon X \to \mathbb{R}$ as follows

$$F(x) = \begin{cases} \frac{k}{n} & if \quad x \in R_{n,k}, \quad (n,k) \in K, \\ \frac{k+1}{n} & if \quad x \in S_{n,k}, \quad (n,k) \in K, \\ g(x) & if \quad x \in X \setminus \bigcup_{(n,k)\in K} (R_{n,k} \cup S_{n,k}). \end{cases}$$

We shall show that (1) and (2) hold. Fix $x_0 \in X$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$ and $f(x_0) < n_0 + 1$. For every $n \geq n_0$ we may find $-n^2 \leq k_n < n^2$ for which $\frac{k_n}{n_0} \leq f(x_0) < \frac{k_n+1}{n_0}$. Thus $x_0 \in A_{n,k_n}$. From (d) for every $n \geq n_0$ there exists $y_n \in R_{n,k_n}$ such that $d(x_0, y_n) < \frac{2}{n}$. Hence $\lim_{n\to\infty} y_n = x_0$. From this we obtain

$$F(y_n) = \frac{k_n}{n} \quad \text{and} \qquad 0 \leq f(x) - F(y_n) < \frac{1}{n}.$$

This gives $\lim_{n\to\infty} F(y_n) = f(x_0)$. Thus we have proved that

$(*)$ $\qquad \limsup_{x\to x_0} f(x) \geq f(x_0).$

In the same manner we can see that $\liminf_{x\to x_0} f(x) \leq g(x_0)$.

Let $(x_m)_{m\in\mathbb{N}}$ be a sequence of elements of X converging to x_0, $x_m \neq x_0$ for $n \in \mathbb{N}$ and $\lim_{m\to\infty} F(x_m) = \alpha$, $\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$. Without the loss of generality we may assume that all elements of the sequence belong to one of the three sets

$$\bigcup_{(n,k)\in K} R_{n,k}, \qquad \bigcup_{(n,k)\in K} S_{n,k} \quad \text{or} \quad X \setminus \bigcup_{(n,k)\in K} (R_{n,k} \cup S_{n,k}).$$

First, suppose that $x_m \in \bigcup_{(n,k)\in K} R_{n,k}$ for $m \geq 1$. Then for every $m \in \mathbb{N}$ we can find $(n_m, k_m) \in K$ such that $x_m \in R_{n_m,k_m}$. The sets $R_{n,k}$ are closed

and discrete and for fixed $m \in \mathbb{N}$ there is only a finite number $k \in \mathbb{Z}$ for which $(n,k) \in K$. Besides, $(x_m)_{m\in\mathbb{N}}$ is convergent and is not constant. Hence $\lim_{m\to\infty} n_m = +\infty$. From (c) for every $m \in \mathbb{N}$ there exists $z_m \in A_{n_m,k_m}$ such that $d(x_m, z_m) < \frac{2}{n}$. Moreover

$$F(x_m) = \frac{k_m}{n_m} \quad \text{and} \quad \frac{k_m}{n_m} \le f(z_m) < \frac{k_m+1}{n_m}.$$

Since the function f is upper semicontinuous,

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} f(z_m) \le f(x_0).$$

Now, let $x_m \in \bigcup_{(n,k)\in K} S_{n,k}$ for $m \ge 1$. Then for every $m \in \mathbb{N}$ we can find $(n_m, k_m) \in K$ such that $x_m \in S_{n_m,k_m}$. In the same manner as before we can prove that $\lim_{m\to\infty} n_m = +\infty$. From (c) for every $m \in \mathbb{N}$ there exists $z_m \in B_{n_m,k_m}$ such that $d(x_m, z_m) < \frac{2}{n}$. Besides

$$F(x_m) = \frac{k_m+1}{n_m} \quad \text{and} \quad \frac{k_m}{n_m} \le g(z_m) < \frac{k_m+1}{n_m}.$$

Since $g \le f$ and f is upper semicontinuous, it follows that

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} g(z_m) \le \limsup_{m\to\infty} f(z_m) \le f(x_0).$$

.

At the end, if $x_m \in X \setminus \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k})$, then $F(x_m) = g(x_m)$ for $m \in \mathbb{N}$. Therefore

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} g(x_m) \le \limsup_{m\to\infty} f(x_m) \le f(x_0).$$

Thus we have proved that $\alpha \le f(x_0)$. Since α is an arbitrary limit number of f at x_0, $\limsup_{x\to x_0} F(x) \le f(x_0)$. Together, with $(*)$ we get

$$\limsup_{x\to x_0} F(x) = f(x_0)$$

for every $x_0 \in X$.

Applying lower semicontinuity of g in the same way we can prove $\liminf_{t\to x} F(t) = g(x)$ for $x \in X$. The equality $F(x) = g(x)$ for $x \in X \setminus Y$ is obvious, becouse $\bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}) \subset Y$ and $F(x) = g(x)$ for $x \notin \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k})$. The proof is complete. □

Remark 2. If under the notation from the proof of the last theorem we define a function $\widetilde{F}\colon X \to \mathbb{R}$ in the following way

$$\widetilde{F}(x) = \begin{cases} \frac{k}{n} & if \quad x \in R_{n,k}, \quad (n,k) \in K, \\ \frac{k+1}{n} & if \quad x \in S_{n,k}, \quad (n,k) \in K, \\ f(x) & if \quad x \in X \setminus \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}), \end{cases}$$

then it is easily seen that

$$\limsup_{t\to x} \widetilde{F}(t) = f(x) \quad \text{and} \quad \liminf_{t\to x} \widetilde{F}(t) = g(x) \text{ for } x \in X.$$

Hence we get a theorem analogous with Theorem 8.

Theorem 9. Let (X, ϱ) be a dense in itself metric space and let Y be dense subset of X. Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be a pair of functions such that f is upper semicontinuous, g is lower semicontinuous and $g \le f$. Then there exists a function $F\colon X \to \mathbb{R}$ for which

[1] $\limsup_{t\to x} F(t) = f(x)$ and $\liminf_{t\to x} F(t) = g(x)$ for $x \in X$,

[2] $F(x) = f(x)$ for $x \in X \setminus Y$.

C Let (X, ϱ) be a dense in itself metric space. For every upper semicontinuous function $f\colon X \to [0, \infty)$ there exists a function $F\colon X \to \mathbb{R}$ such that $\omega^*(F, x) = f(x)$ for $x \in X$.

For upper and lower Baire functions M_f and m_f theorem analogous to Theorem 2 is not true.

Example 2. Let $X = \{\frac{2k-1}{2^n} : k = 1 \dots, 2^{n-1}, n \ge 0\} \subset \mathbb{R}$ and $f\colon X \to \mathbb{R}$, $f(\frac{2k-1}{2^n}) = 1 + \frac{1}{2^n}$ for $\frac{k}{2^n} \in X$. Then X is dense in itself, f is upper semicontinuous. Suppose, that there exists a function $F\colon X \to \mathbb{R}$ such that $M_F(x) = f(x)$ and $m_F(x) = 0$ for $x \in X$. Then $0 \le F \le f$. It is easy to prove that $\limsup_{t\to x} f(t) = 1$ for every $x \in X$. Hence $\limsup_{t\to x} F(t) \le 1$ for every $x \in X$. Since $M_F(x) = \max\{F(x), \limsup_{t\to x} F(x)\}$, it have to be $F(x) = f(x)$ for every $x \in X$. But then $m_F = 1$. Thus we have proved that there is **no** a function $F\colon X \to \mathbb{R}$ such that $M_F(x) = f(x)$ and $m_F(x) = 0$ for $x \in X$.

At the end we will consider problems of the existence of ω-primitives and ω^*-primitives for nonmetrizable topological spaces. The problem of the existence of ω-primitive has a positive solution for some nonmetrizable topological spaces, for example:

Theorem 10. Let $(X, \mathcal{T})$ be a regular separable topological space. Then for every upper semicontinuous function $f\colon X \to [0, +\infty]$ vanishing at isolated points of X there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$.

Theorem 11 ([?]). Let $(X, \mathcal{T})$ be a regular Baire space. Then for every upper semicontinuous function $f\colon X \to [0, +\infty]$ vanishing at isolated points of X there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$.

The problem of the existence of ω^*-primitives for nonmetrizable topological spaces is more complicated.

Example 3. Let $(X, \mathcal{T})$, $X = \mathbb{R} \times [0, +\infty)$ be a Niemytzky plane. Then X is a "nice" nonmetrizabe, separable, Tychonoff, Baire topological space. Define $f\colon X \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & if \quad x \in \mathbb{Q} \times \{0\}, \\ 0 & if \quad x \notin \mathbb{Q} \times \{0\}. \end{cases}$$

We will show that ω^*-primitive for f does **not** exist. Let $F\colon X \to \mathbb{R}$ be any function such that $\omega^*(F, x) = f(x)$ for $x \in X \setminus (\mathbb{Q} \times \{0\})$. Then the function F has a limit at $(x, 0)$ for every $x \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$A_{n,k} = \left\{x \in \mathbb{R} \setminus \mathbb{Q} : F(v) \in (\tfrac{k}{4} - \tfrac{1}{4}, \tfrac{k}{4} + \tfrac{1}{4}) \ \text{ for } \ v \in (x - \tfrac{1}{n}, x + \tfrac{1}{n}) \times (0, \tfrac{1}{n})\right\}$$

for every $n, k \in \mathbb{N}$. Then $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n,k \in \mathbb{N}} A_{n,k}$ and by Baire Theorem there exist $n_0, k_0 \in \mathbb{N}$ and an open interval (a, b) such that A_{n_0,k_0} is dense in (a, b). But then for every $x_0 \in (a, b) \cap \mathbb{Q}$ there exists a neighbourhood U of $(x_0, 0) \in X$ such that $\sup_{u,v \in U \setminus \{(x_0,0)\}} |F(u) - F(v)| \leq \frac{1}{2}$. Therefore $\omega^*(F, x_0) \leq \frac{1}{2}$. Thus $\omega^*(F, x_0) \neq f(x_0) = 1$ and $\omega^*(F, \cdot) \neq f$. So, we have proved that ω^*-primitive for f does **not** exists.

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APPLYING THE IDEA OF FUSIONISM IN THE PROBABILITY THEORY

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Abstract. The solutions presented in this paper may serve as an illustration of „the principle of internal integration", know as the idea of fusionism. In the paper we consider some problem. From an urn containing b white balls and c black ones are selected simultaneously some balls. If the balls are of the same colours one of the players wins, otherwise the other player is the winner. For which values of b and c is the game fair?

Let us consider an urn containing b white balls and c black balls. Let us assume, k balls ($k \geq 2$) are selected simultaneously from our urn. If both balls are of the same colour, then one of the players wins and if the balls are of different colours, then the other player is the winner. For what values of b and c is this game fair?

Some solutions of this problem for $k = 2$ are presented in [2].

In the present paper we suggest three other solutions.

The conditions of the problem imply that $b \geq k$ and $c \geq 1$ or $b \geq 1$ and $c \geq k$. Let us treat all the white balls and all the black ones as distinct objects. Under such assumptions the outcome of such an experiment is a combination of k elements out of the set of $b + c$ balls and the model of this experiment is a classic sample space (Ω, p).

Let us consider the following events:

$A = \{$*both selected balls are of the same colour*$\}$,

$B = \{$*the selected balls are of different colours*$\}$.

Solution problem for $k = 2$.

Therefore

$$P(A) = \frac{b(b-1) + c(c-1)}{(b+c)(b+c-1)},$$

$$P(B) = \frac{2bc}{(b+c)(b+c-1)}.$$

In the sample space (Ω, p) the system $\{A, B\}$ is a complete system of events and therefore, the game is fair if the following condition holds:

$$P(A) = 1/2.$$

These condition is equivalent to the condition

$$b^2 + c^2 - b - c - 2bc = 0. \tag{1}$$

Let us consider the equation

$$x^2 + y^2 - x - y - 2xy = 0, \tag{2}$$

where $x \in \mathbf{R}$ i $y \in \mathbf{R}$.

It means that equation (2) describes the curve which is symmetrical to the straight line $y = x$.

We can find the natural solutions of the equation (2). It can be easily verified that pairs $(0,0)$, $(1,0)$, $(1,0)$ satisfy equation (2). The process of finding consecutive natural solutions of the equations (2) is presented in Fig. 1.

Solution of the problem for $k = 3$.

Therefore

$$P(A) = \frac{b^3 - 3b^2 + 2b + 2c - 3c^2 + c^3}{b^3 - 3b^2 + 2b + 3b^2c - 6bc + 3bc^2 + 2c - 3c^2 + c^3}.$$

The game is fair if:

$$P(A) = 1/2.$$

This condition is equivalent to the condition

$$b^3 - 3b^2 + 2b - 3b^2c + 6bc - 3bc^2 + 2c - 3c^2 + c^3 = 0. \tag{3}$$

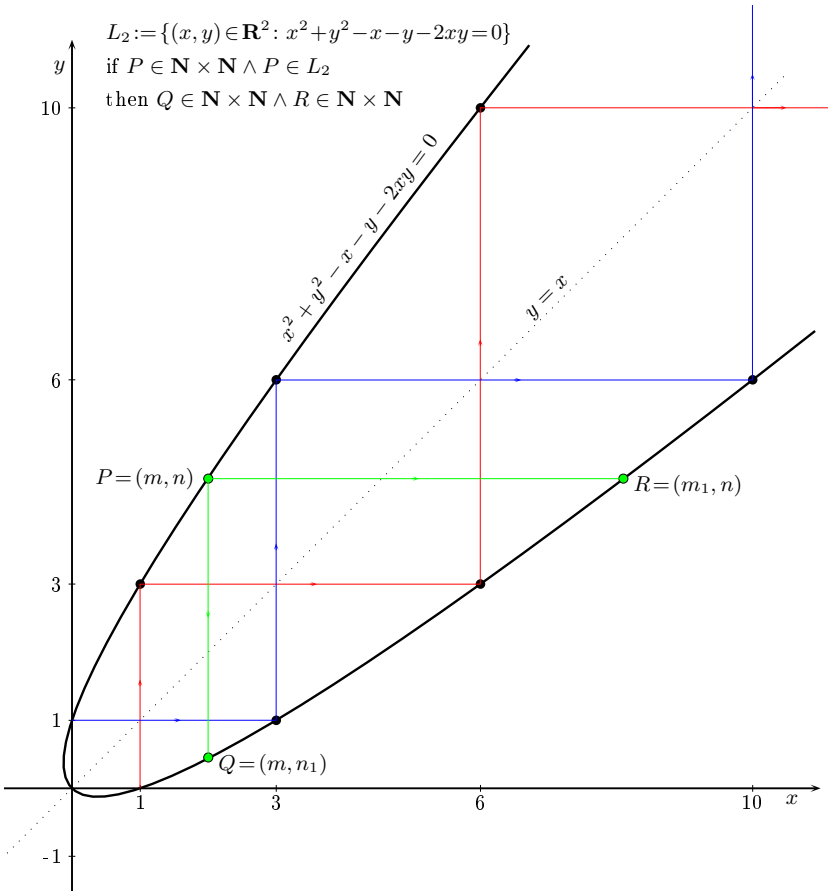

Figure 1:

Let us consider the equation

$$x^3 - 3x^2 + 2x - 3x^2y + 6xy - 3xy^2 + 2y - 3y^2 + y^3 = 0, \tag{4}$$

where $x \in \mathbf{R}$ i $y \in \mathbf{R}$.

Thus condition (4) and

$$(y + x - 2)(x^2 - x - 4xy - y - y^2) = 0 \tag{5}$$

are equivalent.

Fig. 2 shows the curves representing condition 4 (see [3], pp. 133-138).

We can find the natural solutions of the equation (5). It can be easily verified that pairs $(0,0)$, $(1,0)$, $(1,0)$ satisfy equation (5). The process of finding consecutive natural solutions of the equations (5) is presented in Fig. 3.

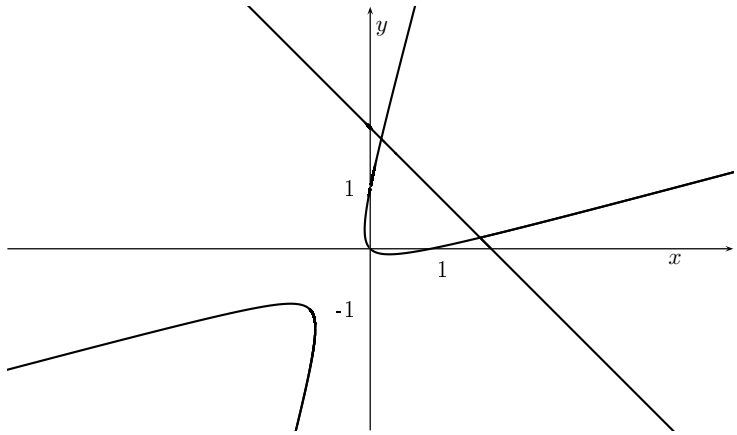

Figure 2:

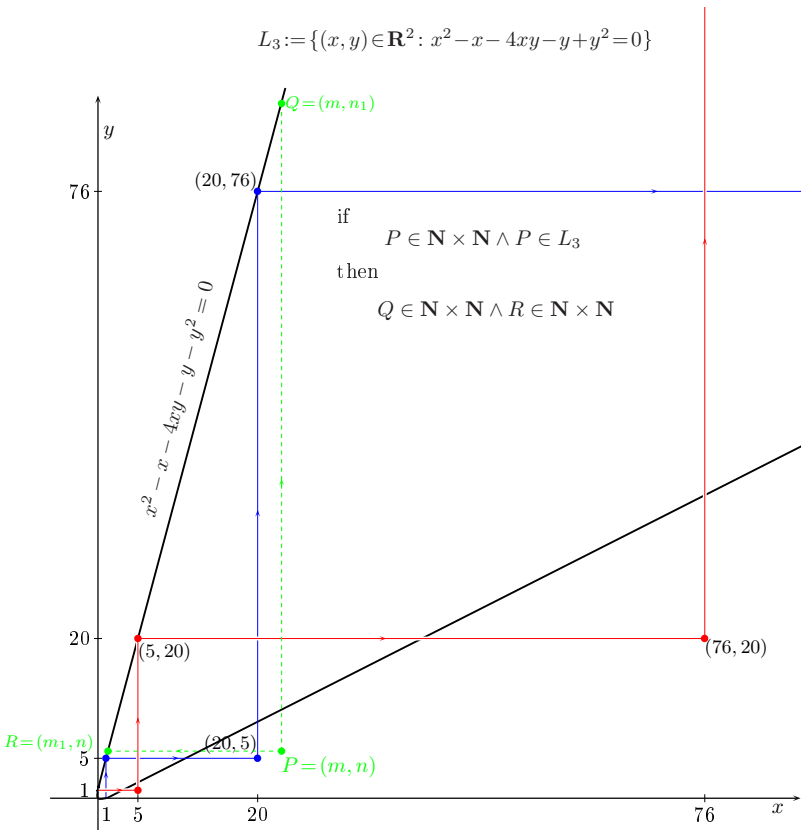

Figure 3:

Some remarks for $k \geq 2$.

Let $k \geq 4$.

Because the game is fair if $P(A) = \frac{1}{2}$ then

$$\begin{aligned} &2[b(b-1)\cdot\ldots\cdot(b-k+1)+c(c-1)\cdot\ldots\cdot(c-k+1)] \\ &-(b+c)(b+c-1)\cdot\ldots\cdot(b+c-k+1)=0. \end{aligned} \tag{6}$$

Let us consider the equation

$$\begin{aligned} &2[x(x-1)\cdot\ldots\cdot(x-k+1)+y(y-1)\cdot\ldots\cdot(y-k+1)] \\ &\quad -(x+y)(x+y-1)\cdot\ldots\cdot(x+y-k+1)=0. \end{aligned} \tag{7}$$

where $x \in \mathbf{R}$ i $y \in \mathbf{R}$.

Let

$$\begin{aligned} L_k = \{(x,y) \in \mathbf{R}^2 \colon\, &2[x(x-1)\cdot\ldots\cdot(x-k+1)+y(y-1)\cdot\ldots\cdot(y-k+1)] \\ &-(x+y)(x+y-1)\cdot\ldots\cdot(x+y-k+1)=0 \end{aligned}$$

for $k = 2, 3, 4, \ldots$

Figures 4-6 shows the curves representing condition (7) for $k = 4$, 5, 6 respectively.

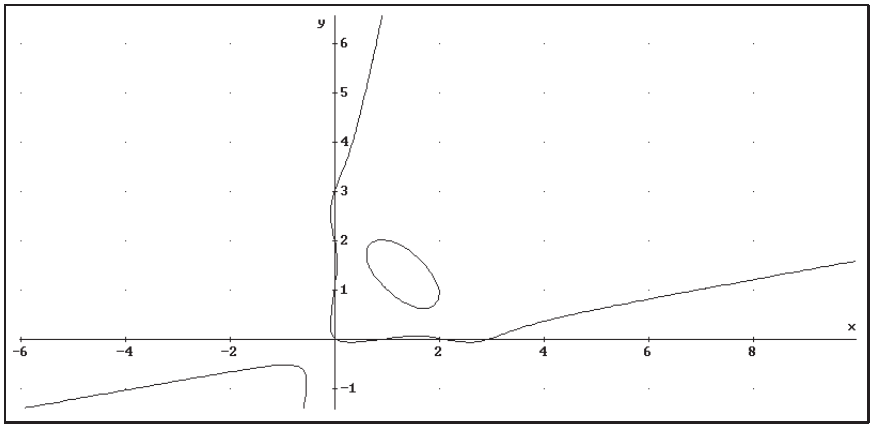

Figure 4:

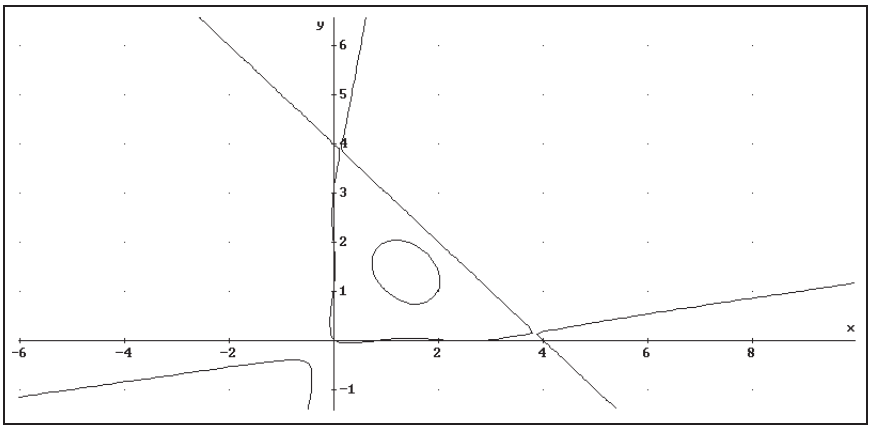

Figure 5:

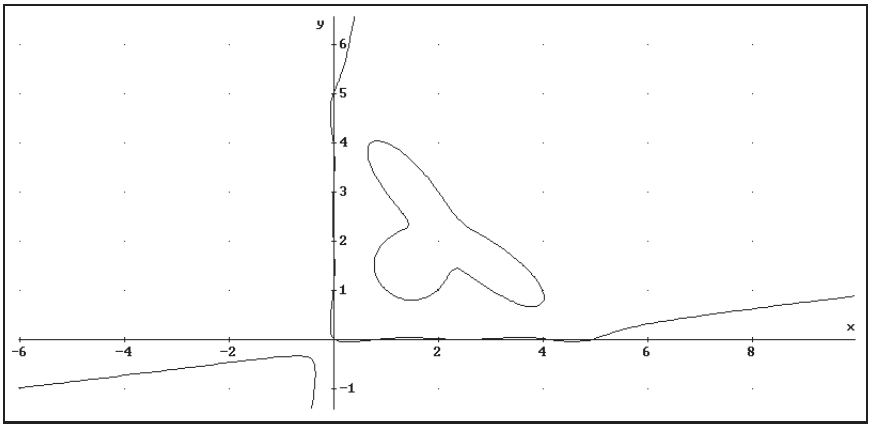

Figure 6:

In the paper a method of finding the solutions of (1) and (3) was given. Unfortunately, this method does not give the complete solution of (6).

It can be easily verified that pairs $(0,0)$, $(1,0)$, $(1,0)$ satisfy equation (7).

Applying a similar reasoning to this used for finding the solutions of (2) and (4) yields

– points $(1,7)$ and $(7,1)$ belong to L_4,

– points $(1,9)$ and $(9,1)$ belong to L_5,

– points $(1,11)$ and $(11,1)$ belong to L_6.

Obtained results lead to the following conclusion. The game is fair if the urn contains 1 white ball and $2k-1$ black balls (or $2k-1$ white balls and 1 black ball).

We shall prove the above remark.

If the urn contains 1 white ball and b black balls, then the condition

$P(A) = P(B)$ is equivalent to

$$\binom{b}{k} = \binom{b}{k-1}.$$

It follows that

$$b = 2k - 1.$$

Some generalization of the above game

Now consider a generalization of the above game.

Let us consider an urn containing b white balls, c black balls and z green bals. Assume that 2 balls are selected simultaneously from our urn. If both balls are of the same colour, then one of the players wins and if the balls are of different colours, then the other player is the winner. For what values of b, c and z is this game fair?

The conditions of the problem imply that ($b \geq 2$ and $c \geq 1$ and $z \geq 1$) or ($b \geq 1$ and $c \geq 2$ and $z \geq 1$) or ($b \geq 1$ and $c \geq 1$ and $z \geq 2$). Let us treat all the white balls and all the black balls and all the green ones as distinct objects. Under such assumptions the outcome of such an experiment is a combination of 2 elements out of the set of $b+c+z$ balls and the model of this experiment is a classic sample space (Ω, p).

Let us consider the following events:

$A = \{$*both selected balls are of the same colour*$\}$,

$B = \{$*the selected balls are of different colours*$\}$.

Therefore

$$P(A) = \frac{b(b-1) + c(c-1) + z(z-1)}{(b+c+z)(b+c+z-1)}.$$

The game is fair if the following condition holds:

$$P(A) = \frac{1}{2}.$$

This condition is equivalent to the condition

$$b^2 + c^2 + z^2 - b - c - z - 2bc - 2cz - 2bz = 0. \tag{8}$$

Let us consider the equation

$$x^2 + y^2 + z^2 - x - y - z - 2xy - 2yz - 2xz = 0, \tag{9}$$

where $x \in \mathbf{R}$, $y \in \mathbf{R}$, $z \in \mathbf{R}$.

It means that the equation (9) describes the paraboloida.

In the paper [2] it was proved that all solutions of (1) are of the form

$$b = \frac{k^2+k}{2}, \quad c = \frac{k^2-k}{2}, \quad \text{for } k \in \mathbf{Z} \setminus \{-1, 0, 1\}.$$

Substituting these equalities to (8) we get

$$z = 2k^2 + 1.$$

Hence, the game is fair if:
$b = \frac{k^2+k}{2}, \quad c = \frac{k^2-k}{2}, \quad z = 2k^2+1$; or
$b = \frac{k^2-k}{2}, \quad c = 2k^2+1, \quad z = \frac{k^2+k}{2}$; or
$b = 2k^2+1, \quad c = \frac{k^2+k}{2}, \quad z = \frac{k^2-k}{2}$
for $k \in \mathbf{Z} \setminus \{-1, 0, 1\}$.

The solutions presented above may serve as an illustration of "the principle of internal integration", known as the idea of fusionism. According to this principle, the process of teaching various modules of the school curriculum on mathematics should be conducted so that they could support one another and play a certain role in one another's creation (see [3], p. 39). Particular sections of mathematics appear in the process of teaching as separate threads. At various stages of this process these threads may be linked in order to create a certain unity. Questions offered by the problems on probability may serve as a considerable source of such opportunities.

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EXPONENT IN ONE OF THE VARIABLES

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Abstract. A periodicity functional equation of one complex variable which characterizes the exponential function is discussed. This functional equation can be generalized to equation for functions depending on two complex variables. It is conjectured that the second functional equation also characterizes the exponent. Applications to representations of complex continuous elementary functions are discussed.

1. A functional equation

Let ζ denote a complex variables, $i = \sqrt{-1}$. A function $f(\zeta)$ is called entire, if it is analytic in the complex plane, i.e., $f(\zeta)$ is represented in the form of the absolutely convergent series for all $\zeta \in \mathbb{C}$

$$f(\zeta) = \sum_{k=0}^{\infty} f_k \zeta^k.$$

The functional equation

$$\varphi(\zeta + 2\pi i) = \varphi(\zeta) \tag{1}$$

in the class of entire functions has the general solution of the form

$$\varphi(\zeta) = \psi(\exp \zeta) = \sum_{k=0}^{\infty} \psi_k e^{\zeta k}, \tag{2}$$

where ψ is an arbitrary entire function. In order to prove (2) we consider equation (1) in the strip $D = \{z \in \mathbb{C} : 0 \leq Im\ z \leq 2\pi\}$. The conformal mapping $t = \exp z$ maps D onto $\mathbb{C}$ with cut along the positive half–axis.

The functional equation (1) implies that the limit values of $\psi(t) = \varphi(\zeta)$ at the different edges of the cut coincide. Hence, $\psi(t)$ is an arbitrary function analytic in $\mathbb{C}$. This proves (2).

2. Exponent on the second variable

In the present section, we discuss a functional equation similar to (1). We state just a conjecture about solutions of new functional equation.

Consider a class $\mathcal{A}$ of functions entire in the variables z and w. Let φ satisfy the functional equation

$$\varphi(\zeta, \zeta + 2\pi i) = \varphi(\zeta, \zeta), \quad \zeta \in \mathbb{C}. \tag{3}$$

Conjecture 1. *φ is the exponential function with respect to the second variable w, i.e.,*

$$\varphi(\zeta, w) = h(\zeta, \exp w) = \sum_{k=0}^{\infty} h_k(\zeta) e^{wk} \tag{4}$$

for some $h \in \mathcal{A}$.

Remark 1. The function $\varphi(\zeta, w) = w$ does not satisfy (3). This function satisfies (4) with $h(\zeta, u) = \ln u$. One can see that h does not belong to $\mathcal{A}$, since it has a jump across the cut of the complex logarithm.

Remark 2. The function equation

$$\varphi(w, \zeta + 2\pi i) = \varphi(w, \zeta), \quad (\zeta, w) \in \mathbb{C}^2, \tag{5}$$

is equivalent to equation (1), but it is not equivalent to equation (3), since only (5) $\Longrightarrow$ (3) by substitution of $w = \zeta$ in (5).

3. Elementary functions

The functional equation (3) has applications to topologically non–elementary functions introduced by Arnold [1, 2].

For brevity we denote rational functions by Q. The functions Q, exp, log, sin, cos, tg, ctg, arcsin, arccos, arctg, arcctg are called by the basic elementary functions. Real elementary functions are such functions which can be built from a finite number of the basic elementary functions through composition and combinations using the four elementary operations. Usually, the radical x^α is referred to the basic elementary functions. However, it can be expressed through the above functions $x^\alpha = \exp(\alpha \log x)$.

All complex elementary functions can be built by the basic functions Q, exp, log, since all other basic in the real case functions are express through these functions by the formulas

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}),$$

$$\arcsin z = -i \log[iz + \sqrt{1 - z^2}], \quad \operatorname{arctg} z = \frac{i}{2}[\log(1 - iz) - \log(1 + iz)],$$

and so forth.

As it is noted before, log is a discontinuous function. Hence, if we wish to discuss only complex continuous elementary functions, we can take just Q, exp as the basic functions. But it is only a conjecture which can be formulated more precisely as follows.

Conjecture 2. *All complex continuous elementary functions can be built by the basic functions* Q, exp. *More precisely, any complex continuous elementary function (even it contains terms of the type* $\exp \circ \log$*) can be generated from a finite number of the basic functions* Q, exp.

Conjecture 2 is related to Conjecture 1, if we do not care about singularities in Conjecture 1. Let a complex continuous elementary function $f(z)$ contain $\log z$, hence it has the form $f(z) = F(z, \log z)$. Introduce the variable $\zeta = \log z$. Then $f(e^\zeta) = F(e^\zeta, \zeta)$. The function $G(w, \zeta) = F(e^\zeta, w)$ satisfies the functional equation (3). If Conjecture 1 is true, $F(e^\zeta, w)$ is an exponential function in the variable w (in the first variable it is already exponential). This implies that log in $F(z, \log z)$ disappears after simplifications.

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PRESENTATION OF A NEW BILINGUAL MATHEMATICAL DICTIONARY

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Abstract. In the contribution I am presenting a new English–Czech–English mathematical dictionary, which was prepared for university students in the Czech Republic, in the form of an on-line application with unrestricted access.

1. Background and justification

Given the massive use of information technologies at all school levels and types, the expansion of the Internet, the possibilities of student exchange, etc. the importance of English – both the general one and English for specific purposes – has been rising. A modern university student in a non-English speaking country cannot avoid his/her contact with the English environment. Yet first-year university students obviously are not (and cannot be) familiar with mathematical terminology in a foreign language, which is a serious obstacle. We have studied this issue e.g. in [4], [5] or [6]. As a result of the rather unfavourable situation in the Czech context, a new bilingual mathematical dictionary was prepared.

2. Basic information

The new dictionary is available as an on-line application [2] at:

- `http://www.umat.feec.vutbr.cz/~novakm/`
  `/slovnik_matematicke_terminologie/`

3. The structure of a dictionary entry

Each of the approx. 2400 entries contains the following information:*

Fig. 1. Structure of a dictionary entry.

The grammar notes and related terms items are displayed as hyperlinks* leading to respective entries.

4. Modes of work: single term translation

The user can work with the application in two modes: a single term translation and making lists. A form is available for each mode. The *single term translation form* is rather simple: it contains an input text field and a *Translate* button. The language of the string entered is not relevant as the application searches both Czech and English entries simultaneously. Therefore, no additional controls for selecting language are necessary, which adds to the straightforwardness of use. After clicking the *Translate* button, the respective dictionary entry(ies) is (are) displayed as shown in Fig. 1, i.e. with all information available.

5. Modes of work: making lists

The dictionary was designed with a student in mind. The option of providing students with customized lists of words was a priority.

*The respective items are displayed only when relevant.
*Not all terms given in grammar notes are included in the database.

Fig. 2. Making lists.

Fig. 3. Customizing the list.

Fig. 2 shows all criteria, which can be used when making lists. Even though some of them are linked to the faculty context (e.g. subject or level*), users from other faculties or universities can use the dictionary as well. This is possible due to the number of the options available, and the construction of the dictionary corpus, which includes terms used not only at our faculty but also elsewhere.

*The level options include: *secondary school*, *Bachelor*, *Master*, *post-graduate*.

In contrast to the output of a single term translation, the information included in the lists can be customized, as seen in Fig. 3.* The respective checkboxes correspond to the details of a dictionary entry displayed in Fig. 1.

6. Linking the dictionary to mathematical software

Increasing students' awareness of mathematical software and increasing their motivation for using it was one of the reasons for preparing the dictionary.* The dictionary is linked to Maple and MATLAB in the following ways:

- via the *with [software] commands* selector in the filtering options,
- via the *relevant Maple commands* and *relevant Matlab commands* checkboxes, which control the information included in the output lists of terms.

A Maple or MATLAB user can therefore enter a string and get a suggestion of relevant commands. Obviously, this strategy can be used even without the intention to translate.

In order to establish yet another link to other applications, we have included the *automatic translation function*:* a text string of less than 20 characters not containing characters other than letters copied into the Windows clipboard is automatically translated using the single term translation mode, when the dictionary window receives focus. This increases efficiency of using e.g. Maple or MATLAB help files as any unknown word copied in the Windows clipboard is translated automatically.

7. Printing

Any dictionary output can be printed easily. By clicking the *Printable version* button, the output list is formatted into a printer-friendly version and the *Print* dialog is displayed. Each list starts with a brief summary of filtering options and a survey of selected entry details. This enables users to make their own sets of lists, e.g. each for a given subject, area of mathematics, etc.

*The *relevant Mathematica commands* option is currently not active as Mathematica is not used in our courses. The *usual meaning* checkbox does not give any results apart from a pop-up window suggesting using the *general notes & commands* checkbox – the issue of displaying meaning is planned to be treated separately in a more complex way.

*See e.g. [5], where the necessity of such a solution is advocated.

*The automatic translation function works in the Windows + Internet Explorer 5.5+ environment only.

Fig. 4. A summary of filtering options and list details.

8. Possible enhancements

The application relies on the combination of PHP and MySQL. Apart from providing new terms and making corrections of the existing entries, the design of the dictionary corpus (i.e. its database) enables the enhancements such as *editing or adding new areas of mathematics*, *editing or adding new subjects*, *editing or adding new levels*, *supplementing Mathematica commands*, or *adding new filtering options*. Most importantly, the construction of the database makes *adding new languages* possible. Therefore, new bilingual versions can be prepared based on the existing code and database. This even holds for multilingual dictionaries. Furthermore, visual information can be included in the application.*

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*Adding Maple created images is scheduled for 2008, if funding is available.

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UNDER AND EXACT ESTIMATES OF COMPLEXITY OF ALGORITHMS FOR MULTI-PEG TOWER OF HANOI PROBLEM

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Abstract. It is proved under and exact estimates of complexity of algorithms for the multi-peg Tower of Hanoi problem with the limited number of discs.

1. Introduction

In 1884 famous French mathematician Edouard Lucas formulated and solved a mathematical problem "The Tower of Hanoi" [1]. To solve the problem E. Lucas proposes a recurrence algorithm. The complexity of this algorithm was described with the help of the formula

$$H_3(n) = 2^n - 1, \tag{1}$$

where $H_3(n)$ is the minimum number of moves needed to solve the puzzle with n discs for three pegs.

Later different generalizations of the classical Tower of Hanoi problem were published, one of which is "The multi-peg Tower of Hanoi". The multi-peg Tower of Hanoi problem consists $k > 3$ pegs $(B_1, B_2, \ldots, B_k)$ mounted on a board together with n discs of different sizes $(1, 2, \ldots, n)$. Initially these discs are placed on one peg (B_1) in order of size, with the largest (n-disc) on the bottom. The rules of the problem allow discs to be moved one at a time from one peg to another as long as a largest disc is never placed on top of a smaller disc. The goal of the problem is to transfer all n discs to another peg (B_2) with the minimum number of moves, denoted $H_k(n)$. The function $H_k(n)$ characterizes the complexity of the algorithm for the solution to the multi-peg Tower of Hanoi problem. For an optimal solution to the k-peg version of the classic Tower of Hanoi problem also is needed recurrence algorithms.

Algorithms of moving discs from one of k pegs $k > 3$ to another peg with the help of the more than one subsidiary pegs are used today in many computer science textbooks to demonstrate how to write a recursive algorithm or program. Also these algorithms are often proposed from programming on different olympiads and competitions for informaticians. It is known the difficulty of proofs of the complexity of recursive algorithms for the multi-peg Tower of Hanoi problem. Therefore this problem is interesting for mathematicians.

The Bibliography of P.K.Stockmeyer, which is devoted to the Tower of Hanoi problem maintains more than 200 relevant positions [2] However, an optimal solution to the k-peg version of the classic Tower of Hanoi problem is unknown for each $k \geq 4$.

In [3] algorithms with upper estimates of complexity for the k-peg Tower of Hanoi problem were published (cases: $k = 4, k = 5$ for any $n \geq 2$ and the case: any k for $k \leq n \leq \frac{k*(k-1)}{2}$).

The main aim of this paper is presentation of algorithms with exact estimates of complexity for the multi-peg Tower of Hanoi problem with the limited number of discs. To prove the optimality of the algorithm $A*$ we must take out the upper estimate (non-recursive) formula of the complixity for this algorithm $H(A*, k, n)$ and to prove that any other algorithm does not allow to solve our problem better (with smaller number of moves for the same parameters k, n).

It is obvious that for any $A*$ is evenly the inequality $H(A*, k, n) \geq H_k(n)$.

If we can estimate from below function $H_k(n)$ with the help of the general argumentations we get the under estimate $H(Min, k, n)$, where $H(Min, k, n) \leq H_k(n)$.

The algorithm $A*$ allows to find the optimal solution of our problem if it is proved that $H(A*, k, n) = H(Min, k, n)$. Then we have $H(A*, k, n) = H_k(n)$ and $A*$ is the optimal algorithm.

2. An upper estimate of the function $H_k(n)$

For investigation of our problem is interesting a next algorithm for transporting n discs from the first peg to B_2 for the case of $k \geq 3$ pegs.

Algorithm A_0

1. Move $k - 2$ smallest discs from the first peg to the peg B_k.
2. Move $k - 2$ next discs from the first peg to the peg B_{k-1}.
3. Move discs from the B_k to the peg B_{k-1}.
4. Move $k - 2$ next discs from the B_1 to the peg B_k.
5. Move discs from the B_{k-1} to the peg B_k.

At last (on step $2l - 1$, where $l =]n/(k - 2)[$) we move rest largest discs from the first peg to the peg B_2.

$2l$. Move all discs from the B_{k-1} (if l is not an even number) or from the B_k (if l is an even number) to the peg B_2.

In accordance with the algorithm we transport the stack of $k-2$ largest discs (from B_1 to the peg B_2) only one time. The next stack of $k-2$ largest discs we transport twice (from B_1 to B_k or B_{k-1} and then from B_k or B_{k-1} to the B_2). The next stack of $k-2$ discs we transport four times. At last the stack of $k-2$ smallest discs we transport 2^{l-1} times, where $l =]n/(k-2)[$.

We sum our moves of stacks for all steps and obtain $2^l - 1$ moves of stacks. For transporting of one stack with $k-2$ discs is needed $2k-5$ moves. Only for one stack (largest discs) less moves may be needed.

Then the explicit estimate of the comlexity of this algorithm is equal to

$$H(A_0, k, n) = (2^{]\frac{n}{k-2}[} - 1)(2k-5) \tag{2}$$

Formula (2) for the case $k = 3$ changes into formula (1). However for cases $k \geq 4$ the solution to our problem by the algorithm A_0 is not optimal. For example in the case $k = 7, n = 21$ we have 279 moves with the help of formula (2) and we have 71 moves with the help of our formula (6) from [3].

Then formula (2) is the upper estimate of the function $H_k(n)$. The algorithm A_0 has one quality. This algorithm may be used for our problem without restrictions for parameters k and n.

3. Domains for investigations

The Tower of Hanoi (in classic version) is a well known NP problem. Obviously the problem multi-peg Tower of Hanoi is also NP problem.

Upper estimates of the complexity of all known algorithms are functions exponential (sum of power of 2). In [4] seven algorithms for multi-peg Tower of Hanoi problem were analysed. All of them have the equivalent complexity, which is a function exponential. Upper estimates for functions $H_4(n)$ and $H_5(n)$ from [3] are also exponential functions.

However for some interesting cases, described by correlation between parameters k and n, we can propose easy algorithms for the multi-peg Tower of Hanoi problem, where the calculating complexity is not an exponential function.

Let's consider next cases:

A) $n \leq C^1_{k-1}$,
B) $C^1_{k-1} < n \leq C^2_k$,
C) $C^2_k < n \leq C^3_{k+1}$,
D) $C^3_{k+1} < n \leq C^4_{k+2}$.

4. Estimations of the function $H_k(n)$ for the case A)

Theorem 1. If $k \geq 3$ and $n \leq C_{k-1}^1$, then the exact estimation of the function $H_k(n)$ is equal to

$$H_k(n) = 2n - 1. \tag{3}$$

Proof. Let's infer the under estimate $H(Min, k, n)$ of the function $H_k(n)$ for our case A).

In this case for transferring each disc (except for the largest n-disc) from the first peg to the peg B_2 in according with the rules of the problem Tower of Hanoi two moves indepently of an algorithm of transferring are needed. The largest n-disc may be transferred from the first peg to the peg B_2 with one move. We sum our moves in this case and obtain the estimation

$$H(Min, k, n) = 2(n-1) + 1 = 2n - 1$$

For getting the upper estimate of the function $H_k(n)$ we use the next simple algorithm of transferring of discs.

Algorithm A_1

[1] Move n discs from the first peg to B_2, B_3, ... , B_{n+1} so that on the each peg one disc is placed and the largest n-disc is placed on B_2.

[2] Move $n-1$ discs from temporal pegs to the peg B_2, where one n-disc is already placed.

We sum our moves and obtain

$$H(A_1, k, n) = n + n - 1 = 2n - 1.$$

Therefore, for the case A) we have $H(Min, k, n) = H(A_1, k, n)$ and our algorithm A_1 is the optimal algorithm. Then the exact estimate for the case A) is equal to $H_k(n) = 2n - 1$.

5. Estimations of the function $H_k(n)$ for the case B)

Theorem 2. If $k \geq 3$ and $C_{k-1}^1 < n \leq C_k^2$, then the exact estimation of the function $H_k(n)$ is equal to

$$H_k(n) = 4n - 2k + 1. \tag{4}$$

Proof. Let's infer the under estimate $H(Min, k, n)$ of the function $H_k(n)$ for our case B).

In this case for transferring each of some smallest discs, which are marked as n_s, (except largest $n_l = k-1$ discs) from the first peg to the peg B_2 in according with rules of the problem Tower of Hanoi four moves independly of an algorithm of transferring are needed (from B_1 to B_h, from B_h to B_i, from B_i to B_j,from B_j to B_2). All largest $k-1$ discs may be transferred from the first peg to the peg B_2 no less than with $2k-3$ moves (with the help of (3)).

We sum our moves in this case and obtain the estimation

$$H(Min, k, n) = 4(n-(k-1)) + 2k - 3 = 4n - 2k + 1.$$

For getting the upper estimate of the function $H_k(n)$ we use the next algorithm [3] of transferring of discs.

Algorithm A_2

1. Move $k-1$ smallest discs from the first peg to the peg B_k ($2k-3$ moves) by the algorithm A_1.
2. Move $k-2$ next discs from the first peg to the peg B_{k-1} ($2k-5$ moves) by the algorithm A_1.
3. Move $k-3$ next discs from the first peg to the peg B_{k-2} ($2k-7$ moves) by the algorithm A_1.

   At last on stage $k-1$ we move one n-disc from B_1 to B_2.

k. Move two discs ($(n-2)$-disc and $(n-1)$-disc) from B_3 to B_2.

$k+1$. Move three next discs from B_4 to B_2.

At last on stage $2k-3$ we move rest smallest $k-1$ discs from B_k to B_2 by the algorithm A_1.

We sum our moves and obtain [3]

$$H(A_2, k, n) = 4n - 2k + 1.$$

Then

$$4n - 2k + 1 \leq H_k(n) \leq 4n - 2k + 1.$$

This yields our theorem.

6. Estimations of the function $H_k(n)$ for the case C)

We will prove the next statement

Theorem 3. If $k \geq 3$ and $C_k^2 < n \leq C_{k+1}^3$, then the exact estimation of the function $H_k(n)$ is equal to

$$H_k(n) = 8n - 2k^2 + 1. \tag{5}$$

Proof. Let's infer the under estimate $H(Min, k, n)$ of the function $H_k(n)$ for our case C).

In this case for transferring each of some smallest n_s discs from the first peg to the peg B_2 in according with rules of the problem Tower of Hanoi 2^3 moves independly of an algorithm of transferring are needed. Obviously, the number of such discs is equal to

$$n_s = n - n_l = n - \frac{k(k-1)}{2} = \frac{2n - k(k-1)}{2}.$$

All largest $n_l = C_k^2$ discs may be transferred from the first peg to the peg B_2 no less than with $4n_l - 2k + 1$ moves (with the help of Theorem 2).

Then for transferring all n discs in the case C) independently of used algorithm $H(Min, k, n)$ moves are needed, where $H(Min, k, n)$ is equal to

$$H(Min, k, n) = 8\frac{2n - k(k-1)}{2} + 4\frac{k(k-1)}{2} - 2k + 1 =$$

$$= 8n - 4k^2 + 4k + 2k^2 - 2k - 2k + 1 = 8n - 2k^2 + 1.$$

For getting the upper estimate of the function $H_k(n)$ we use the next algorithm.

Algorithm A_3

1. Move C_k^2 smallest discs from the first peg to the peg B_k by the algorithm A_2.
2. Move C_{k-1}^2 next (greater according to the size) discs from the first peg to the peg B_{k-1} by the algorithm A_2.
3. Move C_{k-2}^2 next (greater according to the size) discs from the first peg to the peg B_{k-2} by the algorithm A_2.

   At last on stage $k-1$ we move one n-disc from B_1 to B_2.

k. Move all discs from pegs B_3,$B_4, \ldots, B_k$ to the peg B_2 by the algorithm A_2.

In this case C) we can transfer the maximum number of discs n, which is equal to

$$n = C_2^2 + C_3^2 + \cdots + C_{k-1}^2 + C_k^2 = C_{k+1}^3.$$

For getting the upper estimate of the function $H_k(n)$ in the case C) we use the following formula

$$H(A_3, k, n) = H_2(n_2) + 2(H_3(n_3) + H_4(n_4) + \cdots + H_k(n_k)),$$

where n_i is a number of discs placed on the peg B_i, where $i \in \{2, \cdots, k\}$ and $n_2 = 1, n_3 = 3$.

We get with the help of formulas (1), (4)

$$H(A_3,k,n) = 1+2(7+(4n_4-2*4+1)+(4n_5-2*5+1)+\cdots+(4n_k-2k+1)) =$$

$$= 1+2(4(n_4+n_5+\cdots+n_k) - \frac{2(k+4)(k-3)}{2} + k - 3 + 7) =$$

$$= 1+2(4(n-4)-k^2+3k-4k+12+k+4) = 1+2(4n-k^2) = 8n-2k^2+1.$$

Then A_3 is the optimal algorithm and in the case C) we have

$$8n-2k^2+1 \le H_k(n) \le 8n-2k^2+1.$$

Theorem 3 is proved.

7. Estimations of the function $H_k(n)$ for the case D).

Theorem 4. If $k \ge 3$ and $C_{k+1}^3 < n \le C_{k+2}^4$, then the exact estimation of the function $H_k(n)$ is equal to

$$H_k(n) = 16n - \frac{2k((k+1)(2k+1)-3)}{3} + 1. \tag{6}$$

Proof. Let's infer the under estimate $H(Min,k,n)$ of the function $H_k(n)$ for our case D).

It is impossible (by Theorem 3) to transfer a smallest disc (1-disc) from B_1 to B_2 with 2^3 moves, if $n > C_{k+1}^3$. In the case D) we have n_s discs for transferring of each $2^4 = 16$ moves are needed.

Obviously, the number of such (smallest) discs (in case D)) is equal to

$$n_s = n - C_{k+1}^3 = n - \frac{(k+1)k(k-1)}{6}.$$

All largest $n_l = C_{k+1}^3$ discs may be transferred from the first peg to the peg B_2 no less than with $8n_l - 2k^2 + 1$ moves (with the help of Theorem 3).

Then we have

$$H(Min,k,n) = 16(n - \frac{(k+1)k(k-1)}{6}) + 8C_{k+1}^3 - 2k^2 + 1 =$$

$$= 16n - 4\frac{(k+1)k(k-1)}{3} - 2k^2 + 1.$$

Then for the case D) we have

$$H(Min,k,n) = 16n - 2(k^2-1)\frac{2k+3}{3} - 1. \tag{7}$$

For getting the upper estimate of the function $H_k(n)$ we use the next algorithm.

Algorithm A_4

1. Move C_{k+1}^3 smallest discs from the first peg to the peg B_k by the algorithm A_3. In this order to transfer from the B_1 to the peg B_2 smallest C_k^2 discs, to the peg B_3 next C_{k-1}^2 discs, ... to the B_k one disc and later move all discs from pegs B_{k-1},B_{k-2}, ... ,B_2 to the peg B_k by the algorithm A_2.
2. Move C_k^3 next (greater according to the size) discs from the first peg to the peg B_{k-1} by the algorithm A_3.
3. Move C_{k-1}^3 next (greater according to the size) discs from the first peg to the peg B_{k-2} by the algorithm A_3.

   At last on stage $k-1$ we move one n-disc from B_1 to B_2.

k. Move all discs from pegs B_3,B_4, ... , B_k to the peg B_2 by the algorithm A_3.

In this case D) we can transfer the maximum number of discs n , where

$$n = C_3^3 + C_4^3 + \cdots + C_k^3 + C_{k+1}^3 = C_{k+2}^4.$$

For getting the upper estimate of the function $H_k(n)$ in the case D) we use the following formula

$$H(A_4, k, n) = H_3(n_3) + 2(H_4(n_4) + H_5(n_5) + \cdots + H_k(n_k)),$$

where n_i means a number of discs placed on the peg B_i, where $i \in \{3, \cdots, k\}$.

With the help of formulas (1), (5) and with the condition $n_3 = 5$ we obtain

$$H(A_4, k, n) = 31+2((8n_4-2*4^2+1)+(8n_5-2*5^2+1)+\cdots+(8n_k-2*k^2+1)) =$$

$$= 31 + 16(n-5) - 4\sum_{i=4}^{k} i^2 + 2(k-3) =$$

$$= 16n + 31 - 80 + 56 + 2(k-3) - \frac{2k(k+1)(2k+1)}{3} =$$

$$= 16n + 7 + \frac{6(k-3) - 2k((k+1)(2k+1))}{3} =$$

$$= 16n + 7 + \frac{2k((k+1)(2k+1) - 6(k-3)}{3} =$$

$$= 16n + 7 - 6 - \frac{2k((k+1)(2k+1) - 3)}{3} =$$

$$= 16n + 1 - \frac{2k((k+1)(2k+1) - 3)}{3}.$$

So we get

$$H(A_4, k, n) = 16n + 1 - \frac{2k((k+1)(2k+1)-3)}{3}. \tag{8}$$

Now we will compare our under estimate (7) with our upper estimate (8). We have

$$H(A_4, k, n) - H(Min, k, n) = 16n + 1 - \frac{2k((k+1)(2k+1)-3)}{3} - 16n +$$

$$+2(k^2-1)\frac{2k+3}{3} + 1 = 2(k^2-1)\frac{2k+3}{3} - \frac{2k((k+1)(2k+1)-3)}{3} + 2 =$$

$$= \frac{2(2k+3)(k^2-1) - 2k((k+1)(2k+1)-3) + 6}{3} =$$

$$= \frac{2(2k^3+3k^2-2k-3) - 2k(2k^2+3k-2) + 6}{3} =$$

$$= \frac{2k(2k^2+3k-2) - 6 - 2k(2k^2+3k-2) + 6}{3} = 0.$$

Therefore, for the case D) $H(Min, k, n) = H(A_4, k, n)$ and our algorithm A_4 is the optimal algorithm. Then the exact estimate for the case D) is equal to

$$H_k(n) = 16n - \frac{2k((k+1)(2k+1)-3)}{3} + 1 = 16n - 2(k^2-1)\frac{2k+3}{3} - 1.$$

Theorem 4 is proved.

8. Conclusions

Our method for getting of under estimates of the function $H_k(n)$ may be generalized for any cases, where $k > 3$ and

$$C_{k+t-3}^{t-1} < n \leq C_{k+t-2}^{t}.$$

The parameter t may be defined always with the help of the Pascal's triangle for concrete values n and k.

We use from [5] the formula

$$tri_d(n+1) = \sum_{i=1}^{n} tri_{d-1}(i),$$

where $tri_d(j)$ is a "d-triangle" number.

It is known $tri_1(j) = C_j^1 = j$ natural numbers, $tri_2(j) = C_j^2 = 1 + 2 + \cdots + (j-1)$ triangular numbers, $tri_3(j) = C_j^3$ tetrahedral (pyramidal) numbers, which are the sum of consecutive triangular numbers.

We can write this formula as

$$C_{n+1}^d = \sum_{i=1}^{n} C_i^{d-1}.$$

Then our under estimate for the function $H_k(n)$ is equal to

$$H(Min, k, n) = 2^t(n - C_{k+t-3}^{t-1}) + \sum_{i=0}^{t-1} 2^i C_{(k-3)+i}^i. \tag{9}$$

Formula (9) allows to obtain values $H(Min, k, n)$ for any n and k. In particular $H(Min, 4, 64) = 18433, H(Min, 5, 64) = 1535, H(Min, 6, 64) = 673$, which coincide with corresponding values from [3]. Howevere this fact does not prove that our formula (9) is the exact estimate of the function $H_k(n)$.

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ANOMALOUS DIFFUSION EQUATION AND DIFFUSIVE STRESSES

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Abstract. Essentials of the Riemann-Liouville fractional calculus are recalled. Nonlocal generalizations of the Fourier law of the classical theory of heat conduction relating the heat flux vector to the temperature gradient and of the Fick law of the classical theory of diffusion relating the matter flux vector to the concentration gradient lead to nonclassical theories. The time-nonlocal dependence between the flux vectors and corresponding gradients with "long-tale" power kernel can be interpreted in terms of fractional integrals and derivatives and yields the time-fractional diffusion equation.

1. Essentials of the Riemann-Liouville fractional calculus

In this section we recall the main ideas of fractional calculus (see [1, 2], among others). It is common knowledge that integrating by parts $n-1$ times the calculation of the n-fold primitive of a function $f(t)$ can be reduced to the calculation of a single integral of the convolution type

$$I^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau, \tag{1}$$

where n is a positive integer.

The Riemann–Liouville fractional integral is introduced as a natural generalization of the convolution type form (1):

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad \alpha > 0, \tag{2}$$

where $\Gamma(\alpha)$ is the gamma function.

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to I^α

$$D^\alpha_{RL} f(t) = D^n I^{n-\alpha} f(t) \tag{3}$$

and for its Laplace transform rule requires the knowledge of the initial values of the fractional integral $I^{n-\alpha} f(t)$ and its derivatives of the order $k = 1, 2, \ldots, n-1$.

An alternative definition of the fractional derivative was proposed by Caputo [3]:

$$D^\alpha_C f(t) = I^{n-\alpha} D^n f(t). \tag{4}$$

For its Laplace transform rule the Caputo fractional derivative requires the knowledge of the initial values of the function $f(t)$ and its integer derivatives of order $k = 1, 2, \ldots, n-1$.

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions. The major utility of Caputo fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the govering equation is of fractional order [4].

2. Nonlocal generalizations of the Fick and Fourier laws

The classical theory of diffusion is based on the Fick law

$$\mathbf{J} = -\kappa \operatorname{grad} c \tag{5}$$

relating the matter flux vector $\mathbf{J}$ to the concentration gradient, where κ is the diffusion conductivity. In combination with the balance equation for mass the Fick law leads to the classical diffusion equation

$$\frac{\partial c}{\partial t} = a \, \Delta \, c, \tag{6}$$

where a is the diffusivity coefficient.

The classical theory of heat conduction is based on the Fourier law

$$\mathbf{q} = -k \operatorname{grad} T \tag{7}$$

relating the heat flux vector $\mathbf{q}$ to the temperature gradient, where k is the thermal conductivity of a solid. In combination with the law of conservation of energy, this equation leads to the parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a_{\mathrm{T}} \Delta \, T, \tag{8}$$

where a_{T} is the thermal diffusivity coefficient, t is time, Δ is the Laplace operator.

During the past three decades, nonclassical theories, in which the Fourier law and the Fick law as well as the heat conduction equation and the diffusion equation were replaced by more general equations, have been proposed. Some of these theories were formulated in terms of the theory of heat conduction, other in terms of the diffusion theory.

In time-nonlocal theories the Fourier law is generalized to integral dependence between the heat flux vector and the temperature gradient

$$\mathbf{q}(t) = -k \int_0^t K(t-\tau)\,\mathrm{grad}\,T(\tau)\,\mathrm{d}\tau \tag{9}$$

or in terms of diffusion

$$\mathbf{J}(t) = -\kappa \int_0^t K(t-\tau)\,\mathrm{grad}\,c(\tau)\,\mathrm{d}\tau. \tag{10}$$

The time-nonlocal dependence between the flux vectors and corresponding gradients with "long-tale" power kernel can be interpreted in terms of fractional integrals and derivatives and yields the time-fractional diffusion (or heat conduction) equation

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta\,c, \qquad 0 < \alpha < 2. \tag{11}$$

This equation is usually referred to "anomalous diffusion". Other terms used in this context are: "anomalous transport", "fractional diffusion", "paradoxal diffusion", "strange kinetics".

Various types of anomalous transport can be distinguished. The limiting case $\alpha = 0$ corresponding to the Helmholtz equation is associated with localized diffusion. The slow diffusion regime is characterized by the value $0 < \alpha < 1$. The power-law tails make it possible to have very long waiting times, and particles move slower than in the ordinary diffusion which corresponds to $\alpha = 1$. In the fast diffusion regime ($1 < \alpha < 2$) it is possible to have very long jumps, and particles move faster than in the ordinary diffusion. The limiting case $\alpha = 2$ corresponding to the wave equation is known as ballistic diffusion.

Equation (11) is a mathematical model of important physical phenomena ranging from amorphous, colloid, glassy and porous materials through fractals, percolation clusters, random and disordered media to comb structures, dielectrics and semiconductors, polymers and biological systems.

3. Theory of diffusive stresses based on anomalous diffusion equation

A quasi-static uncoupled theory of diffusive (or thermal) stresses based on Eq. (11) was proposed by the author [5–7]. A quasi-static uncoupled theory of diffusive stress is governed by the equilibrium equation in terms of displacements

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\mathbf{u} = \beta_c K_c\,\mathrm{grad}\,c, \tag{12}$$

the stress-strain-concentration relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_c K_c c)\mathbf{I}, \tag{13}$$

and the time-fractional diffusion equation

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta\,c + Q, \qquad 0 \le \alpha \le 2, \tag{14}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, $\mathbf{e}$ the linear strain tensor, c the concentration, Q the mass source, a the diffusivity coefficient, λ and μ are Lamé constants, $K_c = \lambda + 2\mu/3$, β_c is the diffusion coefficient of volumetric expansion, $\mathbf{I}$ denotes the unit tensor.

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PROPERTIES OF FOURIER COEFFICIENTS OF SPLINE WAVELETS

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Abstract. Periodic B-spline functions have got many useful properties. Especially it is the property of its Fourier coefficients. In this article it is introduced and proved a similar property of Fourier coefficients of spline wavelets.

1. Spline wavelets

In this section, we shall briefly summarize the essences of the theory of the wavelet expansion. We start by defining of a multiresolution analysis.

Definition: The multiresolution analysis of $L^2(\mathbb{R})$ is a sequence of closed subspaces V_j of $L^2(\mathbb{R})$, $j \in \mathbb{Z}$, with the following properties:

[1] $V_j \subset V_{j+1}$

[2] $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$

[3] $f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$

[4] $\bigcup\limits_{j=-\infty}^{+\infty} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap\limits_{j=-\infty}^{+\infty} V_j = \{0\}$

[5] A scaling function $\phi \in V_0$, with a non-vanishing integral, exists such that the collection $\{\phi(x-l) | l \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Since $\phi \in V_0 \subset V_1$, a sequence $(h_k) \in \ell^2(\mathbb{Z})$ exists such that the scaling function satisfies the dilation equation

$$\phi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \phi(2x-k) \tag{1}$$

It is immediately to view that the collection of functions $\{\phi_{j,k}| \quad k \in \mathbb{Z}\}$, with $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$, is a Riesz basis of V_j.

We will define a space W_j complementing V_j in V_{j+1}, i.e. a space that satisfies

$$V_{j+1} = V_j \oplus W_j,$$

where symbol $\oplus$ stands for direct sum. From this follows the relation $\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R})$. This subspace W_j is called "wavelet subspace" and is generated by $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, where function $\psi(x)$ is called the "wavelet" and collection of functions $\{\psi(x-k)|k \in \mathbb{Z}\}$ forms a Riesz basis of W_0.

Since the wavelet ψ is an element of V_1, a sequence $(g_k) \in \ell^2(\mathbb{Z})$ exists such that

$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_k \phi(2x-k). \tag{2}$$

We described wavelets and scaling functions defined on the real line. For many applications it is necessary, or at least more natural, to work on a subset of the real line. Many of these cases can be dealt with by introducing periodized scaling function and wavelets, which we define as follows:

$$\phi_j^{\langle 0,1\rangle}(x) = 2^{\frac{j}{2}} \sum_{n\in\mathbb{Z}} \phi(2^j(x-n)) \tag{3}$$

$$\psi_j^{\langle 0,1\rangle}(x) = 2^{\frac{j}{2}} \sum_{n\in\mathbb{Z}} \psi(2^j(x-n)). \tag{4}$$

$$\tag{5}$$

Functions $\phi_j^{\langle 0,1\rangle}(x - k2^{-j})$ and $\psi_j^{\langle 0,1\rangle}(x - k2^{-j})$ for $k = 0, 1, .., 2^j - 1$ are linear independent and generate the spaces $V_j^{\langle 0,1\rangle}$ eventually $W_j^{\langle 0,1\rangle}$. Let us define for $j \in \mathbb{N}, j \geq j_0$ and $n_j = 2^j$ the set $K_j = \{0, 1, .., n_j - 1\}$ and take equidistant partition of the interval $\langle 0, 1\rangle$ with step width $h_j = \frac{1}{n_j}$. The points x_k^j are defined by $x_k^j = \frac{k}{n_j}$ for $k \in K_j$.

Characteristic function on the interval $\langle 0, 1)$ is defined as

$$\chi = \begin{cases} 1 & x \in \langle 0,1\rangle \\ 0 & \text{elsewhere} \end{cases}$$

For $r > 1$ we define χ^r as the convolution

$$\chi^r(x) = \int_0^1 \chi^{r-1}(x-y)\chi(y)dy.$$

This function is B-spline on $\mathbb{R}$ of the order r. If we take $\phi(x) = \chi^r(x)$, where $\phi(x)$ is scaling function introduced above, then $\phi_j^{\langle 0,1\rangle}$ for $k \in K_j$ is B-spline of the order r (piecewise polynomial of order $r-1$) on the interval $\langle 0,1\rangle$. To this scaling function we can build (as was shown earlier) functions $\psi_j^{\langle 0,1\rangle}(x-k2^{-j})$.

2. Property of Fourier coefficient of B-spline and spline wavelets

Fourier series of one-periodic function $\phi_j^{\langle 0,1\rangle}$ have form

$$\phi_j^{\langle 0,1\rangle}(x) = \sum_{p\in\mathbb{Z}} \widehat{\phi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x},$$

where $\widehat{\phi}_j^{\langle 0,1\rangle}(p)$ are Fourier coefficient defined by formula

$$\widehat{\phi}_j^{\langle 0,1\rangle}(p) = \int_0^1 \phi_j^{\langle 0,1\rangle}(x) e^{-2\pi i p x} dx. \tag{6}$$

Theorem 1. Let S_n^d is the space of one-periodic B-spline of order $d+1$ (piecewise polynomial of degree d) with knots $x_k = \frac{k}{n}$, where n is arbitrary natural number, $j = 0, ..., n-1$. Let us define a set $\Lambda_n = \{p \in \mathbb{Z}; -\frac{n}{2} < p \leq \frac{n}{2}\}$. Then for Fourier coefficients of one-periodic B-spline $\widehat{\phi}$ equality holds

$$(-1)^{l(d+1)}\widehat{\phi}(p+ln)(p+ln)^{d+1} = \widehat{\phi}(p)p^{d+1}, \qquad l \in \mathbb{Z}, \phi \in S_n^d, p \in \Lambda_n. \tag{7}$$

This equality was proved in [1].

Relation between Fourier coefficients of functions $\phi_j^{\langle 0,1\rangle}$ and $\phi_{j+1}^{\langle 0,1\rangle}$ arise from dilation equation by the following way:

$$\begin{aligned}
\sum_{p\in\mathbb{Z}} \widehat{\phi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x} &= \phi_j^{\langle 0,1\rangle}(x) \\
&= \sum_{k\in\mathbb{Z}} h_k \phi_{j+1}^{\langle 0,1\rangle}(x - k2^{-j-1}) \\
&= \sum_{k\in\mathbb{Z}} h_k \left(\sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{2\pi i p (x-k2^{-j-1})} \right) \\
&= \sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) \left(\sum_{k\in\mathbb{Z}} h_k e^{-2\pi i p k 2^{-j-1}} \right) e^{2\pi i p x}.
\end{aligned}$$

By comparing of Fourier coefficients we obtain

$$\widehat{\phi}_j^{\langle 0,1\rangle}(p) = \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) m_{j+1}(p), \tag{8}$$

where $m_{j+1}(p) = \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i p k 2^{-j-1}}$, is function with period 2^{j+1}. It is valid for this function

$$m_{j+1}(2p) = m_j(p) \tag{9}$$

and

$$m_{j+1}(2^j) = 0. \tag{10}$$

Using this relation, we can prove following theorem.

Theorem 2. Let $\phi_j^{\langle 0,1\rangle}$ is a member of space $V_j^{\langle 0,1\rangle} \equiv S_{n_j}^d$ one-periodic B-spline of order $d+1$, where $n_j = 2^j$ and $\psi_j^{\langle 0,1\rangle}$ is corresponding spline wavelet from the space $W_j^{\langle 0,1\rangle}$, a set $\Lambda_{n_j} = \{p \in \mathbb{Z}; -\frac{n_j}{2} < p \leq \frac{n_j}{2}\}$. Then for Fourier coefficients of spline wavelets $\psi_j^{\langle 0,1\rangle}$ in the space $W_j^{\langle 0,1\rangle}$ the relation holds

$$(-1)^{l(d+1)}\widehat{\psi}(p+ln)(p+ln)^{d+1} = \widehat{\psi}(p)p^{d+1}, \quad l \in 2\cdot\mathbb{Z}, \psi \in W_j, p \in \Lambda_{n_j}, \tag{11}$$

where the set $2\cdot\mathbb{Z}$ is the set of even integral numbers.

Proof We can write

$$\psi_j^{\langle 0,1\rangle}(x) = \sum_{p\in\mathbb{Z}} \widehat{\psi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x},$$

where $\widehat{\psi}_j^{\langle 0,1\rangle}(p)$ are Fourier coefficients. It follows from (2)

$$\begin{aligned}
\sum_{p\in\mathbb{Z}} \widehat{\psi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x} &= \psi_j^{\langle 0,1\rangle}(x) \\
&= \sum_{k\in\mathbb{Z}} g_k \phi_{j+1}^{\langle 0,1\rangle}(x - k2^{-j-1}) \\
&= \sum_{k\in\mathbb{Z}} g_k \sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{2\pi i p (x - k2^{-j-1})} \\
&= \sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) \left(\sum_{k\in\mathbb{Z}} g_k e^{-2\pi i p k 2^{-j-1}} \right) e^{2\pi i p x}.
\end{aligned}$$

Now we compare Fourier coefficients.

$$\begin{aligned}
\widehat{\psi}_j^{\langle 0,1\rangle}(p) &= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} g_k e^{-2\pi i p k 2^{-j-1}} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} (-1)^k h_{1-k} e^{-2\pi i p k 2^{-j-1}} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} e^{k\pi i} h_{1-k} e^{-2\pi i p k 2^{-j-1}} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} h_k e^{-2\pi i p (1-k) 2^{-j-1}} e^{(1-k)\pi i} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{\pi i (1-p2^{-j})}\sum_{k\in\mathbb{Z}} h_k e^{2\pi i p k 2^{-j-1}} e^{-k\pi i} \\
&= -\widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{-2\pi i p 2^{-j-1}}\sum_{k\in\mathbb{Z}} h_k e^{-2\pi i k (\frac{1}{2}-p2^{-j-1})} \\
&= -\widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{-2\pi i p 2^{-j-1}}\overline{m_{j+1}(p-2^j)}
\end{aligned}$$

For $m_{j+1}(p) \neq 0$ we can write

$$\widehat{\psi}_j^{\langle 0,1\rangle}(p) = \widehat{\phi}_j^{\langle 0,1\rangle}(p) m_{j+1}^{-1}(p)\left[-e^{-2\pi i p 2^{-j-1}}\overline{m_{j+1}(p-2^j)}\right]. \tag{12}$$

When we propound $n = 2^j$ and $\Lambda_n = \{p \in \mathbb{Z}; -\frac{n}{2} < p \leq \frac{n}{2}\}$ as in (7) and signify $\widehat{\phi}_j^{\langle 0,1\rangle}(p) \equiv \widehat{\phi}(p)$. For $\Lambda_n = \{p \in \mathbb{Z}; -2^{j-1} < p \leq 2^{j-1}\}$ follows this $m_{j+1}(p) \neq 0$. Further with using the relation (7) we can write

$$\begin{aligned}
&(-1)^{l(d+1)}\widehat{\psi}_j^{\langle 0,1\rangle}(p+ln)(p+ln)^{d+1} = \\
&= (-1)^{l(d+1)}\widehat{\phi}_j^{\langle 0,1\rangle}(p+ln)(p+ln)^{d+1} m_{j+1}^{-1}(p+ln)\cdot \\
&\cdot \left[-e^{\frac{-2\pi i(p+ln)}{2^{j+1}}}\overline{m_{j+1}(p+ln-2^j)}\right].
\end{aligned}$$

Since it holds

$$\begin{aligned}
m_{j+1}(p+ln) &= \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i (p+ln) k 2^{-j-1}} \\
&= \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i (p) k 2^{-j-1}} e^{-2\pi i (ln) k 2^{-j-1}}
\end{aligned}$$

for $n = 2^j$ we obtain

$$m_{j+1}(p+ln) = \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i (p) k 2^{-j-1}} (-1)^{lk}$$

When we write

$$e^{-2\pi i(ln)k2^{-j-1}} = e^{-2\pi ilk}$$

we obtain for even numbers l

$$\begin{aligned}
&(-1)^{l(d+1)}\widehat{\psi}_j^{\langle 0,1\rangle}(p+ln)(p+ln)^{d+1} = \\
&= \widehat{\phi}_j^{\langle 0,1\rangle}(p)p^{d+1}m_{j+1}^{-1}(p)\left[-e^{-2\pi ip2^{-j-1}}\overline{m_{j+1}(p-2^j)}\right] \\
&= \widehat{\psi}_j^{\langle 0,1\rangle}(p)p^{d+1}.
\end{aligned}$$

From preceding it follows the property (11) for spline wavelets.

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GENERAL APPROACH TO DETERMINING THE BASIC CHARACTERISTICS OF QUEUEING SYSTEMS WITH FINITE TOTAL CAPACITY

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Abstract. We discuss a general view of solutions for characteristics of non-classical queueing systems with random capacity customers (demands), i.e. we suppose that each customer is characterized by some random capacity (volume) and the whole capacity (total volume) of customers present in the queueing system is bounded by a constant walue $V > 0$. We determine the general view of the stationary number distribution and loss probability in the systems under consideration as compared with corresponding classical queueing systems. It's turned that in some cases we can write expressions for non-classical characteristics of finite total capacity queues if corresponding classical characteristics are known.

1. Introduction

In present work we investigate non-classical queueing systems with random volume demands and finite total (demands) capacity. It means that 1) each demand is characterized by some non-negative random space requirement (capacity or volume) ζ; 2) the total sum $\sigma(t)$ of space requirements (volumes) of all demands present in the system at arbitrary time moment t is limited by some constant value V, which is named the memory volume of the system; 3) we also assume that service time ξ of the demand and it's volume ζ are generally dependent.

Such systems have been used to model and solve various practical problems occurring in the design of computer and communicating systems.

Let $F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$ be the distribution function of the random vector (ζ, ξ). Then $L(x) = \mathbf{P}\{\zeta < x\} = F(x, \infty)$, $B(t) = \mathbf{P}\{\xi < t\} = F(\infty, t)$

are the distribution functions of the demand volume and service time, respectively. The space is occupied by the demand at the epoch it arrives and is released entirely at the epoch it completes service. The process $\sigma(t)$ is called the total (demands) volume.

Total volume limitation leads to additional losses of demands. A demand having the space requirement x, which arrives at the epoch τ, when there are idle servers or waiting positions, will be admitted to the system if $\sigma(\tau - 0) + x > V$. Otherwise ($\sigma(\tau - 0) + x \leq V$), the demand will be lost.

Various queueing systems with limited memory volume were analyzed in the papers [1–8]. It follows from the papers that it's possible to determine a stationary demands number distribution and loss probability for the following queueing systems:

1) $M/M/n/(m, V)$ ($M/M/n/m$-type system in which a demand has an arbitrary distributed volume, but service time is independent of the demand volume and the total volume is limited by the value $V > 0$, $1 \leq n \leq \infty$, $0 \leq m \leq \infty$);

2) $M/G/n/(0, V)$ system (generalized Erlang system or $M/G/n/0$-type system with an arbitrary joint distribution of service time and demand volume and limited total volume);

3) processor-sharing system with an arbitrary joint distribution of service time and demand volume and limited total volume.

Our aim is to show that it is possible to determine some characteristics of non-classical (in the above sense) queueing systems, if the similar characteristics of classical systems are determined. In other words, we want to show the relation between similar classical and non-classical characteristics.

We'll demonstrate this approach by some examples.

2. M/M/n/m and M/M/n/(m, V) systems

Let a be the intensity of input flow, μ be the parameter of service time. Denote as $p_k = \mathbf{P}\{\eta = k\}$ the stationary probability of presence of k demands in the classical system, $k = \overline{0, n+m}$. Then we have the following well known equations for p_k:

$$0 = -ap_0 + \mu p_1; \tag{1}$$

$$0 = ap_0 - (a + \mu)p_1 + 2\mu p_2; \tag{2}$$

$$0 = ap_{k-1} - (a + k\mu)p_k + (k+1)\mu p_{k+1},\ k = \overline{1, n-1}; \tag{3}$$

$$0 = ap_{k-1} - (a + n\mu)p_k + n\mu p_{k+1},\ k = \overline{1, n+m-1}; \tag{4}$$

$$0 = ap_{n+m-1} - n\mu p_{n+m}. \tag{5}$$

Suppose now that each demand in the non-classical system is characterized by some random volume ζ, and service time ξ doesn't depend on its volume. Let $L(x)$ be the distribution function of the random variable ζ. Suppose that the total demands volume is limited by value $V > 0$.

We suggest a hypothesis that probabilities $r_k = \mathbf{P}\{\eta = k\}$, $k = 0, 1, \ldots,$ for the second system have the following form:

$$r_k = Cp_k L_*^{(k)}(V),\ k = 1, 2, \ldots,$$

where $L_*^{(k)}(x)$ is the kth order Stieltjes convolution of the function $L(x)$, i.e.

$$L_*^{(0)}(x) \equiv 1,\ L_*^{(k)}(x) = \int_0^x L_*^{(k-1)}(x-u)dL(u),\ k = 1, 2, \ldots.$$

To conform this hypothesis, we have introduce the following functions having (for $k \geq 1$) the following probability sense:

$$g_k(x) = \mathbf{P}\{\eta = k, \sigma < x\}, \tag{6}$$

where σ is the stationary total volume of the demands present in the system. It's clear that $r_k = g_k(V)$, $k = 1, 2, \ldots.$ According to our hypothesis we assume that

$$g_k(x) = Cp_k L_*^{(k)}(x),\ k = 1, 2, \ldots. \tag{7}$$

We obtain the following equations for introduced functions:

$$0 = -ar_0 L(V) + \mu r_1; \tag{8}$$

$$0 = ar_0 L(V) - a\int_0^V g_1(V-x)dL(x) - \mu r_1 + 2\mu r_2; \tag{9}$$

$$0 = a\int_0^V g_{k-1}(V-x)dL(x) - a\int_0^V g_k(V-x)dL(x) - k\mu r_k + (k+1)\mu r_{k+1},$$
$$k = \overline{1, n-1}; \tag{10}$$

$$0 = a\int_0^V g_{k-1}(V-x)dL(x) - a\int_0^V g_k(V-x)dL(x) - n\mu r_k + n\mu r_{k+1},$$
$$k = \overline{n, n+m-1}; \tag{11}$$

$$0 = a\int_0^V g_{n+m-1}(V-x)dL(x) - n\mu r_{n+m}. \tag{12}$$

It's easy to see that, if we substitute functions (7) to the equations (8)–(12), we obtain the equations (1)–(5) for p_k. So, our hypothesis is truthful, and we have

$$r_k = \begin{cases} \dfrac{(n\rho)^k}{k!} r_0 L_*^{(k)}(V), \ k = \overline{1,n}, \\ \dfrac{n^n \rho^k}{n!} r_0 L_*^{(k)}(V), \ \ k = \overline{n+1, n+m}. \end{cases}$$

From the normalization condition we obtain [9]

$$r_0 = \left[\sum_{k=0}^{n} \frac{(n\rho)^k}{k!} L_*^{(k)}(V) + \frac{n^n}{n!} \sum_{k=n+1}^{n+m} \rho^k L_*^{(k)}(V) \right]^{-1}.$$

For the loss probability we have [9]

$$p_l = 1 - (n\rho)^{-1} \sum_{k=1}^{n-1} k r_k - \rho^{-1} \left(1 - \sum_{k=0}^{n-1} r_k \right).$$

3. M/M/1/(∞, V) system with preemptive discipline

Let us consider $M/M/1/(\infty, V)$ system with two Poisson input flows: the first is the flow of the first priority with parameter a_1 and the second is the flow of the second priority with parameter a_2. Demands from the first flow gain an advantage over demands from the second one in accordance with preemptive resume discipline. Each demand is characterized by some random volume ζ with the distribution function $L(x)$ for the both priorities. The total demands volume $\sigma(t)$ is limited by the value $V > 0$. A demand arriving to the system will be lost in accordance with the above agreement. Demands service time doesn't depend on its volume. Let μ_1 and μ_2 be the parameter of service time of the first and second priority demands, respectively.

Our aim is to determine the stationary joint distribution of numbers of both priority demands present in the system and stationary loss probability for demands of each priority.

Let $\eta_1(t)$ and $\eta_2(t)$ be the number of demands of the first and second priority accordingly present in the system at time moment t, $\zeta_i^j(t)$ be the volume of jth demand of ith priority ($i = 1, 2$). Then system behavior can be described by the following Markov process:

$$\left(\eta_1(t); \eta_2(t); \zeta_1^i(t), i = \overline{1, \eta_1(t)}; \zeta_2^j(t), j = \overline{1, \eta_2(t)} \right). \tag{13}$$

It's obvious that $\sigma(t) = \sum\limits_{i=1}^{\eta_1(t)} \zeta_1^i(t) + \sum\limits_{j=1}^{\eta_2(t)} \zeta_2^j(t)$.

We shall characterize the process (13) by the following functions:

$$P(0,0,t) = \mathbf{P}\{\eta_1(t) = \eta_2(t) = 0\};$$

$$G(i,j,x,t) = \mathbf{P}\{\eta_1(t) = i, \eta_2(t) = j, \sigma(t) < x\},\ i,j = 0,1,..., \max(i,j) \geq 1;$$

$$P(i,j,t) = \mathbf{P}\{\eta_1(t) = i, \eta_2(t) = j\} = G(i,j,V,t),\ i,j = 0,1,..., \max(i,j) \geq 1.$$

If stationary condition takes place ($V < \infty$), we have $\sigma(t) \Rightarrow \sigma$, $\eta_i(t) \Rightarrow \eta_i$, $i = 1,2$, in the sense of a weak convergence. Then the following limits exist:

$$p(0,0) = \lim_{t\to\infty} P(0,0,t) = \mathbf{P}\{\eta_1 = \eta_2 = 0\}; \tag{14}$$

$$g(i,j,x) = \lim_{t\to\infty} G(i,j,x,t) = \mathbf{P}\{\eta_1 = i, \eta_2 = j, \sigma < x\},$$
$$i,j = 0,1,..., \max(i,j) \geq 1; \tag{15}$$

$$p(i,j) = \lim_{t\to\infty} P(i,j,t) = \mathbf{P}\{\eta_1 = i, \eta_2 = j\} = g(i,j,V),$$
$$i,j = 0,1,..., \max(i,j) \geq 1. \tag{16}$$

It can be easy shown that the functions (14)–(16) satisfy the following equations:

$$0 = -(a_1 + a_2)p(0,0)L(V) + \mu_1 p(1,0) + \mu_2 p(0,1); \tag{17}$$

$$0 = a_1 p(0,0)L(V) - (a_1 + a_2)\int_0^V g(1,0,V-x)dL(x) - \mu_1 p(1,0) + \mu_1 p(2,0); \tag{18}$$

$$0 = a_1 \int_0^V g(i-1,0,V-x)dL(x) - (a_1 + a_2)\int_0^V g(i,0,V-x)dL(x) -$$
$$-\mu_1 p(i,0) + \mu_1 p(i+1,0),\ i = 2,3,\ldots; \tag{19}$$

$$0 = a_2 p(0,0)L(V) - (a_1 + a_2)\int_0^V g(0,1,V-x)dL(x) -$$
$$-\mu_2 p(0,1) + \mu_1 p(1,1) + \mu_2 p(0,2); \tag{20}$$

$$0 = a_2 \int_0^V g(0,j-1,V-x)dL(x) - (a_1 + a_2)\int_0^V g(0,j,V-x)dL(x) -$$
$$-\mu_2 p(0,j) + \mu_1 p(1,j) + \mu_2 p(0,j+1),\ j = 2,3,\ldots; \tag{21}$$

$$0 = a_1 \int_0^V g(i-1,j,V-x)dL(x) + a_2 \int_0^V g(i,j-1,V-x)dL(x) -$$
$$-(a_1 + a_2)\int_0^V g(i,j,V-x)dL(x) - \mu_1 p(i,j) + \mu_1 p(i+1,j),\ i,j = 1,2,\ldots, \tag{22}$$

and the following equilibrium equations take place:

$$a_1 p(0,0)L(V) = \mu_1 p(1,0),\ a_2 p(0,0)L(V) = \mu_2 p(0,1), \tag{23}$$

$$(a_1 + a_2)\int_0^V g(i,j,V-x)dL(x) = \mu_1 p(i+1,j),\ i = 1,2,\dots,\ j = 0,1,\dots, \tag{24}$$

$$a_2 \int_0^V g(0,j,V-x)dL(x) = \mu_2 p(0,j+1),\ j = 1,2,\dots. \tag{25}$$

For similar classical preemptive discipline system $M/M/1/\infty$ with two priority classes we can obtain for stationary functions $r(i,j) = \mathbf{P}\{\eta_1 = i, \eta_2 = j\}$, $i,j = 0,1,\dots$, the following known [10] equations:

$$0 = -(a_1 + a_2)r(0,0) + \mu_1 r(1,0) + \mu_2 r(0,1); \tag{26}$$

$$0 = a_1 p(0,0) - (a_1 + a_2 + \mu_1)r(1,0) + \mu_1 r(2,0); \tag{27}$$

$$0 = a_1 r(i-1,0) - (a_1 + a_2 + \mu_1)r(i,0) + \mu_1 r(i+1,0),\ i = 2,3,\dots; \tag{28}$$

$$0 = a_2 r(0,0)L(V) - (a_1 + a_2 + \mu_2)r(0,1) + \mu_1 r(1,1) + \mu_2 r(0,2); \tag{29}$$

$$0 = a_2 r(0,j-1) - (a_1 + a_2 + \mu_2)r(0,j) + \mu_1 r(1,j) + \mu_2 r(0,j+1),\ j = 2,3,\dots; \tag{30}$$

$$0 = a_1 r(i-1,j) + a_2 r(i,j-1) - (a_1 + a_2 + \mu_1)r(i,j) + \mu_1 r(i+1,j),\ i,j = 1,2,\dots, \tag{31}$$

and the the following equilibrium conditions take place:

$$a_1 r(0,0) = \mu_1 r(1,0); \tag{32}$$

$$(a_1 + a_2)r(i,j) = \mu_1 r(i+1,j),\ i = 1,2,\dots,\ j = 0,1,\dots; \tag{33}$$

$$a_2 r(0,j) = \mu_2 p(0,j+1),\ j = 0,1,\dots. \tag{34}$$

Assume that numbers $r(i,j)$ sutisfy equations (26)–(34) and normalization condition $\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} r(i,j) = 1$. We suggest a hypothesis that

$$g(i,j,x) = Cr(i,j)L_*^{i+j}(x),\ i,j = 0,1,...,\ \max(i,j) \geq 1, \tag{35}$$

where C is some constant value. This hypothesis is truthful, as it follows from the direct substitution of the function (35) to equations (26)–(34). By this way we obtain equations (17)–(25) for the functions $r(i,j)$. So, for probabilities $p(i,j)$ we have

$$p(i,j) = Cr(i,j)L_*^{i+j}(V),\ i,j = 0,1,...,$$

where C can be obtained from the normalization condition $\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} p(i,j) = 1$.

It's known [10] that equations (26)–(34) can be solved by using the generation function $R(z_1, z_2) = \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} p(i,j) z_1^i z_2^j$. Then we have [10]

$$R(z_1, z_2) = \frac{(1-\rho)(1-\rho_1 z^*)}{(1-\rho_1 z^* - \rho_2 z_2)(1-\rho_1 z^* z_1)},$$

where $\rho_1 = a_1/\mu_1$, $\rho_2 = a_2/\mu_2$, $\rho = \rho_1 + \rho_2$,

$$z^* = \frac{a_1 + a_2(1-z_2) + \mu_1 - \sqrt{[a_1 + a_2(1-z_2) + \mu_1]^2 - 4a_1\mu_1}}{2a_1}.$$

Now we can calculate the numbers $r(i,j)$:

$$r(0,0) = 1-\rho,$$
$$r(i,j) = \frac{1}{i!j!} \cdot \frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j} R(z_1, z_2)\Big|_{z_1=z_2=0}, \quad i,j = 0,1,\ldots, \ \max(i,j) \geq 1.$$

So, the probabilities $p(i,j)$ can be determined as

$$p(0,0) = C(1-\rho),$$
$$p(i,j) = \frac{C}{i!j!} \cdot \frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j} R(z_1, z_2)\Big|_{z_1=z_2=0} L_*^{(i+j)} L(V), \quad i,j = 0,1,\ldots, \ \max(i,j) \geq 1,$$

where the constant value C can be calculated from the normalization condition, i.e.

$$C = \left[1 - \rho + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{i!j!} \cdot \frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j} R(z_1, z_2)\Big|_{z_1=z_2=0} L_*^{(i+j)} L(V)\right]^{-1}.$$

Now we can determine loss probabilities p_l^1 and p_l^2 for both priority demands accordingly from the followin equilibrium equations:

$$a_1(1-p_l^1) = \mu_1 \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} p(i,j), \ a_2(1-p_l^2) = \mu_2 \sum_{j=1}^{\infty} p(0,j),$$

whence we have

$$p_l^1 = 1 - \frac{\mu_1}{a_1} \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} p(i,j), \ p_l^2 = 1 - \frac{\mu_2}{a_2} \sum_{j=1}^{\infty} p(0,j).$$

Note that for more general case when demands from different priority have generally different volume distribution (with distribution functions $L_1(x)$ and $L_2(x)$ accordingly), the hypothesis

$$g(i,j,x) = Cr(i,j)L_{1*}^{(i)} * L_{2*}^{(j)}(x)$$

is not truth.

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A NEW ITERATIVE METHOD FOR SOLUTION OF THE DUAL PROBLEM OF GEOMETRIC PROGRAMMING

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Abstract. In this article a new method of optimal solution of the dual problem is proposed. This method is based on Newton's attraction theorem. An estimate of iteration convergence is given. The method uses some new procedure of correction of the current iteration. It is shown that the method uses matrix operations at each step of calculations and has a quadric speed of the convergence.

Let us consider initial geometric programming problem without constraints:

$$y = f(x) = u(x)_1 + u(x)_2 + \ldots + u(x)_n \to \min, \tag{1}$$

where $x > 0$, $q_i = u(x)_i = C_i \prod_{j=1}^{m} x_j^{a_{ij}}$,
and also the dual problem

$$V(\delta) = \prod_{i=1}^{n} \left(\frac{C_i}{\delta_i}\right)^{\delta_i} \to \max, \tag{2}$$

$\delta \in \delta_+ = \{\delta > 0 : \ \delta^T A = 0^T, \ \delta_1 + \delta_2 + \ldots + \delta_n = 1\}$.
In (1) and (2) we have denoted: x_j and δ_i are correspondingly direct and dual variables, C_i and a_{ij} are given coefficients, at that $C_i > 0$,

$$A = (a_{ij}) = \begin{pmatrix} B \\ H \end{pmatrix},$$

where B is given squarte submatrix with determinant (by assumption) $|B| \neq 0$, and H is submatrix from such d strings of matrix A, which do not belong to the basis B. The difficulty level is characterized by this number d:$d = n - m$. Let $d > 1$.

In initial problem we need to find vector $x = x_* > 0$ with components x_{j*}, such that the value $y_* = f(x_*)$ of the criterion function $f(x)$ from (1) is minimal.

In dual problem it is necessary to find vector $\delta = \delta^* \in \delta_+$ with components $\delta_j = \delta_j^*$, which gives the maximal value V^* of the criterion function $V(\delta)$ from (2). Conditions (3) can be rewritten in the following way

$$\delta_{(m)}^T = \delta_{(d)}^T Q \quad \text{and} \quad \delta_{(d)}^T \mu = 1. \tag{3}$$

Here $\delta_{(m)}^T = (\delta_1, \ldots, \delta_m)$ and $\delta_{(d)}^T = (\delta_{m+1}, \ldots, \delta_{m+d})$ are strings from the so named basic and free elements of the vector δ, $Q = -HB^{-1}$, and μ is a vector with components μ_i, which are equal to sums of string elements in some matrix $S = (Q, I)$.

For the extra difficulty level, i.e. when d >1, in [1] the solution of the problem is given in the following analytical form:

$$y_* = \prod_{i=1}^{n} \left(\frac{C_i}{\delta_i^*} \right)^{\delta i^*}, x_{j*} = \prod_{\nu=1}^{m} \left(\frac{\delta_\nu^* y_*}{C_\nu} \right)^{k_{j\nu}}, \tag{4}$$

where numbers $\delta_i = \delta_i{}^*$ are obtained by optimal choice as a result of the solution of the dual problem (2).

At that we consider two problems: 1) to find the set δ_+ of admissible vectors δ in dual problem, 2) to find the maximum of the dual function.

Function $V(\delta)$, where the dependence $\delta^T = \delta_{(d)}^T S$ is used, is called transformed dual function and is denoted by $V(\delta_{(d)})$, where $\delta_{(d)} \in \delta_{(d)+} = \{\delta_{(d)} > 0$: $\delta_{(m)} = Q^T \delta_{(d)} > 0,\ \mu^T \delta_{(d)} = 1\}$

In [1] we have proved, that logarithm of the transformed dual function is strictly concave when $\delta_{(d)} \in \delta_{(d)+}$. If there exists a vector $\delta_{(d)} \in \delta_{(d)+}$, then the dual problem has a unique solution $\delta_{(d)} = \delta_{(d)}{}^* \in \delta_{(d)+}$, which may be determined from the optimality condition $S\tilde{z} = 0$.

To the solution $\delta_{(d)}^* \in \delta_{(d)+}$ corresponds the maximal value $V^* = V(\delta_{(d)}^*)$ of the dual function in the domain $\delta_{(d)+}$.Vector $\tilde{z}$ has elements $\tilde{z}_j = \ln(C_j/\lambda\delta_j)$, $\lambda = V^*$.

Thus, the problem is reduced to the solution of the equation $S\tilde{z} = 0$ with respect to vector $\delta_{(d)} = \delta_{(d)}^*$. In order to get it the method of fixed point may be used. However in this method the information on vector's $\delta_{(d)} = \delta_{(d)}^*$ existence and uniqueness is not taken into consideration.

In the present paper a new iterative, based on this information, method for solution of the dual problem (2) is proposed.

Following [2], consider iterations

$$x^k = G(x^{k-1}), \quad k = 1, 2, \dots, \tag{5}$$

where G: $D \subset R^n \to R^n$.

Vector x^* is called an attraction point of the iteration (8), if there exists open neighborhood $\Re$ of this point such, that $\Re \subset D$, and for any point $x^0 \in \Re$ all the iterations, determined by formula (8), belong to D and converge to x^*.

Note that in given definition there are no conditions specifying the set $\Re$. Therefore in every particular case (when we need to use the concept of attractive point) the set $\Re$ is specifying according to some features of the particular problem.

Let for initial data of the dual problem vector $\delta_{(d)+} \neq \varnothing$, and the initial admissible solution is already founded: $\delta^0_{(d)} \in \delta_{(d)+}$ and also the meaning $V_0 = V(\delta^0_{(d)})$. Then, as it was noted before, there exists unique optimal solution $\delta^*_{(d)} \in \delta_{(d)+}$ of the dual problem.

At that this is straight forward task to prove that the set $\delta_{(d)+}$ is open and hence can be considered as an open neighborhood of the optimal point $\delta^*_{(d)}$.

Let us input new variables with the help of the following parities

$$w_i = \delta_i V^*, \quad i = 1, \dots, n,$$

where vector of basic variables and vector of free variables:

$$w_{(m)} = (w_1, w_2, \dots, w_m)^T, \quad w_{(d)} = (w_{m+1}, w_{m+2}, \dots, w_{m+d})^T.$$

Since $w_{(m)} = V^* \delta_{(m)}$, $w_{(d)} = V^* \delta_{(d)}$, then the set

$$\Re_+ = \{w_{(d)} > 0 : \ w_{(m)} = Q^T w_{(d)} > 0\} = \{\delta_{(d)} > 0, \delta_{(m)} = Q^T \delta_{(d)} > 0\}$$

represents itself as a variety of all positive solutions of the equation $A^T \delta = 0$ and can be considered as an open neighborhood of the optimal point $w^*_{(d)} = V^* \delta^*_{(d)}$.

If the set $\delta_{(d)+} \neq \varnothing$, then by criterion of optimality the unique solution $\delta^*_{(d)}$ of the dual problem contains in $\delta_{(d)+}$, moreover, $\delta_{(d)+} \subset \Re_+$.

In case $\Re_+ \neq \varnothing$ consider equation $S\tilde{z} = 0$, rewriting it in the form

$$G(w_{(d)}) \equiv \lambda (L(w_{(d)})')^T = -Sz(w_{(d)}) = 0 \quad \text{when} \quad w_{(d)} \in \Re_+,$$

where number $\lambda = V^*$, and $L(w_{(d)})'$ — is the derivative of Lagrange's function, i.e. $L(w_{(d)})'$ is the string having elements

$$\frac{\partial L}{\partial \delta_{m+i}} = -s_i^T z, \quad i = 1, 2, \dots, d,$$

and s_i^T is the string with number i of matrix S, at that vector $z = z(w_{(d)})$ has components

$$z_i = \ln(\lambda\delta_j/C_j) = \ln(w_j/C_j), \quad j = 1, 2, \dots, n.$$

In optimality criterion a transformed Lagrange's function is considered. Therefore the vector $w_{(m)}$ is expressed by $w_{(d)} > 0$ by formula $w_{(m)} = Q^T w_{(d)}$ in assumption that $w_{(m)} = Q^T w_{(d)} > 0$, and hence $G : R_+^d \to R^d$.

Here R_+^d is the set of positive vectors from R^d.

It is evident that $\Re_+ \subset R_+^d$, and the mapping G: $R_+^d \to R^d$ is differentiable in $w_{(d)}$ by Gateaux in open neighborhood $\Re_+ \subset R_+^d$, if $w_{(m)} = Q^T w_{(d)} > 0$.

By optimality criterion in point $w_{(d)}^* \in \Re_+$ the following parity is true

$$G(w_{(d)}^*) = Sz(w_{(d)}^*) = 0, \tag{6}$$

and hence vector function $G(w_{(d)})$ has components $g(w)_\nu = -s_\nu^T z(w)$, and the derivative $(g(w)_\nu)_i'$ of $g(w)_\nu$ in w_{m+i} is

$$(g(w)_\nu)_i' = -(s_{\nu 1}\ln(w_1/C_1) + \ldots + s_{\nu n}\ln(w_n/C_n))_i' = \\ = -s_\nu^T D_w s^i, \quad s^i = S^T e^i,$$

where (when $w_{(m)} = \mathrm{Q}^T w_{(d)} > 0$) the following parities hold

$$\frac{\partial w_j}{\partial w_{m+i}} = \frac{\partial \delta_j}{\partial \delta_{m+i}} = s_{ij}, \quad D_w = \mathrm{diag}(1/w_1, \dots, 1/w_n), \quad w_i = V^*\delta_i > 0.$$

At that derivative $G(w_{(d)})' = -SD_wS^T$, is continuous at point $w_{(d)}^* \in \Re_+$, and matrix $G'(w_{(d)}^*)$ is not degenerate, because by supposition the rank of the matrix S is $r(S) = m$, and

$$(r(S) = m) \Rightarrow (r(SD_wS^T) = m) \Leftrightarrow (|SD_wS^T| \neq 0).$$

By theorem from [2] and taking into consideration the optimality criterion come to the conclusion: when $\Re_+ \neq \varnothing$ the solution $w_{(d)}^*$ of the equation (9) exists, it is unique and is the point of attractions for Newton's iterations

$$w_{(d)}^k = w_{(d)}^{k-1} - G'(w_{(d)}^{k-1})^{-1} G(w_{(d)}^{k-1}), \quad k = 1, 2, \dots,$$

where $G(w_{(d)}^{k-1}) = -Sz(w_{(d)}^{k-1})$, $G'(w_{(d)}^{k-1})^{-1} = -(SD(w_{(d)}^{k-1})S^T)^{-1}$, if for each vector $w_{(d)}^k$ holds

$$w_{(m)}^k = Q^T w_{(d)}^k > 0.$$

In other transcript Newton's iterations are

$$w_{(d)}^k = w_{(d)}^{k-1} - A_{k-1} z(w_{(d)}^{k-1}), \quad k = 1, 2, \dots , \tag{7}$$

where

$$A_{k-1} = (SD(w_{(d)}^{k-1}) S^T)^{-1} S,$$

at that $D(w_{(d)}^{k-1}) = \mathrm{diag}(1/w_{1,k-1}, \dots , 1/w_{n,k-1})$, $z_{i,k-1} = \ln(w_{i,k-1}/C_i)$, $w_{n,k-1} > 0$, $i = 1, 2, \dots , n$, and it is assumed that

$$w_{(m)}^k = Q^T w_{(d)}^k > 0, \quad k = 1, 2, \dots . \tag{8}$$

By optimality criterion when $\Re_+ \neq \varnothing$ it is known the fact of existence of the solution $w_{(d)}^*$ of the equation $Sz(w_{(d)}) = 0$. Therefore the mentioned conclusion on adequacy of attractive point definition means that all the iterations $w_{(d)}^k$, defined by formula (10) when conditions (11) holds, for any initial point $w_{(d)}^0 \in \Re_+$ belong to R_+^d and converge to the solution $w_{(d)}^* \in \Re_+$.

Thus, iterations (10) allow to get values $w_{(d)}^k > 0$, after that it is necessary to verify inequality (11).

If $w_{(m)}^k = Q^T w_{(d)}^k > 0$, then vector $w_{(d)}^k > 0$ is admissible, i.e. $w_{(d)}^k \in \Re_+$. In this case come to the next iteration.

If the condition $w_{(m)}^k = Q^T w_{(d)}^k > 0$ does not hold, then the vector $w_{(d)}^k > 0$ is not admissible in the sense that $w_{(d)}^k \in R_+^d$, but $w_{(d)}^k \notin \Re_+$. Then we use correction method proposed in [4]. This method allows to find an admissible vector $\tilde{\delta}_{(d)}^k \in \delta_{(d)+}$ with the help of meaning $w_{(d)}^k \notin \Re_+$ if the column $\mu > 0$, and each sum of elements of every string in matrix Q^T is positive.

Theorem 1. Bearing in mind the set $\Re_+$, let us account that the vector $\mu > 0$, and each sum of elements of every string in matrix Q^T is positive.

Let the admissible initial value $\delta_{(d)}^0$ of the vector of free dual variables and also vector $w_{(d)}^0 = V_0 \delta_{(d)}^0 \in \Re_+$ are already founded. Here $V_0 = V(\delta_{(d)}^0)$ is the initial value of the dual function.

1. For any initial point $w_{(d)}^0 = V_0 \delta_{(d)}^0 \in \Re_+$, iterations defined by formula (10) of the type

$$w_{(d)}^k = w_{(d)}^{k-1} - A_{k-1} z(w_{(d)}^{k-1}), \quad k = 1, 2, \dots ,$$

converge to the unique value $w_{(d)}^* = V_{(d)}^* \delta_{(d)}^* \in \Re_+$ when the iteration's correction method is used at each iteration and the following conditions hold:

$w^k_{(m)} = Q^T w^k_{(d)} > 0, \quad k = 1, 2 \ldots$. Here $\delta^*_{(d)}$ is the unique solution of the dual problem, which corresponds to the maximal value

$$V^* = V(\delta^*_{(d)}) = \mu^T w^*_{(d)} = w^*_1 + w^*_2 + \ldots + w^*_n$$

of the dual function $V = V(\delta_{(d)})$.

In formulas for iterations the matrix

$$A_{k-1} = (SD(w^{k-1}_{(d)})S^T)^{-1}S, \quad \text{at that } S = (Q, I) \quad \text{and}$$

$$D(w^{k-1}_{(d)}) = \mathrm{diag}(1/w_{1,k-1}, 1/w_{2,k-1}, \ldots, 1/w_{n,k-1}), \quad w_{i,k-1} > 0,$$

where

$$(w_{1k}, \ldots, w_{mk}) = (w_{m+1,k}, \ldots, w_{n,k})Q, \quad z(w^{k-1}_{(d)}) = (z_{1,k-1}, \ldots, z_{n,k-1})^T,$$

$$z_{i,k-1} = \ln(w^{k-1}_i / C_i), \quad i = 1, 2, \ldots, n.$$

2. Let we got (as a result of calculations): optimal vector $w^*_{(d)}$ and the maximal value V^* of the dual function $V = V(\delta_{(d)})$.

The optimal vector δ^* of dual variables may be computed by formulas

$$\delta^*_{(d)} = w^*_{(d)}/V^*, \quad \delta^*_{(m)} = Q^T \delta^*_{(d)}, \quad \delta^* = \begin{pmatrix} \delta^*_{(m)} \\ \delta^*_{(d)} \end{pmatrix}.$$

3. Convergence speed of iteration process (10) is quadric. This means that there exists constant such that the following inequality holds

$$\|w^k_{(d)} - w^*_{(d)}\| \leqslant c\|w^{k-1}_{(d)} - w^*_{(d)}\|^2$$

for all sufficiently large numbers k.

References

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[3] A.I. Yatsko. System Interval Analysis. Elements of the Theory and Applications. Znanie, Moscow, 2005.